Characteristic Polynomial

Definition: Let A be an $n \times n$ matrix. the characteristic polynomial of A, denoted by $P_A(x)$, is the polynomial defined by

$$P_A(x) = \det(A - xI),$$

where I denotes the n-by-n identity matrix.

Sometimes, we put

$$P_A(x) = \det(xI - A).$$

Example (1): Compute the characteristic polynomial of the matrix

$$A = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$$

Solution: From definition, we have

$$P_A(x) = \begin{vmatrix} 2-x & 1 \\ -1 & -x \end{vmatrix}$$

$$= (2-x)(-x) + 1 = x^2 - 2x + 1$$

Thus, $P_A(x) = x^2 - 2x + 1$

Second Method: We have

$$P_A(x) = \begin{vmatrix} 2-x & 1 \\ -1 & -x \end{vmatrix} \qquad \begin{array}{c} C_1 \\ \downarrow \\ \text{becomes} \\ C_1 - C_2 \end{array}$$

$$= \begin{vmatrix} 1-x & 1 \\ -(1-x) & -x \end{vmatrix}$$

$$= (1-x) \cdot \begin{vmatrix} 1 & 1 \\ -1 & -x \end{vmatrix}$$

$$= (1-x)\left[-x+1\right]$$

Then, $P_A(x) = (1-x)^2$

proposition: Let A be a square matrix of order 2, that is,

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Then

$$P_A(x) = x^2 - tr(A) + det(A)$$

proof: By definition, we have

$$P_A(x) = \begin{vmatrix} a-x & b \\ c & d-x \end{vmatrix}$$

$$= (a-x)(d-x) - cb$$

$$= x^2 - (a+d)x + (ad-cb)$$

$$= x^2 - tr(A) + det(A).$$

* $tr(A) \equiv$ the trace of A.
* $det(A) \equiv$ the determinant of A.

Test N° 01

Exercise 1 : We Calculate the characteristic polynomial of the matrix A_3.
In fact, we have

$$P_{A_3}(x) = \begin{vmatrix} 1-x & 1 & 1 \\ 1 & 1-x & 1 \\ 1 & 1 & 1-x \end{vmatrix} \quad \begin{array}{cc} C_1 & C_2 \\ \downarrow & \downarrow \\ \text{becomes} & \\ C_1 - C_2 & C_2 - C_3 \end{array}$$

$$= \begin{vmatrix} -x & 0 & 1 \\ x & -x & 1 \\ 0 & x & 1-x \end{vmatrix}$$

$$= x^2 \cdot \begin{array}{ccc} + & - & + \end{array}$$

$$\begin{vmatrix} -1 & 0 & 1 \\ 1 & -1 & 1 \\ 0 & 1 & 1-x \end{vmatrix}$$

$$= x^2 \cdot \left[-\left(-1 + x - 1\right) + 1 \right]$$

$$= x^2 \cdot (3 - x)$$

Thus, $f_{A_3}(x) = x^2 (3 - x)$

- Calculate the characteristic polynomial of A_4:

We have

$$f_{A_4}(x) = \begin{vmatrix} 1-x & 1 & 1 & 1 \\ 1 & 1-x & 1 & 1 \\ 1 & 1 & 1-x & 1 \\ 1 & 1 & 1 & 1-x \end{vmatrix}$$

$$= \begin{vmatrix} -x & 0 & 0 & 1 \\ x & -x & 0 & 1 \\ 0 & x & -x & 1 \\ 0 & 0 & x & 1-x \end{vmatrix}$$

$$= x^3 \cdot \begin{array}{cccc} + & - & + & - \\ \left|\begin{array}{cccc} -1 & 0 & 0 & 1 \\ 1 & -1 & 0 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 1-x \end{array}\right| \end{array}$$

$$= x^3 (-1) \cdot \begin{vmatrix} -1 & 0 & 1 \\ 1 & -1 & 1 \\ 0 & 1 & 1-x \end{vmatrix} + x^3 (-1) \cdot \begin{vmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= -x^3 \left[-(-1 + x - 1) + 1 \right] - x^3 \left[1 \right]$$

$$= -x^3 (3 - x) - x^3$$

$$= x^3 (x - 4)$$

then, $P_{A_4}(x) = x^3 (x-4)$.

