Oct 07 - 2019

Homework N° 01

Consider the following matrix

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$$

① Calculate the characteristic polynomial of A, say $P_A(x)$.

② Verify that $P_A(A) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O_{M_2(\mathbb{R})}$.

③ Find the inverse matrix A^{-1} (Use 2 methods)

④ Find the eigenvalues and eigenvectors of A

⑤ Setting $D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$ and $P = \begin{bmatrix} u_x & v_x \\ u_y & v_y \end{bmatrix}$,

where (λ_1, λ_2) are the eigenvalues of A and (u_x, u_y), (v_x, v_y) are the corresponding eigenvectors.
Compute PDP^{-1} and deduce the expression of A^n.

⑥ Solve the system of linear recurrence sequences

$$\begin{cases} a_{n+1} = a_n + 2b_n & a_0 = 1 \\ b_{n+1} = 3a_n + 2b_n & b_0 = 0, \end{cases}$$

That is, find a_n and b_n in terms of n.

end

problems with Solution.

problem 01:

Consider the following matrix

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$$

① Calculate the characteristic polynomial of A (we denote by $P_A(x)$).

② Verify that

$$P_A(A) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

this is the 2×2 zero matrix.

③ Find the inverse matrix A^{-1} (use two methods)

④ Find the eigenvalues and eigenvectors of A.

⑤ Setting $D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$

and $P = \begin{bmatrix} u_x & v_x \\ u_y & v_y \end{bmatrix}$,

where λ_1, λ_2 are the eigenvalues of A and

(u_x, u_y), (v_x, v_y) are the Corresponding eigenvectors. Compute $P D P^{-1}$ and deduce the expression of A^n, $n \geq 0$.

⑥ Solve the system of linear recurrence sequences

$$\begin{cases} a_{n+1} = a_n + 2b_n , & a_0 = 1 \\ b_{n+1} = 3a_n + 2b_n , & b_0 = 0 \end{cases}$$

that is, find a_n and b_n in terms of n.

Solution:

① We find the characteristic polynomial:

$$P_A(x) = \begin{vmatrix} 1-x & 2 \\ 3 & 2-x \end{vmatrix} = \begin{vmatrix} -(1+x) & 2 \\ (1+x) & 2-x \end{vmatrix}$$

$$= (1+x) \cdot \begin{vmatrix} -1 & 2 \\ 1 & 2-x \end{vmatrix}$$

$$= (1+x) \cdot \left[-2 + x - 2 \right]$$

$$= (1+x)(x-4)$$

Then, $P_A(x) = (1+x)(x-4)$.

② We show that $P_A(A)$ is equal to the zero matrix.

In fact, we see that

$$P_A(x) = x^2 - 3x - 4$$

It follows that

$$P_A(A) = A^2 - 3A - 4I,$$

where $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is the identity matrix.

Therefore,

$$P_A(A) = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} - 3\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 6 \\ 9 & 10 \end{bmatrix} - \begin{bmatrix} 3 & 6 \\ 9 & 6 \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O_{M_2(\mathbb{R})}$$

③ Finding A^{-1} by two methods :

1st Method :

We have : $A^{-1} = \dfrac{1}{\det(A)} C^t$,

where

$$C' = \begin{bmatrix} 2 & -3 \\ -2 & 1 \end{bmatrix}$$

is the Comatrix of A. Thus,

$$A^{-1} = \frac{-1}{4} \begin{bmatrix} 2 & -3 \\ -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{3}{4} & \frac{-1}{4} \end{bmatrix}$$

2nd Method:

Since $P_A(A) = 0$, then

$$4 I_2 = A^2 - 3 A$$

$$= A(A - 3I)$$

Then
$$A^{-1} = \frac{1}{4}\left(A - 3I_2\right)$$
$$= \frac{1}{4}\begin{bmatrix} -2 & 2 \\ 3 & -1 \end{bmatrix}$$
$$= \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{3}{4} & -\frac{1}{4} \end{bmatrix}$$

4 We find the eigenpairs of A :

- Eigenvalues of A :

Since $P_A(x) = (1+x)(x-4)$, then
$$\begin{cases} \lambda_1 = -1 \\ \lambda_2 = 4 \end{cases}$$ are the eigenvalues of A.

