Diagonalizable matrices

By

Bellaouar Djamel

University 08 Mai 1945 Guelma

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Definition

Let $A = (a_{ij}) \in \mathcal{M}_n(\mathbb{R})$ be a square matrix. A is said to be **diagonal**, if and only if

$$a_{ij} = 0, \ \ \forall\ i \neq j.$$

Or, equivalently

$$A = \begin{pmatrix} a_{11} & & & \\ & a_{22} & & \\ & & \ddots & \\ & & & a_{nn} \end{pmatrix}.$$

In this case, A is denoted by D. We also write $D = diag\,\{a_{11}, a_{22}, ..., a_{nn}\}$.

Definition

Let A be a square matrix. We say that A is **diagonalizable** if A is similar to a diagonal matrix D. That is, there exists an invertible matrix P such that $P^{-1}AP$ is diagonal, say D. That is,

$$A \text{ is diagonalizable} \Leftrightarrow \exists\ P \in \mathbb{GL}_n(\mathbb{R}) \text{ such that } A = PDP^{-1},$$

where $D = diag\{\lambda_1, \lambda_2, ..., \lambda_n\}$ and $\lambda_1, \lambda_2, ..., \lambda_n$ are the eigenvalues of A.

Example

Consider the following matrices

$$A = \begin{pmatrix} 5 & -4 \\ 2 & -1 \end{pmatrix},\ D = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \text{ and } P = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}.$$

Compute PDP^{-1}. What can we conclude?

Example

By computation, we obtain

$$\begin{aligned} PDP^{-1} &= \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 6 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 5 & -4 \\ 2 & -1 \end{pmatrix} = A. \end{aligned}$$

Thus, $A = PDP^{-1}$ and so A is diagonalizable.

But the question posed is how to determine P and D if they exist? How to diagonalize a matrix?. Here is the following theorem.

Theorem (Necessary and sufficient condition for diagonalization)

Let $A \in \mathcal{M}_n(\mathbb{R})$ be a square matrix. A is diagonalizable, if and only if, there exists a basis B of $\mathbb{R}^n$ formed by n eigenvectors of A.

Proof.

Assume that A is diagonalizable. That is, there exists an invertible matrix P such that

$$A = PDP^{-1}.$$

Or, equivalently

$$P^{-1}AP = D.$$

Setting

$$P = \begin{bmatrix} y_1 & y_2 & \dots & y_n \end{bmatrix} = \begin{bmatrix} Pe_1 & Pe_2 & \dots & Pe_n \end{bmatrix},$$

where $(e_i)_{1 \le i \le n}$ is the canonical basis of $\mathbb{R}^n$, $\square$

and let

$$
\begin{aligned}
D &= \begin{pmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{pmatrix} = diag\left\{d_{1,}d_2, ..., d_n\right\} \\
&= \left[\begin{array}{cccc} d_1 e_1 & d_2 e_2 & \dots & d_n e_n \end{array}\right].
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\left[\begin{array}{cccc} Ay_1 & Ay_2 & \dots & Ay_n \end{array}\right] &= AP = I_n AP = PP^{-1}AP = PD \\
&= P\left[\begin{array}{cccc} d_1 e_1 & d_2 e_2 & \dots & d_n e_n \end{array}\right] \\
&= \left[\begin{array}{cccc} d_1 Pe_1 & d_2 Pe_2 & \dots & d_n Pe_n \end{array}\right] \\
&= \left[\begin{array}{cccc} d_1 y_1 & d_2 y_2 & \dots & d_n y_n \end{array}\right].
\end{aligned}
$$

We deduce that for each $i \in \overline{1, n}$, $Ay_i = d_i y_i$. Then y_i is an eigenvector of A corresponding to d_i. Since P is invertible, then the familly $B = \left\{y_1, y_2, ..., y_n\right\}$ is a basis of $\mathbb{R}^n$.

Conversely, assume that $\mathbb{R}^n$ has a basis $B = \{x_1, x_2, ..., x_n\}$ formed by n eigenvectors of A. In this case, we put

$$P = \left[\begin{array}{cccc} x_1 & x_2 & \ldots & x_n \end{array}\right].$$

It follows that

$$\begin{aligned} AP &= \left[\begin{array}{cccc} Ax_1 & Ax_2 & \ldots & Ax_n \end{array}\right] \\ &= \left[\begin{array}{cccc} \lambda_1 x_1 & \lambda_2 x_2 & \ldots & \lambda_n x_n \end{array}\right], \end{aligned}$$

where $(\lambda_i)_{1\leq i\leq n}$ are the eigenvalues of A associated with $(x_i)_{1\leq i\leq n}$, respectively.