Conclusion: In the general Case, we have

$$P_{A_n}(x) = \begin{cases} x^{n-1}(x-n) \; ; & \text{if } n \text{ is } \underline{\text{even}} \\ x^{n-1}(n-1-x) \; ; & \text{if } n \text{ is } \underline{\text{odd}} \end{cases}$$

Homework : Find the characteristic polynomial of the matrix :

$$A_n(\alpha) = \begin{bmatrix} \alpha & 1 & 1 & \cdots & 1 \\ 1 & \alpha & 1 & \cdots & 1 \\ \cdots & & & & \\ \cdots & & & & \\ 1 & 1 & 1 & \cdots & \alpha \end{bmatrix},$$

where $\alpha \in \mathbb{R}$.

Exercise 02 : For the matrix

$$A = \begin{bmatrix} 7 & -6 & -2 \\ 2 & 0 & -1 \\ 2 & -3 & 2 \end{bmatrix}$$

We show that $P_A(x) = (x-3)^3$.

By definition, we can write

$$P_A(x) = \begin{vmatrix} 7-x & -6 & -2 \\ 2 & -x & -1 \\ 2 & -3 & 2-x \end{vmatrix} \quad L_1 \to L_1 + (-2)L_2$$

$$= \begin{vmatrix} 3-x & -2(3-x) & 0 \\ 2 & -x & -1 \\ 2 & -3 & 2-x \end{vmatrix}$$

$$= (x-3) \cdot \begin{vmatrix} -1 & 2 & 0 \\ 2 & -x & -1 \\ 2 & -3 & 2-x \end{vmatrix} \qquad L_2 \to (-1)L_2 + L_3$$

$$= (x-3) \cdot \begin{vmatrix} -1 & 2 & 0 \\ 0 & x-3 & 3-x \\ 2 & -3 & 2-x \end{vmatrix}$$

$$= (x-3)^2 \cdot \begin{array}{c} \begin{matrix} + & \quad - \quad & + \end{matrix} \\ \begin{vmatrix} -1 & 2 & 0 \\ 0 & 1 & -1 \\ 2 & -3 & 2-x \end{vmatrix} \end{array}$$

$$= (x-3)^2 \cdot \left[-(2-x-3) - 2(-2) \right]$$

$$= (x-3)^3 .$$

Then, $P_A(x) = (x-3)^3$

Exercise 3 : Let $A \in \mathcal{M}_n(\mathbb{R})$ be a square matrix. We prove the result:

$$\forall r \neq 0 : P_{rA}(x) = r^n P_A\left(\frac{x}{r}\right).$$

In fact, from definition we have

$$P_{rA}(x) = \det(rA - xI_n)$$

$$= \begin{vmatrix} ra_{11}-x & ra_{12} & \cdots & ra_{1n} \\ ra_{21} & ra_{22}-x & & ra_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ ra_{n1} & ra_{n2} & \cdots & ra_{nn}-x \end{vmatrix}$$

$$= \begin{vmatrix} r\left(a_{11}-\frac{x}{r}\right) & ra_{12} & \cdots & ra_{1n} \\ ra_{21} & r\left(a_{22}-\frac{x}{r}\right) & \cdots & ra_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ ra_{n1} & ra_{n2} & \cdots & r\left(a_{nn}-\frac{x}{r}\right) \end{vmatrix}$$

$$= r^n \cdot P_A\left(\frac{x}{r}\right); \quad r \neq 0.$$

Exercise 04 : (i) Let A, B be two matrices such that $A^2 = B^2 = (AB)^2 = I_n \cdots\cdots (*)$

It follows from $(*)$ that

$$\begin{cases} A = A^{-1} \\ B = B^{-1} \\ AB = (AB)^{-1} = B^{-1} \cdot A^{-1} \end{cases}$$

Thus, $AB = BA$.

(ii) Let $A, B \in \mathcal{M}_n(\mathbb{R})$ with A is invertible

We will show that $P_{AB}(x) = P_{BA}(x)$.

