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# Algebra 3

By. Dr. Dj-  
Bellacour. Univ. Guelma.

- The ring of polynomials
- Square matrices
- Eigenvalues and Eigenvectors, eigenspaces,...
- Similar matrices
- Similarity and diagonalizability
- System of linear recurrence sequences
- System of linear differential equations
  - \* Finding the power of square matrices
  - \* Finding the exponential of square matrices
- Non-diagonalizable matrices
- Minimal polynomial
- Nilpotent matrices
- trigonalization of square matrices
- Jordan Canonical form
- ...

□

## Basic expressions in Algebra 3.

- Consider the matrix  $A$  given by
- the matrix  $A$  is nonsingular since  $\det(A) \neq 0$ .
- $\det(A) \equiv$  the determinant of  $A$ .
- We find the characteristic polynomial of the matrix  $A$  :
- Then by the hypothesis, we have
- According to the Cayley-Hamilton's theorem we have
- Let  $A = (a_{ij})$  be an  $n \times n$  matrix.
- Let  $A$  be a square matrix.
- From definition, we obtain
- By definition, we have
- By using Cayley-Hamilton theorem, we get
- Hence, therefore, Thus, it follows that
- which gives, and so, and hence,
- and therefore
- where

- We Compute the characteristic polynomial of the matrix  $A$  :
- In fact, we have
- Thus, we have
- Consider the following Counterexample :
- Let  $A$  be an  $n \times n$  matrix and  $h$  is an eigenvalue of  $A$ .
- The Subspace  $E_h$  is called the eigenspace of  $h$ .
- Let  $A$  be a square matrix of size  $n$ .
- A scalar  $h$  is an eigenvalue of  $A$  if and only if  $\det(A - hI) = 0$ , where  $I$  denotes the identity matrix.
- By definition,  $h$  is an eigenvalue of  $A$  if and only if, for some nonzero  $x$ , we have  

$$Ax = hx = h \cdot I \cdot x$$

$$\Leftrightarrow (A - hI) \cdot x = 0$$

$$\Leftrightarrow \det(A - hI) = 0.$$
- there are two cases, we distinguish two cases

- Let  $A$  be a square matrix of size  $n$ .  
Then the equation:

$$\det(A - \lambda I) = 0$$

is called the **characteristic equation** of  $A$ .

- So, the characteristic equation is

$$(\lambda + 1)^2 (\lambda - 3) = 0.$$

- $+$  : plus     $-$  : minus,     $\times$  : times

- $\frac{a}{b}$  : a over b

- Solving the system  $X' = AX$ :

- Solving the system of linear differential equations  $X' = AX$ :

- Solving the system of linear recurrence sequences  $X_n = AX_{n-1}$ :

- In particular,

- We find the dimension of the eigenspace corresponding to the eigenvalue  $\lambda = 1$ .

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Definition: Suppose  $A, B$  are two square matrices of size  $n \times n$ . We say  $A, B$  are **similar**, if  $A = P B P^{-1}$  for some invertible matrix  $P$ .

Definition: Suppose  $A$  is a square matrix of size  $n \times n$ . We say that  $A$  is **diagonalizable**, if there exists an invertible matrix  $P$  such that  $P^{-1} A P$  is a diagonal matrix.

Example: Let

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 3 \end{bmatrix}, \quad P = \begin{bmatrix} 1 & 1 & 5 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

Verify that  $A$  is diagonalizable, by computing  $P^{-1} A P$ .

**Solution**: After computation, we obtain

$$P^{-1} A P = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

④

- $A^2$  : A squared
- $A^3$  : A cubed
- $A^{4,5,\dots,k}$  : A to the power of 4, 5, ..., k
- $A^k$  : A to the k (to the power of k)
- $P^{-1}$  : P inverse

- x : times, multiplied by
- Therefore A is not diagonalizable.

- Next, we show that A is not diagonalizable. To do this, we have find and Count the dimensions of all the eigenspaces  $E_h$  :

- the dimension of  $E_h$  is called the geometric multiplicity of  $h$ . That is,  

$$\dim E_h = G_m(h).$$

- So,  $h = -1, 2$  are the only eigenvalues of  $A$ .

- Now, we compute the dimension  $\dim E_h$ :

- First, we find all the eigenvalues:

To do this, we solve  $\det(A - \lambda I) = 0$ .

- Therefore, by theorem in the course,  $A$  is diagonalizable.