Therefore,

$$\begin{aligned} AP &= \begin{pmatrix} \lambda_1 x_{11} & \lambda_2 x_{21} & \dots & \lambda_n x_{n1} \\ \lambda_1 x_{12} & \lambda_2 x_{22} & \dots & \lambda_n x_{n2} \\ \vdots & \vdots & \dots & \vdots \\ \lambda_1 x_{1N} & \lambda_2 x_{2N} & \dots & \lambda_n x_{nn} \end{pmatrix} = \begin{pmatrix} x_{11} & x_{21} & \dots & x_{n1} \\ x_{12} & x_{22} & \dots & x_{n2} \\ \vdots & \vdots & \dots & \vdots \\ x_{1N} & x_{2N} & \dots & x_{nn} \end{pmatrix} \times \\ & \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix} \\ &= PD. \end{aligned}$$

Hence $A = PDP^{-1}$, where D is diagonale and P is invertible. The proof is finished.

Corollary

Let $A \in \mathcal{M}_n(\mathbb{R})$ be a diagonalizable matrix. There exists a basis $B = \{x_1, x_2, ..., x_n\}$ of $\mathbb{R}^n$ formed by n eigenvectors A.

Proof.

Assume that $A = PDP^{-1}$. We know that $\{e_1, e_2, ..., e_n\}$ are eigenvectors of D associated with $diag(D)$, i.e.,

$$De_i = P^{-1}APe_i = \lambda_i e_i, \text{ for } i = 1, 2, ..., n.$$

Hence

$$APe_i = \lambda_i Pe_i, \text{ for } i = 1, 2, ..., n.$$

That is, $\{Pe_i\}_{1 \leq i \leq n}$ are eigenvectors of A. Since P is invertible, then $\{Pe_i\}_{1 \leq i \leq n}$ is a basis of $\mathbb{R}^n$. $\square$

Conclusion. Let $A \in \mathcal{M}_n(\mathbb{R})$ be a square matrix and let $\lambda_1, \lambda_2, ..., \lambda_k$ be its eigenvalues. Let $A_m(\lambda_i)$ and $G_m(\lambda_i)$ denote the algebraic multiplicity and the geometric multiplicity of λ_i, respectively. Then A is diagonalizable if and oly if

$$A_m(\lambda_i) = G_m(\lambda_i), \text{ for } i = 1, 2, ..., k.$$

Corollary

Let $A \in \mathcal{M}_n(\mathbb{R})$ be a square matrix. Assume that

$$p_A(x) = (x - \lambda_1)^{\alpha_1} (x - \lambda_2)^{\alpha_2} \dots (x - \lambda_k)^{\alpha_k}, \text{ where } k \leq n.$$

Then A is diagonalizable if and only if $\dim E_{\lambda_i} = \alpha_i$, *for* $i = 1, 2, ..., k$.

Example

For the following matrices, by calculating the eigenpairs one has:

Matrix	$p_A(x)$	$Sp(A)$	$A_m(\lambda)$	$G_m(\lambda)$
$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$	$x(x-2)^2$	0 2	1 2	1 2
$B = \begin{pmatrix} 2 & 1 & 1 \\ 2 & 1 & -2 \\ -1 & 0 & -2 \end{pmatrix}$	$(x+1)^2(x-3)$	-1 3	2 1	1 1
$C = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & -3 \\ 1 & -1 & 0 \end{pmatrix}$	$(x+1)(x-1)(x-3)$	-1 1 3	1 1 1	1 1 1

We deduce that A and B are diagonalizable, but B is not.

We see also the following example:

Example

Show that the following matrix is diagonalizable.

$$A = \begin{pmatrix} 4 & 1 & 1 & 1 \\ 1 & 4 & 1 & 1 \\ 1 & 1 & 4 & 1 \\ 1 & 1 & 1 & 4 \end{pmatrix}$$

Solution. The characteristic polynomial is $p_A(x) = (x-7)(x-3)^3$. The eigenvalues of A are $\lambda_1 = 7$ (simple), and $\lambda_2 = 3$ (triple). The associated eigenvectors are $v_1 = (1,1,1,1)$ for λ_1, $v_2 = (-1,1,0,0)$, $v_3 = (-1,0,1,0)$ and $v_4 = (-1,0,0,1)$ for λ_2. The matrix A is therefore diagonalizable since $\dim E_{\lambda_i} = A_m(\lambda_i)$, for $i = 1, 2$.