In fact, we see that

$$\begin{aligned} P_{AB}(x) &= \det\left(AB - x I_n\right) \\ &= \det\left(ABAA^{-1} - xAA^{-1}\right) \\ &= \det\left[A\left(BA - xI_n\right)A^{-1}\right], \text{ where } x \in \mathbb{R} \\ &= \det(A) \det\left(BA - xI_n\right) \det\left(A^{-1}\right) \\ &= \det\left(BA - xI_n\right) \\ &= P_{BA}(x). \end{aligned}$$

This Completes the proof. $\square$

Exercise 05 : Consider the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$$

- We Calculate the eigenvalues of A :

From definition, we have

$$P_A(x) = \det(A - x I_2)$$

$$= \begin{vmatrix} 1-x & 2 \\ 3 & 2-x \end{vmatrix} \quad \begin{array}{c} C_1 \\ \downarrow \\ C_1 - C_2 \end{array}$$

$$= \begin{vmatrix} -1-x & 2 \\ 1+x & 2-x \end{vmatrix}$$

$$= (1+x) \cdot \begin{vmatrix} -1 & 2 \\ 1 & 2-x \end{vmatrix}$$

$$= (1+x) \cdot (-2 + x - 2)$$

$$= (1+x) \cdot (x-4)$$

that is $P_A(x) = (1+x)(x-4)$

Since the eigenvalues of A are the roots of $P_A(x)$, then $\lambda_1 = -1$ and $\lambda_2 = 4$.

That is, $sp(A) = \{-1, 4\}$

Next, we compute the eigenvectors of A:

From definition, we have

$$E_{\lambda_1} = \left\{ (x,y) \in \mathbb{R}^2 \Big/ \begin{array}{l} x + 2y = -x \\ 3x + 2y = -y \end{array} \right\},$$

$$= \left\{ (x,y) \in \mathbb{R}^2 \big/ \; y = -x \right\},$$

$$= \left\{ (x, -x) \; ; \; x \in \mathbb{R} \right\},$$

$$= \left\{ x(1,-1) \; ; \; x \in \mathbb{R} \right\},$$

$$= \text{Vect}\left\{ (1,-1) \right\}.$$

i.e E_{λ_1} is a vector space equipped by the vector $v_1 = (1,-1)$.

Using the same manner, we obtain

$$E_{\lambda_2} = \left\{ (x,y) \in \mathbb{R}^2 \Big/ \begin{array}{l} x + 2y = 4x \\ 3x + 2y = 4y \end{array} \right\},$$

$= \{ (x, y) \in \mathbb{R}^2 / \ y = \frac{3}{2} x \},$

$= \{ (x, \frac{3}{2} x) \ ; \ x \in \mathbb{R} \},$

$= \{ x (1, \frac{3}{2}) \ ; \ x \in \mathbb{R} \},$

$= \text{Vect} \{ (1, \frac{3}{2}) \},$

$= \text{Vect} \{ (2, 3) \}.$

That is $v_2 = (2, 3)$.

Exercise 06 : Let $A = \begin{bmatrix} 1 & 2 \\ 0 & 5 \end{bmatrix}$

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Since A is an upper triangular matrix, then the eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = 5$.

- The eigenvectors of A :

$E_{\lambda_1} = \{ (x, y) \in \mathbb{R}^2 / \begin{array}{l} x + 2y = x \\ 5y = y \end{array} \}$

$= \{ (x, 0) \in \mathbb{R}^2 / \ x \in \mathbb{R} \}$

$= \text{Vect} \{ (1, 0) \}$

then, $v_1 = (1, 0)$ is the Corresponding eigenvector of λ_1.

$E_{\lambda_2} = \{ (x, y) \in \mathbb{R}^2 / \begin{array}{l} x + 2y = 5x \\ 5y = 5y \end{array} \}$

$= \{ (x, y) \in \mathbb{R}^2 / \ y = 2x \}$

$= \{ (x, 2x) \ / \ x \in \mathbb{R} \} = \text{Vect} \{ (1, 2) \}.$

$\boxed{2}$ For $A = \begin{bmatrix} 2 & 6 \\ 0 & 2 \end{bmatrix}$

This is an upper triangular matrix. The eigenvalues of A are $\lambda = 2$ (double).

* The eigenvectors of A:

$$E_\lambda = \left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; \begin{array}{l} 2x + 6y = 2x \\ 2y = 2y \end{array} \right\}$$

$$= \left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; y = 0 \right\}$$

$$= \operatorname{Vect}\{(1,0)\}$$

Thus, $v = (1,0)$ is the corresponding eigenvector of λ.

$\boxed{3}$ For the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & -5 \end{bmatrix}$$

Also, A is an upper triangular matrix.