- Eigenvectors of A :

From definition, we have
$$E_{\lambda_1} = \left\{ (x,y) \in \mathbb{R}^2 \,/\, \begin{array}{l} x + 2y = -x \\ 3x + 2y = -y \end{array} \right\}$$
$$= \left\{ (x,y) \in \mathbb{R}^2 \,/\, y = -x \right\}$$
$$= \text{Vect}\left\{ (1,-1) \right\}.$$

In the same way, we obtain

$$E_{\lambda_2} = \left\{ (x,y) \in \mathbb{R}^2 \;/\; \begin{array}{l} x + 2y = 4x \\ 3x + 2y = 4y \end{array} \right\}$$

$$= \left\{ (x,y) \in \mathbb{R}^2 \;/\; y = \frac{3}{2} x \right\}$$

$$= \operatorname{Vect}\left\{ \left(1, \frac{3}{2}\right) \right\}$$

$$= \operatorname{Vect}\left\{ (2,3) \right\}.$$

5 We put

$$D = \begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix} \text{ and } P = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$$

- We compute PDP^{-1}:

In fact

$$PDP^{-1} = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} \frac{3}{5} & -\frac{2}{5} \\ \frac{1}{5} & \frac{1}{5} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$$

$= A.$ Thus, $PDP^{-1} = A.$

Since $PDP^{-1} = A$, then

$$A^2 = P D \underbrace{P^{-1} \cdot P}_{I} D P^{-1} = P D^2 P^{-1}$$

and therefore,

$$A^n = PDP^{-1} \cdot PDP^{-1} \cdots PDP^{-1}$$

$$= P D^n P^{-1}$$

$$= \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} (-1)^n & 0 \\ 0 & 4^n \end{bmatrix} \begin{bmatrix} \frac{3}{5} & -\frac{2}{5} \\ \frac{1}{5} & \frac{1}{5} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3(-1)^n + 2 \cdot 4^n}{5} & \frac{2 \cdot 4^n - 2(-1)^n}{5} \\ \frac{3 \cdot 4^n - 3(-1)^n}{5} & \frac{2(-1)^n + 3 \cdot 4^n}{5} \end{bmatrix}$$

(6) Consider the system:

$$\begin{cases} a_{n+1} = a_n + 2b_n, & a_0 = 1 \\ b_{n+1} = 3a_n + 2b_n, & b_0 = 0. \end{cases} \quad \cdots (S')$$

How to solve the system (S')?

or How to find a_n and b_n in terms of n?

- The matrix form of (S'):

$$\begin{bmatrix} a_{n+1} \\ b_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} a_n \\ b_n \end{bmatrix}$$

$$\downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow$$

$$U_{n+1} \qquad\qquad A \qquad\qquad U_n$$

avec $U_0 = \begin{bmatrix} a_0 \\ b_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Since $U_{n+1} = A U_n$, then

$$\begin{aligned} U_n &= A\, U_{n-1} \\ &= A^2\, U_{n-2} \\ &= \dots \\ &= A^n \cdot U_0 . \end{aligned}$$

Finally, we get

$$U_n = \begin{bmatrix} \dfrac{3(-1)^n + 2\cdot 4^n}{5} & \dfrac{2\cdot 4^n - 2(-1)^n}{5} \\ \dfrac{3\cdot 4^n - 3(-1)^n}{5} & \dfrac{2(-1)^n + 3\cdot 4^n}{5} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

with $U_n = \begin{bmatrix} a_n \\ b_n \end{bmatrix}$.

Thus,

$$\begin{cases} a_n = \frac{3}{5}(-1)^n + \frac{2}{5}\cdot 4^n \\ b_n = \frac{3}{5}\cdot 4^n - \frac{3}{5}(-1)^n \end{cases}, \quad n \geq 0$$

End.

Homework N° 02 :

Let

$$A_3 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

- Find $P_{A_3}(x)$.
- Find the eigenvalues of the matrix :

$$A_n = \begin{bmatrix} 0 & 1 & \cdots & 1 \\ 1 & 0 & & 1 \\ \cdots & & & \cdots \\ 1 & 1 & \cdots & 0 \end{bmatrix}$$

Solution :

- We Compute the characteristic polynomial of A_3.

We have

$$P_{A_3}(x) = \det(A_3 - x I)$$

$$= \begin{vmatrix} -x & 1 & 1 \\ 1 & -x & 1 \\ 1 & 1 & -x \end{vmatrix} = \begin{vmatrix} -(1+x) & 0 & 1 \\ (1+x) & -(1+x) & 1 \\ 0 & (1+x) & -x \end{vmatrix}$$

$$= (1+x)^2 \cdot \begin{array}{c} \begin{array}{ccc} + & - & + \end{array} \\ \begin{vmatrix} -1 & 0 & 1 \\ 1 & -1 & 1 \\ 0 & 1 & -x \end{vmatrix} \end{array}$$

$$= (1+x)^2 \cdot \left[-(x-1) + 1 \right]$$

$$= (1+x)^2 (2-x).$$

Hence, $P_A(x) = (1+x)^2 \cdot (2-x)$.