From Theorem 5, we have the following corollary:

Corollary

Let $A \in \mathcal{M}_n(\mathbb{R})$ be a square matrix. If A has n distinct eigenvalues, then A is diagonalizable.

Proof.

Since $A \in \mathcal{M}_n(\mathbb{R})$ and A has n distinct eigenvalues, then $\dim E_{\lambda_i} = 1 = A_m(\lambda_i)$,

for $i = 1, 2, ..., n$. $\square$

Proposition

Let A and B be two diagonalizable matrices with $P^{-1}AP = D_1$ and $P^{-1}BP = D_2$ for some invertible matrix P. Then $AB = BA$.

Proof.

We can easily verify that if $P^{-1}AP = D_1$ and $P^{-1}BP = D_2$, it follows that

$$\begin{cases} A = PD_1P^{-1} \\ B = PD_2P^{-1}. \end{cases}$$

Note that $D_1D_2 = D_2D_1$, and therefore

$$AB = PD_1D_2P^{-1} = PD_2D_1P^{-1} = PD_2P^{-1}PD_1P^{-1} = BA.$$

Hence the result. $\square$

Corollary

Let $A \in \mathcal{M}_n(\mathbb{R})$ be a square matrix, and assume that A has a unique eigenvalue λ. Then A is diagonalizable if and only if $A = \lambda I_n$.

Proof.

It is clear that if $A = \lambda I_n$, then A is diagonalizable. Conversely, assume that $A \in \mathcal{M}_n(\mathbb{R})$ is diagonalizable and has a unique eigenvalue λ, there is therefore an invertible matrix P such $P^{-1}AP$ is diagonal. We put $P^{-1}AP = D$, where $diag\,(D) = Sp\,(A) = \{\lambda\}$. It follows that

$$A = P \begin{pmatrix} \lambda & & \\ & \ddots & \\ & & \lambda \end{pmatrix} P^{-1} = \lambda P \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix} P^{-1} = \lambda P I_n P^{-1} = \lambda I_n.$$

This completes the proof. $\square$

Proposition

Let A be a diagonalizable matrix [a] *with $Sp(A) = \{\lambda_1, \lambda_2, ..., \lambda_n\}$. Then*

$$\det(A) = \lambda_1 \lambda_2 .... \lambda_n. \tag{1}$$

[a] Note that the result of Equation (1) is always true for any matrix $A \in \mathcal{M}_n(\mathbb{C})$ which may or may not be diagonalizable.

Proof.

Assume that $A = PDP^{-1}$, where $D = diag\{\lambda_1, \lambda_2, ..., \lambda_n\}$. Then

$$\begin{aligned}
\det(A) &= \det\left(PDP^{-1}\right) \\
&= \det(P)\det(D)\det\left(P^{-1}\right) \\
&= \det(D) \\
&= \lambda_1 \lambda_2 .... \lambda_n.
\end{aligned}$$

Definition

$\lambda \in \mathbb{R}$ is called the eigenvalue of multiplicity m if and only if

$$p_A(x) = (x - \lambda)^m q(x) \text{ with } q(\lambda) \neq 0.$$

Example

Let

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 2 & 1 & -2 \\ -1 & 0 & -2 \end{pmatrix}$$

Then $p_A(x) = (x-3)(x+1)^2$ and A cannot be diagonalizable on either $\mathbb{R}$ or $\mathbb{C}$. Indeed, we have

$$E_{-1} = Vect\{(1, -2, -1)\}$$

In $\mathbb{R}^3$ or $\mathbb{C}^3$, E_{-1} is a vector space of dimension 1 equipped by $(1, -2, -1)$. Since -1 is an eigenvalue of A of multiplicity 2, A is not diagonalizable.

A classical application is the computing of the powers of a matrix A. Assume that A is given to be diagonalizable. That is, there exist P and D such that

$$D = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ & & \ddots & \\ 0 & 0 & \dots & \lambda_n \end{pmatrix}$$

and $D = P^{-1}AP$. For each $k \geq 0$ we have

$$A^k = PD^kP^{-1}.$$

The preceding formula then generalizes to $k \in \mathbb{Z}$. The matrix A is then invertible if, and only if, D is invertible and

$$A^{-1} = PD^{-1}P^{-1}.$$

Exercise

Consider the matrix

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}.$$

Calculate A^n for every $n \geq 0$.