Then the eigenvalues of A are:

$$\begin{cases} \lambda_1 = 1 \\ \lambda_2 = 2 \\ \lambda_3 = -5 \end{cases}$$

* The eigenvectors of A

$$E_{\lambda_1} = \left\{ (x,y,z) \in \mathbb{R}^3 \;\middle/\; \begin{array}{l} x+2y+3z = x \\ 2y+3z = y \\ -5z = z \end{array} \right\}$$

$$= \left\{ (x,y,z) \in \mathbb{R}^3 \;/\; y=0,\ z=0 \right\}$$

$$= \left\{ (x,0,0) \in \mathbb{R}^3 \;/\; x \in \mathbb{R} \right\}$$

$$= \mathrm{Vect}\left\{ (1,0,0) \right\}.$$

$$E_{\lambda_2} = \left\{ (x,y,z) \in \mathbb{R}^3 \;\middle/\; \begin{array}{l} x+2y+3z = 2x \\ 2y+3z = 2y \\ -5z = 2z \end{array} \right\}$$

$$= \left\{ (x,y,z) \in \mathbb{R}^3 \;\middle/\; \begin{array}{l} z = 0 \\ x = 2y \end{array} \right\}$$

$$= \left\{ (2y,y,0) \in \mathbb{R}^3 \;/\; y \in \mathbb{R} \right\}$$

$$= \mathrm{Vect}\left\{ (2,1,0) \right\}$$

$$E_{\lambda_3} = \left\{ (x,y,z) \in \mathbb{R}^3 \;\middle/\; \begin{array}{l} x+2y+3z = -5x \\ 2y+3z = -5y \\ -5z = -5z \end{array} \right\}$$

$$= \left\{ (x,y,z) \in \mathbb{R}^3 \;\middle/\; \begin{array}{l} z = -\frac{7}{3}y \text{ and} \\ x+2y-7y = -5x \end{array} \right\}$$

$$= \left\{ (x,y,z) \in \mathbb{R}^3 \;\middle/\; \begin{array}{l} y = \frac{6}{5}x \\ z = -\frac{42}{15}x \end{array} \right\}$$

$= \left\{ \left(x, \frac{6}{5}x, \frac{-42}{15}x \right) ; x \in \mathbb{R} \right\}$

$= \text{Vect}\left\{ \left(1, \frac{6}{5}, -\frac{42}{15}\right) \right\}$

$= \text{Vect}\left\{ (5, 6, -14) \right\}.$

$\boxed{4}$ For the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

This is a lower triangular matrix. That is, $\lambda_1 = 1$, $\lambda_2 = 2$ (double) are the eigenvalues of A.

* The eigenvectors of A:

$$E_{\lambda_1} = \left\{ (x, y, z) \in \mathbb{R}^3 \Big/ \begin{array}{l} x = x \\ x + 2y = y \\ x + 2z = z \end{array} \right\}$$

$$= \left\{ (x, y, z) \in \mathbb{R}^3 \Big/ \begin{array}{l} x = -y \\ z = y \end{array} \right\}$$

$$= \left\{ (-y, y, y) \; ; \; y \in \mathbb{R} \right\}$$

$= \text{Vect}\{(-1,1,1)\}$. i.e, $v_1 = (-1,1,1)$.

$$E_{\lambda_2} = \left\{ (x,y,z) \in \mathbb{R}^3 \;/\; \begin{array}{l} x = 2x \\ x + 2y = 2y \\ x + 2z = 2z \end{array} \right\}$$

$$= \left\{ (x,y,z) \in \mathbb{R}^3 \;/\; \begin{array}{l} x = 0 \\ y, z \in \mathbb{R} \end{array} \right\}$$

$$= \{ (0,y,z) \;;\; y,z \in \mathbb{R} \}$$

$$= \{ y(0,1,0) + z(0,0,1) \;/\; y,z \in \mathbb{R} \}$$

$$= \text{Vect}\{ (0,1,0), (0,0,1) \}$$

Then, $v_2 = (0,1,0)$ and $v_2 = (0,0,1)$ are the Corresponding eigenvectors of λ_2.

$\boxed{5}$ For the matrix

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

This is a strictly upper triangular matrix

Then $\lambda = 0$ is the unique eigenvalue of A.