- Exponential

$\pi : \pi \rightarrow \pi$

- polynomial

Sine  $\rightarrow$  Cos  $\pi$

- differential

Cosine  $\rightarrow$  Sin  $\pi$

- We can write by definition,

- on the other hand  $\equiv$  *d'autre part*,

- That is,

- We find the eigenvalues and associated eigenvectors of the matrix  $A$ :

- We compute  $p_A(\lambda)$ :

- We compute  $\dim E_h$ :

(6)

- Hence the set of eigenvectors associated with  $\lambda = 4$  is spanned by  $v_1 = (1, 1)$ .

Spanned  $\equiv$  equipped  $\equiv$  engendrer. i.e

$$E_\lambda = \text{Vect} \{ (1, 1) \}.$$

- $x \in A$  :  $x$  belongs to  $A$
- $x, y \in A$  :  $x, y$  belong to  $A$ .
- $\notin$  : does not belong to
- $A \cap B$  :  $A$  intersection  $B$
- $A \cup B$  :  $A$  union  $B$
- $\emptyset$   $\equiv$  the empty set
- $\mathbb{N}$  : the Set of positive integers  
= the Set of natural numbers
- $\mathbb{Z}$  : the Set of integers
- $\mathbb{Q}$  : the Set of rational numbers
- $\mathbb{R}$  : the Set of real numbers
- $\mathbb{R}^+$  : the Set of nonnegative real numbers
- $\mathbb{C}$  : the Set of Complex numbers
- $\mathbb{R}^n$  :  $n$ -dimensional Euclidean space (7)



$\mathbb{C}^n$ :  $n$ -dimensional Complex linear space

$\operatorname{Re}(z)$ : real part of the Complex number  $z$

$\operatorname{Im}(z)$ : Imaginary part of the Complex number  $z$ .

$|z|$  = modulus of the Complex number  $z$

i.e.  $|x+iy| = \sqrt{x^2 + y^2}$ ,  $x, y \in \mathbb{R}$ .

- $\sqrt{a}$  = the square root of  $a$
- $a^2$  =  $a$  squared.
- $\kappa^t$  = the transpose of  $\kappa$
- $\det(A)$  = determinant of a square matrix  $A$
- $\operatorname{rank}(A)$  = the rank of a matrix  $A$ .
- $\operatorname{Tr}(A)$  = the trace of a square matrix  $A$ .
- $A^t$  = the transpose of  $A$

- $\|A\|$  : the norm of  $A$ .
- $\bar{A}$  : the conjugate of the matrix  $A$ .
- $A^*$  : Conjugate transpose of  $A$ , i.e.  

$$A^* = (\bar{A})^t = \overline{A^t}$$
- $A^{-1}$  : inverse of square matrix  $A$  (if it exists)
- $I_n$  :  $n \times n$  unit matrix  
 = the  $n$ -by- $n$  identity matrix.
- $O_n$  =  $n \times n$  zero matrix

- $A$  is symmetric  $\Leftrightarrow A = A^t$
- $A$  is skew-symmetric  $\Leftrightarrow A = -A^t$

• First, we study the diagonalization of the matrix  $A$ :

$$\lim_{\kappa \rightarrow 0} \frac{e^{\kappa A} - I}{\kappa} = A$$

the limit of exponential  $\kappa A$  minus  $I$  over  $\kappa$  (as  $\kappa$  tends to zero)

- equals  $A$
- is equal to  $A$ .

(9)

- $$\sum_{k=0}^{\infty} \frac{A^k}{k!}$$

the sum for  $k$  from zero to infinity of  $A$  to the  $k$  over  $k$  factorial.

- Since  $A$  is diagonalizable, then

- From (1), (2) and (3), we have

- Let 
$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$a_{ij}$  are called the entries of  $A$   
 entries  $\equiv$  elements

- Then  $(1, 1, 0)$ ,  $(0, 1, -1)$  are the eigenvectors of  $A$  associated with the eigenvalue  $\lambda = -1$ .

- We find the eigenvectors associated with each of the eigenvalues:

- It is sufficient to prove that ...

- (Here  $O$  is the zero matrix).

- Taking the determinant of both sides of the equation, we find

(10)

• Similarity and diagonalizability :

• Then  $A$  is similar to  $B$  because  $A = P B P^{-1}$ ,

where

$$P = \begin{bmatrix} 4 & -3 \\ -1 & -1 \end{bmatrix}$$

• Thus,  $A$  is similar to  $B$ .

• Then, multiplying both sides of this equation on the right by  $P^{-1}$ , we obtain

• Multiplying both sides of this equation on the left by  $P^{-1}$ , we get

• This shows that...

• where  $Q$  is the matrix  $Q = P^{-1}$ , which is invertible.