- Consider the matrix

$$A_4 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

We have $P_{A_4}(x) = \begin{vmatrix} -x & 1 & 1 & 1 \\ 1 & -x & 1 & 1 \\ 1 & 1 & -x & 1 \\ 1 & 1 & 1 & -x \end{vmatrix}$

$$= \begin{vmatrix} -(1+x) & 0 & 0 & 1 \\ (1+x) & -(1+x) & 0 & 1 \\ 0 & (1+x) & -(1+x) & 1 \\ 0 & 0 & (1+x) & -x \end{vmatrix}$$

$$= (1+x)^3 \cdot \begin{array}{c} \begin{array}{cccc} + & - & + & - \end{array} \\ \begin{vmatrix} -1 & 0 & 0 & 1 \\ 1 & -1 & 0 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -x \end{vmatrix} \end{array}$$

$$= (1+x)^3 \cdot (-1) \cdot \begin{vmatrix} -1 & 0 & 1 \\ 1 & -1 & 1 \\ 0 & 1 & -x \end{vmatrix} + (1+x)^3 (-1) \cdot \begin{vmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= (1+x)^3 (x-2) + (1+x)^3 (-1)$$

$$= (1+x)^3 (x-3).$$

We deduce that

$$f_{A_n}(x) = \begin{cases} (1+x)^{n-1} (x-n+1) \,; & \text{if } n \text{ is even} \\ (1+x)^{n-1} (n-1-x) \,; & \text{if } n \text{ is odd.} \end{cases}$$

Thus, $\lambda_1 = -1$ is an eigenvalue of multiplicity $n-1$ and $\lambda_2 = n-1$ is an eigenvalue of multiplicity 1. (simple eigenvalue).

Home Work N° 03

Vandermonde determinant:

Consider the following determinant

$$\Delta_2 = \begin{vmatrix} 1 & 1 & 1 \\ x_0 & x_1 & x_2 \\ x_0^2 & x_1^2 & x_2^2 \end{vmatrix}, \text{ where } x_0, x_1, x_2 \in \mathbb{R}$$

- Prove that $\Delta_2 = (x_1 - x_0)(x_2 - x_0)(x_2 - x_1)$
- Find the general formula for Δ_n.

Solution: We have

$$\Delta_2 = \begin{vmatrix} 1 & 1 & 1 \\ x_0 & x_1 & x_2 \\ x_0^2 & x_1^2 & x_2^2 \end{vmatrix} \quad \begin{matrix} C_1 \\ \downarrow \\ C_2 - C_1 \end{matrix} \quad \begin{matrix} C_2 \\ \downarrow \\ C_3 - C_2 \end{matrix}$$

$$= \begin{vmatrix} 0 & 0 & 1 \\ x_1 - x_0 & x_2 - x_1 & x_2 \\ x_1^2 - x_0^2 & x_2^2 - x_1^2 & x_2^2 \end{vmatrix}$$

$$= (x_1 - x_0) \cdot (x_2 - x_1) \cdot \begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & x_2 \\ x_1 + x_0 & x_2 + x_1 & x_2^2 \end{vmatrix}$$

$$= (x_1 - x_0)(x_2 - x_1) \cdot (x_2 + x_1 - x_1 - x_0)$$

$$= (x_1 - x_0)(x_2 - x_1)(x_2 - x_0).$$

Thus, $\Delta_2 = (x_1 - x_0)(x_2 - x_1)(x_2 - x_0)$

In the general case, we get

$$\Delta_n = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ x_0 & x_1 & \cdots & x_n \\ x_0^2 & x_1^2 & \cdots & x_n^2 \\ \cdots & \cdots & \cdots & \cdots \\ x_0^n & x_1^n & \cdots & x_n^n \end{vmatrix} = \prod_{i>j} (x_i - x_j)$$

Homework N°: 04

Consider the following theorem:

Theorem: Similar matrices have the same characteristic polynomial.

Prove that the Converse is false.

Solution: Consider the following Counterexample:

For $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$

It is clear that

$$P_A(x) = (1-x)^2 = P_B(x).$$

On the other hand, assume that $A \sim B$. Then there exists an invertible matrix P such that

$$\begin{aligned} A &= P B P^{-1} \\ &= P \cdot I_2 \cdot P^{-1} \\ &= I_2 \end{aligned}$$

This is a Contradiction.

Conclusion: In general,

$$P_A(x) = P_B(x) \not\Rightarrow A \sim B.$$

- Let $f_n(x) = a_0 + a_1 x + \cdots + a_n x^n$ be a polynomial of degree n. prove that

$$A \sim B \Longrightarrow f_n(A) \sim f_n(B).$$

Solution: Since $A \sim B$, then $A = P B P^{-1}$ for some invertible matrix P. Then,

$$f_n(A) = a_0 I + a_1 A + a_2 A^2 + \cdots + a_n A^n$$
$$= a_0 P P^{-1} + a_1 (P B P^{-1}) + a_2 (P B P^{-1})^2 + \cdots + a_n (P B P^{-1})^n$$
$$= a_0 P P^{-1} + a_1 P B P^{-1} + a_2 P B^2 P^{-1} + \cdots + a_n P B^n P^{-1}$$
$$= P \left[a_0 + a_1 B + a_2 B^2 + \cdots + a_n B^n \right] P^{-1}$$
$$= P f_n(B) P^{-1}$$

Thus, $f_n(A) \sim f_n(B)$.