Solution

We start by computing the characteristic polynomial of A

$$\begin{aligned} p_A(x) &= \begin{vmatrix} 2-x & -1 \\ -1 & 2-x \end{vmatrix} = \begin{vmatrix} 1-x & -1 \\ 1-x & 2-x \end{vmatrix} \\ &= (1-x) \begin{vmatrix} 1 & -1 \\ 1 & 2-x \end{vmatrix} = (1-x)(3-x). \end{aligned}$$

Then $Sp(A) = \{1, 3\}$.

Next, we find the eigenvectors of A :

$$\begin{aligned} E_1 &= \left\{ (x,y) \in \mathbb{R}^2;\ \begin{array}{c} 2x - y = x \\ -x + 2y = y \end{array} \right\} \\ &= Vect\left\{ (1,1) \right\}. \end{aligned}$$

and also we have

$$\begin{aligned} E_3 &= \left\{ (x,y) \in \mathbb{R}^2;\ \begin{array}{c} 2x - y = 3x \\ -x + 2y = 3y \end{array} \right\} \\ &= Vect\left\{ (1,-1) \right\}. \end{aligned}$$

We put

$$P = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},\ D = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$$

It follows that

$$\begin{aligned} A^n &= \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1^n & 0 \\ 0 & 3^n \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 3^n \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \\ &= \begin{pmatrix} \dfrac{1+3^n}{2} & \dfrac{1-3^n}{2} \\ \dfrac{1-3^n}{2} & \dfrac{1+3^n}{2} \end{pmatrix}. \end{aligned} \tag{2}$$

Example

Consider the matrix

$$A = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix}.$$

Calculate $\lim\limits_{n \to +\infty} A^n$.

First, let us calculate the eigenvalues and eigenvectors of A. From computation, we find

$$\begin{cases} \lambda_1 = 1, \ v_1 = (1, 1), \\ \lambda_2 = \frac{1}{4}, \ v_2 = (-2, 1). \end{cases}$$

Since $A = PDP^{-1}$, then $A^k = PD^kP^{-1}$, where $P = \begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix}$ and $D = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{4} \end{pmatrix}$. It follows that

$$\begin{aligned}
\lim_{n\to+\infty} A^n &= \lim_{n\to+\infty} \begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1^n & 0 \\ 0 & \left(\frac{1}{4}\right)^n \end{pmatrix} \begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ -\frac{1}{3} & \frac{1}{3} \end{pmatrix} \\
&= \begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \lim_{n\to+\infty} \left(\frac{1}{4}\right)^n \end{pmatrix} \begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ -\frac{1}{3} & \frac{1}{3} \end{pmatrix} \\
&= \begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ -\frac{1}{3} & \frac{1}{3} \end{pmatrix} = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix}.
\end{aligned}$$

Example

Consider the mapping

$$\begin{aligned} f \quad &: \quad \mathbb{R}_3[X] \longrightarrow \mathbb{R}_3[X] \\ p \quad &\mapsto \quad f(p) = 3xp - \left(x^2 - 1\right) p' \end{aligned}$$

and let $\mathcal{B} = \left\{1, x, x^2, x^3\right\}$ be the canonical basis of $\mathbb{R}_3[X]$.

1. Calculate $M_f(\mathcal{B})$.
2. Is f diagonalizable? if so, give the diagonalization.

Solution. There are two steps:

▷ The calculation of $M_f(\mathcal{B})$. We see that

$$\left\{ \begin{array}{c} f(1) = 3x = 0 + 3x + 0x^2 + 0x^3 \\ f(x) = 1 + 2x^2 = 1 + 0x + 2x^2 + 0x^3 \\ f\left(x^2\right) = 2x + x^3 = 0 + 2x + 0x^2 + 1x^3 \\ f\left(x^3\right) = 3x^2 = 0 + 0x + 3x^2 + 0x^3 \end{array} \right.$$

Which gives

$$M_f(\mathcal{B}) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 3 & 0 & 2 & 0 \\ 0 & 2 & 0 & 3 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

▷ Let us calculate the characteristic polynomial of $M_f\left(\mathcal{B}\right)$. Indeed, we have

$$p_{M_f\left(\mathcal{B}\right)}\left(x\right) = \begin{vmatrix} -x & 1 & 0 & 0 \\ 3 & -x & 2 & 0 \\ 0 & 2 & -x & 3 \\ 0 & 0 & 1 & -x \end{vmatrix} = x^4 - 10x^2 + 9.$$

The eigenvalues of A are $\{-1, 1, -3, 3\}$. From Corollary **??**, $M_f\left(\mathcal{B}\right)$ is diagonalizable.