Now, we Compute the eigenspace E_λ:

$$E_\lambda = \{ (x,y,z) \in \mathbb{R}^3 \;/\; y + z = 0 \}$$

$$= \{ (x, y, z) \in \mathbb{R}^3 \mid y = -z \}$$

$$= \{ (x, y, -y) \in \mathbb{R}^3 \mid x, y \in \mathbb{R} \}$$

$$= \{ x(1,0,0) + y(0,1,-1) \mid x, y \in \mathbb{R} \}$$

$$= \text{Vect}\{ (1,0,0), (0,1,-1) \}$$

Hence $v_1 = (1,0,0)$ and $v_2 = (0,1,-1)$.

$\boxed{7}$ For the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

- We find the characteristic polynomial of A

$$P_A(x) = \begin{vmatrix} 1-x & 1 & 0 \\ 1 & 1-x & 0 \\ 0 & 0 & 2-x \end{vmatrix} \quad C_1 \downarrow C_1 + C_2$$

$$= \begin{vmatrix} \underline{2-x} & 1 & \underline{0} \\ \underline{2-x} & 1-x & \underline{0} \\ \underline{0} & 0 & \underline{2-x} \end{vmatrix}$$

$$= (2-x)^2 \cdot \begin{array}{c} \begin{array}{ccc} + & - & + \end{array} \\ \begin{vmatrix} 1 & 1 & 0 \\ 1 & 1-x & 0 \\ 0 & 0 & 1 \end{vmatrix} \end{array}$$

$$= (2-x)^2 \cdot \left[1 - x - 1 \right]$$

$$= (2-x)^2 \cdot (-x) = -x(2-x)^2.$$

<u>Conclusion:</u>

$\lambda_1 = 0$ is an eigenvalue of multiplicity 1

$\lambda_2 = 2$ " " " " " 2

* The eigenvectors of A :

$$E_{\lambda_1} = \left\{ (x,y,z) \in \mathbb{R}^3 \,/\, \begin{array}{l} x + y = 0 \\ 2z = 0 \end{array} \right\}$$

$$= \left\{ (x,y,z) \in \mathbb{R}^3 \,/\, \begin{array}{l} y = -x \\ z = 0 \end{array} \right\}$$

$$= \text{Vect}\left\{ (1,-1,0) \right\}.$$

$$E_{\lambda_2} = \left\{ (x,y,z) \in \mathbb{R}^3 \,/\, \begin{array}{l} x + y = 2x \\ x + y = 2y \\ 2z = 2z \end{array} \right\}$$

$= \{(x, y, z) \in \mathbb{R}^3 \mid x = y,\ z \in \mathbb{R}\}$

$= \{(x, x, z) \ ;\ x, z \in \mathbb{R}\}$

$= \{x(1,1,0) + z(0,0,1) \mid x, z \in \mathbb{R}\}$

$= \text{Vect}\{(1,1,0), (0,0,1)\}$.

$\boxed{8}$ For the matrix

$$A = \begin{bmatrix} a & 2 & 3 \\ 0 & 2a & 8 \\ 0 & 0 & 3a \end{bmatrix} \ ; \quad a \in \mathbb{R}$$

It is clear that A is an upper triangular matrix, which has a, $2a$, $3a$ as eigenvalues.

Setting $h_1 = a$, $h_2 = 2a$, $h_3 = 3a$

There are two cases:

Case 1: If $a = 0 = h$.

$$E_h = \left\{(x, y, z) \in \mathbb{R}^3 \;\middle|\; \begin{array}{l} 2y + 3z = 0 \\ 8z = 0 \end{array}\right\}$$

$$= \left\{(x, y, z) \in \mathbb{R}^3 \;\middle|\; \begin{array}{l} z = 0 \\ y = 0 \\ x \in \mathbb{R} \end{array}\right\}$$

$= \{(x, 0, 0) \ ;\ x \in \mathbb{R}\}$

$= \text{Vect}\{(1,0,0)\}$ i.e $v = (1,0,0)$.

Case 2: If $a \neq 0$.

There are three eigenvalues, λ_1, λ_2 and λ_3.