- prove that $A \sim B \Rightarrow e^A \sim e^B$.

Solution: Since A is similar to B, then there exists an invertible matrix P such that $A = P B P^{-1}$. From definition, we have

$$e^A = I + \frac{A}{1!} + \frac{A^2}{2!} + \cdots + \frac{A^n}{n!} + \cdots, \quad \cancel{\text{then}}$$
$$= P P^{-1} + \frac{P B P^{-1}}{1!} + \frac{(P B P^{-1})^2}{2!} + \cdots + \frac{(P B P^{-1})^n}{n!} + \cdots$$
$$= P P^{-1} + \frac{P B P^{-1}}{1!} + \frac{P B^2 P^{-1}}{2!} + \cdots + \frac{P B^n P^{-1}}{n!} + \cdots$$
$$= P \left[I + \frac{B}{1!} + \frac{B^2}{2!} + \cdots + \frac{B^n}{n!} + \cdots \right] P^{-1}$$

That is,

$$e^{A} = P \cdot e^{B} \cdot P^{-1}, \text{ and hence } e^{A} \sim e^{B}.$$

Homework № 05

- For which values of Constants, a, b, c and d is the matrix

$$A = \begin{bmatrix} 1 & b & d \\ 0 & 1 & c \\ 0 & 0 & a \end{bmatrix}$$

diagonalizable?

Solution: It's clear that the eigenvalues of A are:
$$\begin{cases} \lambda_1 = 1, \\ \lambda_2 = a. \end{cases}$$

- We find the Corresponding eigenspaces:

$$E_{\lambda_1} = \left\{ (x, y, z) \in \mathbb{R}^3 \;/\; \begin{matrix} x + by + dz = x \\ y + cz = y \\ az = z \end{matrix} \right\}$$

there are two Cases:

Case 1: If $a \neq 1$, then $z = 0$. Then

$$E_{\lambda_1} = \left\{ (x, y, z) \in \mathbb{R}^3 \;/\; \begin{matrix} by = 0 \\ z = 0 \end{matrix} \right\}$$

- If $b \neq 0$, then $y = 0$

In this Case, we have

$$E_{\lambda_1} = \{ (x, y, z) \in \mathbb{R}^3 \,/\, y = z = 0 \}$$
$$= \mathrm{Vect}\{ (1,0,0) \}$$

Thus, $\begin{cases} A_m(1) = 2 \\ G_m(1) = 1 \end{cases}$

i.e, A is not diagonalizable.

- If $b = 0$, we have

$$E_{\lambda_1} = \{ (x, y, z) \in \mathbb{R}^3 \,/\, z = 0 \}$$
$$= \{ (x, y, 0) \,/\, x, y \in \mathbb{R} \}$$
$$= \mathrm{Vect}\{ (1,0,0), (0,1,0) \}$$

Thus, $\begin{cases} A_m(1) = 2 \\ G_m(1) = 2 \end{cases} \Rightarrow A_m(1) = G_m(1).$

In this case, A is diagonalizable.

Case 2: Assume that $a = 1$:

In this case $\lambda = 1$ is an eigenvalue of multiplicity 3 i.e $A_m(1) = 3$.

- We find the corresponding eigenspace E_λ:

$$E_\lambda = \left\{ (x, y, z) \in \mathbb{R}^3 \,/\, \begin{array}{l} x + by + dz = x \\ y + cz = y \\ \cancel{a}z = z \end{array} \right\}$$

It is clear that if one of the constants b, c, d is nonzero, then $\dim E_h \leq 2$

Thus, $G_m(h) \neq 3 = A_m(h)$

i.e A is not diagonalizable.

In the case when $b = c = d = 0$, then

$$E_h = \{(x, y, z) \in \mathbb{R}^3 \,/\, x, y, z \in \mathbb{R}\}$$

$$= \mathbb{R}^3 = \text{Vect}\{(1,0,0), (0,1,0), (0,0,1)\}$$

$$\Rightarrow G_m(h) = A_m(h) = 3$$

$\Rightarrow A$ is diagonalizable.

Remark : Let A be an $n \times n$ matrix such that $Sp(A) = \{h\}$. Then

$$A \text{ is diagonalizable} \iff A = hI.$$

In view of the above case $(a = 1)$, we see that $Sp(A) = \{1\}$, then A is diagonalizable if and only if $A = I_3$

which gives $b = c = d = 0$.