▷ Diagonalization of $M_f\left(\mathcal{B}\right)$: First, let us calculate the eigenvectors of $M_f\left(\mathcal{B}\right)$, we obtain

$$M_f\left(\mathcal{B}\right) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -3 & -1 & 1 & 3 \\ 3 & -1 & -1 & 3 \\ -1 & 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} -3 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} \frac{1}{8} & -\frac{1}{8} & \frac{1}{8} & -\frac{1}{8} \\ \frac{3}{8} & -\frac{1}{8} & -\frac{1}{8} & \frac{3}{8} \\ \frac{3}{8} & \frac{1}{8} & -\frac{1}{8} & -\frac{3}{8} \\ \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} \end{pmatrix}$$

Ex 01. Let $A \in \mathcal{M}_3(\mathbb{R})$ be a square matrix such that

$$p_A(x) = (x-1)(x-2)^2.$$

Is it diagonalizable ?

Ex 02. Let f be a diagonalizable endomorphism over a vector space E. Prove that

$$E = \ker f \oplus \operatorname{Im} f.$$

Ex 03. Let f be a diagonalizable endomorphism over a vector space satisfying $f^k = id_E$ for some natural integer k. Show that $f^2 = id_E$.

Ex 04. Let A be a 3-by-3 matrix given by

$$A = \begin{pmatrix} 0 & 1 & 0 \\ -4 & 4 & 0 \\ -2 & 1 & 2 \end{pmatrix}.$$

1. Is the matrix A diagonalizable?
2. Calculate $(A - 2I_3)$ and $(A - 2I_3)^n$ for every $n \in \mathbb{N}$. Deduce an explicit formula for A^n.

Ex 05. Let M be a complex square matrix satisfying $M^k = I$ for some positive integer k. Prove that M is diagonalizable.

Ex 06. Study the diagonalization of the matrix

$$A = \begin{pmatrix} 3 & 0 & 0 \\ 4 & 1 & 2 \\ a & 0 & 3 \end{pmatrix}; \ a \in \mathbb{R}$$

Ans. A is diagonalizable $\Leftrightarrow a = 0$.

Ex 07. Verify that the matrix

$$A = \begin{pmatrix} 2 & -2 & 2 \\ 0 & 1 & 1 \\ -4 & 8 & 3 \end{pmatrix}$$

is diagonalizable. **Ans :** $Sp(A) = \{1, 2, 3\}$.

Ex 08. Study the diagonalization of the matrix

$$A = \begin{pmatrix} a & 1 & -1 \\ 0 & a & 2 \\ 0 & 0 & b \end{pmatrix}; \; a, b \in \mathbb{R}.$$

Ex 09. Check that the matrices of the form

$$A = \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}; \; c \neq 0$$

are not diagonalizable.

Ex 10. Consider the two matrices

$$A = \begin{pmatrix} 2 & 1 & -1 \\ 0 & 2 & -1 \\ -3 & -2 & 3 \end{pmatrix} \text{ and } B = \begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix}.$$

- Check that A and B have the same eigenvalues.
- Prove that $A \nsim B$.

Ex 11. Find a matrix $A \in \mathcal{M}_2(\mathbb{R})$ which is not diagonalizable.

Ex 12. Let

$$A = S \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} S^{-1};\ S \in \mathbb{GL}_2(\mathbb{R}) \text{ and } \lambda_{1,}\lambda_2 \in \mathbb{R}.$$

Calculate the determinant of A and A^{-1}.