- $E_{\lambda_1} = \left\{ (x,y,z) \in \mathbb{R}^3 \;/\; \begin{array}{l} ax + 2y + 3z = ax \\ 2ay + 8z = ay \\ 3az = az \end{array} \right\}$

$= \left\{ (x,y,z) \in \mathbb{R}^3 \;/\; \begin{array}{l} z = 0 \quad (\text{because } a \neq 0) \\ y = 0 \\ x \in \mathbb{R} \end{array} \right\}$

$= \{ (x,0,0) \;;\; x \in \mathbb{R} \}$

$= \text{Vect}\{(1,0,0)\}$

i.e $v_1 = (1,0,0)$.

- $E_{\lambda_2} = \left\{ (x,y,z) \in \mathbb{R}^3 \;/\; \begin{array}{l} ax + 2y + 3z = 2ax \\ 2ay + 8z = 2ay \\ 3az = 2az \end{array} \right\}$

$= \left\{ (x,y,z) \in \mathbb{R}^3 \;/\; \begin{array}{l} z = 0 \\ y = \frac{a}{2} x \end{array} \right\}$

$= \{ (x, \frac{a}{2}x, 0) \;;\; x \in \mathbb{R} \}$

$$= \mathrm{Vect}\left\{ \left(1, \frac{a}{2}, 0\right) \right\}$$

$$= \mathrm{Vect}\left\{ (2, a, 0) \right\}, \text{ i.e } v_2 = (2, a, 0)$$

- $$E_{\lambda_3} = \left\{ (x,y,z) \in \mathbb{R}^3 \;\middle/\; \begin{array}{l} ax + 2y + 3z = 3ax \\ 2ay + 8z = 3ay \\ 3az = 3az \end{array} \right\}$$

$$= \left\{ (x,y,z) \in \mathbb{R}^3 \;\middle/\; \begin{array}{l} y = \frac{8}{a} z \\ ax + \frac{16}{a} z + 3z = 3ax \end{array} \right\}$$

$$= \left\{ (x,y,z) \in \mathbb{R}^3 \;\middle/\; \begin{array}{l} y = \frac{8}{a} z \\ x = \frac{1}{2a}\left(\frac{16+3a}{a}\right) z \end{array} \right\}$$

$$= \left\{ \left(\frac{1}{2a^2}(16+3a) z, \frac{8}{a} z, z \right) ; z \in \mathbb{R} \right\}$$

$$= \mathrm{Vect}\left\{ \left(\frac{1}{2a^2}(16+3a), \frac{8}{a}, 1 \right) \right\}$$

$$= \mathrm{Vect}\left\{ \left(16+3a, 16a, 2a^2 \right) \right\}$$

i.e $v_3 = (16+3a, 16a, 2a^2)$.

$\boxed{6}$ For the matrix

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 3 & 2 \end{bmatrix}$$

Note that A is a lower triangular matrix.
Then $h = 2$ is an eigenvalue of multiplicity 3

* The Corresponding eigenvector :

$$E_h = \left\{ (x,y,z) \in \mathbb{R}^3 \;/\; \begin{aligned} 2x &= 2x \\ x + 2y &= 2y \\ 3y + 2z &= 2z \end{aligned} \right\}$$

$$= \left\{ (x,y,z) \in \mathbb{R}^3 \;/\; \begin{aligned} x &= 0 \\ y &= 0 \\ z &\in \mathbb{R} \end{aligned} \right\}$$

$$= \left\{ (0,0,z) \;;\; z \in \mathbb{R} \right\}$$

$$= \text{Vect}\left\{ (0,0,1) \right\} \quad \text{i.e,} \quad v = (0,0,1).$$

* Eigenvalues and Eigenvectors

Definition: Let A be a $n \times n$ matrix. A scalar λ is called an eigenvalue of A if the equation $Ax = \lambda x$ has a nonzero solution x. Such a nonzero solution x is called an eigenvector corresponding to the eigenvalue λ.

$$Ax = \lambda x \iff (A - \lambda I)x = 0,$$

where I is the n-by-n identity matrix and 0 is the zero vector.

Remark: The fundamental theorem of algebra implies that the characteristic polynomial of an n-by n matrix A, being a polynomial of degree n, can be factored into the product of n linear terms. That is,

$$P_A(x) = (x - \lambda_1)^{n_1} (x - \lambda_2)^{n_2} \cdots (x - \lambda_k)^{n_k},$$

where $n_1 + n_2 + \cdots + n_k = n$.

- the algebraic multiplicity of λ_i is n_i.