Conclusion: the matrix A is diagonalizable if and only if :

$$\begin{pmatrix} a \neq 1 \text{ and} \\ b = 0,\ c, d \in \mathbb{R} \end{pmatrix} \text{ or } \begin{pmatrix} a = 1 \text{ and} \\ b = c = d = 0 \end{pmatrix}.$$

• Study the diagonalization of the matrix:

$$A = \begin{bmatrix} 1 & a & b & c \\ 0 & 1 & d & e \\ 0 & 0 & 2 & f \\ 0 & 0 & 0 & 3 \end{bmatrix}, \quad a,b,c,d,e,f \in \mathbb{R}.$$

Solution: We see that

$$\begin{cases} \lambda_1 = 2, & A_m(\lambda_1) = 1 \\ \lambda_2 = 3, & A_m(\lambda_2) = 1 \\ \lambda_3 = 1, & A_m(\lambda_3) = 2 \end{cases}$$

Moreover, we have

$$E_{\lambda_3} = \left\{ (x,y,z,t) \in \mathbb{R}^4 \middle/ \begin{array}{l} x + ay + bz + ct = x \\ y + dz + et = y \\ 2z + ft = z \\ 3t = t \end{array} \right\}$$

$$= \left\{ (x,y,z,t) \in \mathbb{R}^4 \middle/ \begin{array}{l} t = 0 \\ z = 0 \\ ay = 0 \end{array} \right\}$$

We distinguish two Cases:

• If $a = 0$, then

$$E_{\lambda_3} = \left\{ (x,y,z,t) \in \mathbb{R}^4 \middle/ \begin{array}{l} z = 0 \\ t = 0 \end{array} \right\}$$

$$= \{ (x,y,0,0) \mid x,y \in \mathbb{R} \} \Rightarrow \dim E_{\lambda_3} = 2.$$

Thus, A is diagonalizable.

- If $a \neq 0$, then

$$E_{\lambda_3} = \left\{ (x, y, z, t) \in \mathbb{R}^4 \;\middle/\; \begin{array}{l} y = 0 \\ z = 0 \\ t = 0 \end{array} \right\}$$

$$= \{ (x, 0, 0, 0) \mid x \in \mathbb{R} \}$$

Then $\dim E_{\lambda_3} = 1$. In this Case, A is not diagonalizable.

Conclusion: The matrix is diagonalizable $\Longleftrightarrow$ $a = 0$ and $b, c, d, e, f \in \mathbb{R}$.

* Study the diagonalization of the matrix:

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 4 & 1 & 2 \\ a & 0 & 3 \end{bmatrix}, \quad a \in \mathbb{R}.$$

Solution:

1. we find the characteristic polynomial of A

In fact, we have

$$P_A(x) = \begin{vmatrix} 3-x & 0 & 0 \\ 4 & 1-x & 2 \\ a & 0 & 3-x \end{vmatrix}$$

$$= (3-x)^2 (1-x).$$

② the eigenvalues of A :

$$\begin{cases} \lambda_1 = 1 & \text{with } A_m(\lambda_1) = 1, \\ \lambda_2 = 3 & \text{with } A_m(\lambda_2) = 2. \end{cases}$$

③ We find the eigenvectors of A.

Since $A_m(\lambda_1) = 1$, then $\dim E_{\lambda_1} = 1$

<u>Finding E_{λ_2} :</u>

$$E_{\lambda_2} = \left\{ (x,y,z) \in \mathbb{R}^3 \,/\, \begin{array}{l} 3x = 3x \\ 4x + y + 2z = 3y \\ ax + 3z = 3z \end{array} \right\}$$

$$= \left\{ (x,y,z) \in \mathbb{R}^3 \,/\, \begin{array}{l} y = 2x + z \\ ax = 0 \end{array} \right\}$$

there are two Cases:

<u>Case 1</u>: $a = 0$:

$$E_{\lambda_2} = \{ (x,y,z) \in \mathbb{R}^3 \,/\, y = 2x + z \}$$

$$= \{ (x, 2x+z, z) \,/\, x, z \in \mathbb{R} \}$$

$$= \mathrm{Vect}\{ (1,2,0), (0,1,1) \}$$

Thus, $\dim E_{\lambda_2} = 2$

i.e A is diagonalizable.

Case 02: $a \neq 0$:

$$E_{h_2} = \left\{ (x,y,z) \in \mathbb{R}^3 \;\middle/\; \begin{array}{l} y = z \\ x = 0 \end{array} \right\}$$

$$= \{ (0,y,y) \mid y \in \mathbb{R} \}$$

$$= \text{Vect}\{(0,1,1)\}.$$

In this Case, $\dim E_{h_2} = 1$

$\Rightarrow$ A is not diagonalizable.

Homework N°: 06

Consider the following two matrices:

$$A = \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 2 \\ -2 & 5 \end{bmatrix}$$

① prove that $A \sim B$.