Ex 13. Calculate the eigenvalues and the eigenvectors of the following matrices. Are they diagonalizable? If so, determine a basis of eigenvectors.

$$\begin{pmatrix} 4 & 1 \\ 0 & 3 \end{pmatrix}, \begin{pmatrix} 2 & 4 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & -3 \\ 1 & -3 & 1 \end{pmatrix},$$
$$\begin{pmatrix} 1 & -2 & -1 \\ 2 & 1 & -2 \\ 2 & 2 & -3 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$
$$, \begin{pmatrix} -7 & -2 & 1 \\ 28 & 8 & -4 \\ 31 & 10 & -5 \end{pmatrix}, \begin{pmatrix} 7 & 4 & 0 & 0 \\ -12 & -7 & 0 & 0 \\ 20 & 11 & -6 & -7 \\ -12 & -6 & 6 & 6 \end{pmatrix}$$

Ex 14. Let $A \in \mathcal{M}_n(\mathbb{R})$. Prove that A is diagonalozable $\Leftrightarrow$ A^t is diagonalizable.

Ex 15. Study the diagonalization of the following matrix

$$A = \begin{pmatrix} 1 & a & b & c \\ 0 & 1 & d & e \\ 0 & 0 & 2 & f \\ 0 & 0 & 0 & 3 \end{pmatrix}; a \neq 0 \text{ and } b, c, d, e, f \in \mathbb{R}.$$

Ex 16. Study the diagonalization of the following matrices

$$A_1 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \text{ and } A_2 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

Ans. A_1 : yes, A_2 : no

Ex 17. Discuss the diagonalization, according to $a, b \in \mathbb{R}$ of the matrix

$$A = \begin{pmatrix} a & b & a-b \\ b & 2b & -b \\ a-b & -b & a \end{pmatrix};\ ab \neq 0$$

and find α, β and γ for which

$$A^3 = \alpha A^2 + \beta A + \gamma I_3.$$

Ans. $p_A(x) = x(x-3b)(x-2a+b)$.

Ex 18. Determine the real number a for which the matrix

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & a \\ 0 & 0 & 1 & -a \end{pmatrix}$$

is diagonalizable.

Ex 19. Let $A \in \mathcal{M}_n(\mathbb{R})$ be a diagonalizable matrix with $Sp\,(A) = \{-1, 1\}$. Prove that $A = A^{-1}$.

Ex 20. Let

$$A = \begin{pmatrix} 9 & 0 & 0 \\ -5 & 4 & 0 \\ -8 & 0 & 1 \end{pmatrix}.$$

i) Prove that A is diagonalizable and find a matrix $P \in \mathbb{GL}_3\,(\mathbb{R})$ for which $P^{-1}AP$ is diagonal.

ii) Calculate A^n, $n \in \mathbb{N}$ and deduce an explicit formula of e^A.

Ex 21. Let $A \in \mathcal{M}_n(\mathbb{R})$ such that $A^2 = A$. Prove that A is diagonalizable.

Ex 22. Calculate $p(A) = 2A^8 - 3A^5 + A^4 + A^2 - 4I_3$, where A is given by

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Ex 23. Consider the matrix

$$A_\alpha(n) = \begin{pmatrix} 1 & \dfrac{\alpha}{n} \\ \dfrac{-\alpha}{n} & 1 \end{pmatrix}$$

Prove that

$$\lim_{n \to +\infty} A_\alpha(n) = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}.$$

Ex 24. Let A be the matrix given by

$$A = \begin{pmatrix} 0.6 & 0.8 \\ 0.4 & 0.2 \end{pmatrix}$$

Verify that

$$\lim_{n \to +\infty} A^n = \begin{pmatrix} \frac{2}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} \end{pmatrix}$$

Ex 25. Consider the matrix

$$A = \begin{pmatrix} 9 & 0 & 0 \\ -5 & 4 & 0 \\ -8 & 0 & 1 \end{pmatrix}$$

Calculate A^n, for $n \in \mathbb{N}$. **Ans.**

$$A^n = \begin{pmatrix} 9^n & 0 & 0 \\ 4^n - 9^n & 4^n & 0 \\ 1 - 9^n & 0 & 1 \end{pmatrix}.$$

Ex 26. Let

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

1. Diagonalize the matrix B.
2. Is matrix A similar to B?

Ex 27. Let $n \geq 2$. Let A be the real $n \times n$ matrix of coefficients $a_{ij} = 0$ if $i = j$ and $a_{ij} = 1$; otherwise. We put $B = A + I_n$.

1. What is the rank of the matrix B? Deduce that -1 is an eigenvalue of A and determe the dimension of the associated eigenspace.
2. Calculate

$$A \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix},$$

and deduce a new eigenvalue of A.

3. Justify that A is diagonalizable, and give its characteristic polynomial.
4. Give an invertible matrix P and a matrix D such that $A = PDP^{-1}$ (one does not ask to calculate P^{-1}).