• If  $A$  is similar to  $B$ , then there exists an invertible  $n \times n$  matrix  $P$  such that  $A = P B P^{-1}$ .

• which shows that  $A$  and  $B$  have the same characteristic equation and hence the same eigenvalues.

- If  $A = P D P^{-1}$ , then  $A^2 = P D^2 P^{-1}$  and in general,

$$A^k = P D^k P^{-1}.$$

Since  $D^k$  is easy to compute, then so is  $A^k$ .

- Let us show that the matrix  $A$  is nilpotent.
- which can be written as
- or, equivalently,
- We thus see that the eigenvalues of  $A$  are ...
- Since  $P = [v_1 \ v_2 \ \dots \ v_n]$  is invertible, then we know that the vectors  $v_1, v_2, \dots, v_n$  form a linearly independent set.
- We show that the matrix  $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$  is not diagonalizable.
- which means that ...
- After simple computation, we get
- We prove that  $A$  is **idempotent**. That is,  $A^2 = A$ . In fact, we have

- We can use this to compute  $A^k$  quickly for large  $k$ .
- the matrix  $D$  is a diagonal matrix, i.e. entries off the main diagonal are all zero.
- We have

$$A = P D P^{-1}$$

Again,

$$A^2 = P D^2 P^{-1}$$

In general, we have

$$A^k = P D^k P^{-1}$$

- Equivalently,
- Or, equivalently,
- $\Leftrightarrow$  if and only if
- $\Rightarrow$  implies
- An  $n \times n$  matrix with  $n$  distinct eigenvalues is diagonalizable.
- then,  $A$  has three linearly independent eigenvectors and it is therefore diagonalizable.

Theorem: An  $n \times n$  matrix with  $n$  distinct eigenvalues is diagonalizable.

- for some invertible matrix  $P$
- for some positive integer  $k$ .
- For every  $k \geq 0$ , we have
 
$$A^k = P \cdot D^k \cdot P^{-1}$$
- For every  $k \geq 0$ , we have
 
$$A^k = P \cdot T^k \cdot P^{-1}$$
- For each  $k \geq 0$ , we have
- $]a, b[$  : open interval
- $[a, b]$  : closed interval
- $E, F$  : Vector spaces
- $\mathcal{P}_n[x]$  : Vector space of polynomials of degree  $\leq n$ .
- $\leq$  : less or equals
- $\mathcal{M}_{n,m}(\mathbb{R})$  : Vector space of  $n \times m$  matrices
- $\alpha$  : alpha ,  $\beta$  : beta  $\gamma, \Gamma$  : gamma
- $\delta, \Delta$  : delta ,  $\varepsilon$  : epsilon
- $\zeta$  : zeta ,  $\theta$  : theta
- $\lambda$  : Lambda ,  $\mu$  : mu

- $\pi$ : pi,  $\rho$ : rho
  - $\tau$ : tau,  $\chi$ : chi
  - $\psi, \Psi$ : psi
  - $\omega, \Omega$ : omega
  - Explain why the diagonal entries of a skew-symmetric matrix must be zero.
  - $x^3 - 2x^2 + 7x + 2$   
 $x$  cubed minus two  $x$  squared plus seven  $x$  plus two.
  - $x \neq a$ :  $x$  is different from  $a$
  - Let  $p(x) = a_0 + a_1 x + a_2 x^2$  be a polynomial of degree 2 with real coefficients.
- For  $A \in \mathcal{M}_n(\mathbb{R})$  we define
- $$p(A) = a_0 I + a_1 A + a_2 A^2,$$
- where  $I$  is the  $n \times n$  identity matrix.
- It follows from (\*) that...



- $\sum_{i=1}^n A_i$  : the sum for  $i$  from one to  $n$  of  $A_i$ .
- : the sum of  $A_i$  for  $i$  from one to  $n$ .

• We will usually denote the minimal polynomial of  $A$  as  $m_A(x)$ .

• Let  $A = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 2 & -1 \\ 1 & 1 & 0 \end{bmatrix}$

the characteristic polynomial is

$$x^3 - 2x^2 + x = x(x-1)^2$$

- Since the minimal polynomial divides the characteristic polynomial, every root of the minimal polynomial is an eigenvalue.
- The converse is also true.

• Using the Cayley-Hamilton theorem, we obtain.

- Let  $S = \{v_1, v_2, \dots, v_k\}$ . We say that  $S$  is linearly independent if whenever  $c_1 v_1 + \dots + c_k v_k = 0$  we have  $c_1 = c_2 = \dots = c_k = 0$ .