② Find an invertible matrix G such that

$$A = G \cdot B \cdot G^{-1}.$$

Solution: ①

Step 1: We find the characteristic polynomials of A and B.

$$P_A(x) = \begin{vmatrix} 2-x & 1 \\ 2 & 3-x \end{vmatrix} = \begin{vmatrix} (4-x) & (4-x) \\ 2 & 3-x \end{vmatrix}$$

$$= (4-x)\cdot \begin{vmatrix} 1 & 1 \\ 2 & 3-x \end{vmatrix}$$

$$= (4-x).\left[3-x-2\right]$$

$$= (4-x)(1-x).$$

Also, we have

$$P_B(x) = \begin{vmatrix} -x & 2 \\ -2 & 5-x \end{vmatrix} \quad \begin{matrix} L_1 \\ \downarrow \\ L_1 - L_2 \end{matrix}$$

$$= \begin{vmatrix} 2-x & -3+x \\ -2 & 5-x \end{vmatrix} \quad \begin{matrix} L_1 \\ \downarrow \\ L_1 - L_2 \end{matrix}$$

$$= \begin{vmatrix} 4-x & -2(4-x) \\ -2 & 5-x \end{vmatrix}$$

$$= (4-x).\begin{vmatrix} 1 & -2 \\ -2 & 5-x \end{vmatrix}$$

$$= (4-x)\left[5-x-4\right]$$

$$= (4-x)(1-x).$$ Hence $P_A(x) = P_B(x)$.

Then A and B have the same eigenvalues:

$$\begin{cases} h_1 = 1, & A_m(h_1) = 1, \\ h_2 = 4, & A_m(h_2) = 1. \end{cases}$$

Since $G_m(h_1) = 1 = G_m(h_2)$, then A and B are diagonalizable. Then

$$\begin{cases} A = P D P^{-1} & \text{for some } P \text{ invertible} \\ B = Q D Q^{-1} & \text{" " } Q \text{ " "} \end{cases}$$

It follows that

$$\begin{cases} A = P D P^{-1} \\ D = Q^{-1} B Q. \end{cases}$$

Thus,

$$\begin{aligned} A &= P Q^{-1} B Q P^{-1} \\ &= (P Q^{-1}) \cdot B \cdot (P Q^{-1})^{-1} \\ &= C \cdot B \cdot C^{-1}, \text{ where } C = P Q^{-1} \text{ is} \end{aligned}$$

invertible. That is, A is similar to B.

② Finding the matrix G:

From simple computation, we find

$E_{\lambda_1} = \text{Vect}\{(-1,1)\}$ and $E_{\lambda_2} = \text{Vect}\{(1,2)\}$

for the matrix A, and

$\tilde{E}_{\lambda_1} = \text{Vect}\{(2,1)\}$, $\tilde{E}_{\lambda_2} = \text{Vect}\{(1,2)\}$

for the matrix B.

Since $G = P\,Q^{-1}$, where

$$P = \begin{bmatrix} -1 & 1 \\ 1 & 2 \end{bmatrix}, \quad Q = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

and $Q^{-1} = \begin{bmatrix} \frac{2}{3} & \frac{-1}{3} \\ \frac{-1}{3} & \frac{2}{3} \end{bmatrix}$. Then

$$G = \begin{bmatrix} -1 & 1 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Homework N° 07

Consider the matrix

$$A(a) = \begin{bmatrix} 1 & 0 & 0 \\ a & -2 & 3 \\ 1 & -1 & 2 \end{bmatrix}, \quad a \in \mathbb{R}.$$

① Find all the eigenvalues of $A(a)$.

② Find the real parameter a for which the matrix $A(a)$ is diagonalizable, and give in this case the diagonalization formula of $A(a)$.

Solution: Simple computation, we find

$$P_{A(a)}(x) = (x-1)^2(x+1)$$

- the eigenvalues of $A(a)$ are:

$\lambda_1 = 1$ with $A_m(\lambda_1) = 2$ and

$\lambda_2 = -1$ with $A_m(\lambda_2) = 1$

Now, we find the eigenspace E_{λ_1}:

In fact, we have

$$E_{\lambda_1} = \left\{ (x,y,z) \in \mathbb{R}^3 \,\middle/\, \begin{array}{l} x = x \\ ax - 2y + 3z = y \\ x - y + 2z = z \end{array} \right\}$$

$$= \left\{ (x, y, z) \in \mathbb{R}^3 \;\middle/\; \begin{array}{l} ax + 3z = 3y \\ x - y = -z \end{array} \right\}$$

$$= \left\{ (x, y, z) \in \mathbb{R}^3 \;/\; (a-3)x = 0 \right\}$$

There are two Cases:

- Case 1 : $a \neq 3$:

$$E_{\lambda_1} = \left\{ (x, y, z) \in \mathbb{R}^3 \;\middle/\; \begin{array}{l} x = 0 \\ y = z \end{array} \right\}$$

$$= \mathrm{Vect}\left\{ (0, 1, 1) \right\}$$

Thus, $G_m(\lambda_1) = 1$.

In this Cas $A(a)$ is not diagonalizable.

- Case 2 : $a = 3$.

$$E_{\lambda_1} = \left\{ (x, y, z) \in \mathbb{R}^3 \;/\; y = x + z \right\}$$

$$= \left\{ (x, x+z, z) \;/\; x, z \in \mathbb{R} \right\}$$

$$= \left\{ x(1, 1, 0) + z(0, 1, 1) \;/\; x, z \in \mathbb{R} \right\}$$

$$= \mathrm{Vect}\left\{ (1, 1, 0), (0, 1, 1) \right\}$$

Thus, $G_m(\lambda_1) = 2$

$\Longrightarrow$ $A(a)$ is diagonalizable.

② Diagonalization of $A(a)$:

Setting

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \text{ and}$$

$$P = \begin{bmatrix} 1 & 0 & . \\ 1 & 1 & . \\ 0 & 1 & . \end{bmatrix}$$

Also, we compute E_{λ_2}:

$$E_{\lambda_2} = \left\{ (x,y,z) \in \mathbb{R}^3 \,/\, \begin{array}{l} x = -x \\ 3x - 2y + 3z = -y \\ x - y + 2z = -z \end{array} \right\}$$

$$= \left\{ (x,y,z) \in \mathbb{R}^3 \,/\, \begin{array}{l} x = 0 \\ y = 3z \end{array} \right\}$$

$$= \text{Vect}\left\{ (0,3,1) \right\}$$

Therefore

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 3 \\ 0 & 1 & 1 \end{bmatrix}$$

After computation, we obtain :

$$P^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & \frac{-1}{2} & \frac{3}{2} \\ \frac{-1}{2} & \frac{1}{2} & \frac{-1}{2} \end{bmatrix}$$

It follows that

$$A(3) = P \cdot D \cdot P^{-1}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 3 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & \frac{-1}{2} & \frac{3}{2} \\ \frac{-1}{2} & \frac{1}{2} & \frac{-1}{2} \end{bmatrix}$$

Homework N° 08

Solve the system of linear recurrence sequences:

$$(S) \ldots \begin{cases} x_{n+1} = 3x_n - y_n + 1 \\ y_{n+1} = -x_n + 3y_n + 2 \end{cases}, \quad \text{where}$$

$(x_0, y_0) = (1, 0)$.

Compute x_{1000} and y_{1000}.

Solution: We can write the system (S) as the following matrix formula:

$$\underbrace{\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix}}_{U_{n+1}} = \underbrace{\begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_n \\ y_n \end{bmatrix}}_{U_n} + \underbrace{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}_{C}$$

with $U_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

That is, $U_n = A U_{n-1} + C$, and hence

$$U_n = A^n U_0 + \left(A^{n-1} + A^{n-2} + \cdots + A + I\right) \cdot C$$

- We find A^n:

After computation, we get

$$A^n = \begin{bmatrix} \dfrac{2^n + 4^n}{2} & \dfrac{2^n - 4^n}{2} \\ \dfrac{2^n - 4^n}{2} & \dfrac{2^n + 4^n}{2} \end{bmatrix} ; \; n \geq 0$$

Note that

$$A^n = \frac{2^n}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + \left(\frac{+4^n}{2}\right) \cdot \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

• We Compute $A^{n-1} + A^{n-2} + \cdots + A + I$:

Setting

$$U = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad V = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Since $A^n = \dfrac{2^n}{2} U + \dfrac{4^n}{2} V$, then

$$A^{n-1} + A^{n-2} + \cdots + A + I$$

$$= \frac{2^{n-1} + 2^{n-2} + \cdots + 2^1 + 2^0}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + \frac{4^{n-1} + 4^{n-2} + \cdots + 4^1 + 4^0}{2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

$$= \frac{2^n - 1}{2} \cdot \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + \frac{4^n - 1}{6} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2^n}{2} + \frac{4^n}{6} - \frac{2}{3} & \frac{2^n}{2} - \frac{4^n}{6} - \frac{1}{3} \\ \frac{2^n}{2} - \frac{4^n}{6} - \frac{1}{3} & \frac{2^n}{2} + \frac{4^n}{6} - \frac{2}{3} \end{bmatrix}$$

$$\cdots \quad \begin{cases} x_n = 2^{n+1} + \frac{4^n}{3} - \frac{4}{3} \\ y_n = 2^{n+1} - \frac{4^n}{3} - \frac{5}{3} \end{cases} ; \quad n \geq 0.$$

Homework №: 09

Let (x_n) be the sequence given by

$$x_{n+2} = \frac{2}{\frac{1}{x_n} + \frac{1}{x_{n+1}}}, \quad x_0, x_1 \neq 0$$

Find x_n in terms of n.

Solution: We can write

$$x_n = \frac{2}{\frac{1}{x_{n-2}} + \frac{1}{x_{n-1}}}, \quad x_0, x_1 \neq 0$$

It follows that

$$\frac{2}{x_n} = \frac{1}{x_{n-2}} + \frac{1}{x_{n-1}}$$

We put $\frac{1}{x_n} = y_n$. Then

$$2y_n = y_{n-1} + y_{n-2}, \quad \text{and hence}$$

$$y_n = \frac{1}{2} y_{n-1} + \frac{1}{2} y_{n-2} \quad \ldots \quad (*)$$

Also, we can write $(*)$ as follows:

$$\begin{bmatrix} y_n \\ y_{n-1} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y_{n-1} \\ y_{n-2} \end{bmatrix}, \quad y_0 = \frac{1}{x_0}, \quad y_1 = \frac{1}{x_1}$$

That is,

$$\begin{bmatrix} y_n \\ y_{n-1} \end{bmatrix} = A^{n-1} \cdot \begin{bmatrix} y_1 \\ y_0 \end{bmatrix} \quad \text{with} \quad A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & 0 \end{bmatrix}$$

After simple computation, we have

$$A^{n-1} = \begin{bmatrix} \dfrac{2+\left(\frac{-1}{2}\right)^{n-1}}{3} & \dfrac{1-\left(\frac{-1}{2}\right)^{n-1}}{3} \\ \dfrac{2-2\left(\frac{-1}{2}\right)^{n-1}}{3} & \dfrac{1+2\left(\frac{-1}{2}\right)^{n-1}}{3} \end{bmatrix}$$

Therefore,

$$y_n = \frac{1}{3}\left[2+\left(\frac{-1}{2}\right)^{n-1}\right] y_1 + \frac{1}{3}\left[1-\left(\frac{-1}{2}\right)^{n-1}\right] y_0$$

Since $y_n = \dfrac{1}{x_n}$, then

$$x_n = \frac{3}{\left(2+\left(\frac{-1}{2}\right)^{n-1}\right)\frac{1}{x_1} + \left(1-\left(\frac{-1}{2}\right)^{n-1}\right)\frac{1}{x_0}}$$

Thus,

$$\lim_{n \to +\infty} x_n = \frac{3}{\frac{2}{x_1} + \frac{1}{x_0}}.$$

Homework N° 10

Solve the system of differential equations:

$$\begin{cases} x_1' = x_1 + 2x_2 \\ x_2' = 3x_1 + 2x_2 \end{cases} \quad \ldots\ldots \quad (S')$$

Solution:

1. We see that

$$(S') \Leftrightarrow \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$X' = A \cdot X$$

2. We find the eigenvalues of A and corresponding eigenvectors:

$$\begin{cases} \lambda_1 = -1, \quad v_1 = (-1, 1) \\ \lambda_2 = 4, \quad v_2 = (2, 3). \end{cases}$$

Setting

$$D = \begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix} \text{ and } P = \begin{bmatrix} -1 & 2 \\ 1 & 3 \end{bmatrix}$$

③ Since $X' = A \cdot X$ and $A = P D P^{-1}$, then

$$X' = AX \iff \begin{cases} X' = PDY \\ Y = P^{-1} \cdot X \end{cases}$$

Hence

$$\begin{cases} P^{-1} X' = DY \\ Y = P^{-1} X \end{cases}$$

That is,

$$\begin{cases} X' = P Y' & \dots\dots (1) \\ Y' = DY & \dots\dots (2) \\ Y = P^{-1} \cdot X & \dots\dots (3) \end{cases}$$

$$(2) \Rightarrow \begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

i.e

$$\begin{cases} y_1 = -c_1 e^{-t} \\ y_2 = c_2 e^{4t} \end{cases} ; \quad c_1, c_2 \in \mathbb{R}.$$

$$(3) \Rightarrow X = P \cdot Y$$

$$= \begin{bmatrix} -1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} -c_1 e^{-t} \\ c_2 e^{4t} \end{bmatrix}$$

$$= \begin{bmatrix} c_1 e^{-t} + 2 c_2 e^{4t} \\ -c_1 e^{-t} + 3 c_2 e^{4t} \end{bmatrix}, \quad c_1, c_2 \in \mathbb{R}.$$

Homework N°: 11

Let
$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

- Find e^{tA} for every $t \in \mathbb{R}$.
- Solve the system of differential equations:

$$\begin{cases} x' = x + y \\ y' = x + y \\ z' = 2z \end{cases} \quad \cdots\cdots \quad (S)$$

Solution: Note that A is diagonalizable with

$$A = \underbrace{\begin{bmatrix} -1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}}_{P^{-1}}$$

Then
$$e^{tA} = e^{t(PDP^{-1})} = P \cdot e^{tD} \cdot P^{-1}$$

$$= \begin{bmatrix} -1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{2t} \end{bmatrix} \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{e^{2t}}{2} & \frac{e^{2t}}{2} & 0 \\ \frac{e^{2t}}{2} & \frac{e^{2t}}{2} & 0 \\ 0 & 0 & e^{2t} \end{bmatrix}$$

- the Solution of the system of differential equations (S) is given by :

$$X(t) = e^{tA} \cdot C$$

$$= \begin{bmatrix} \frac{e^{2t}}{2} & \frac{e^{2t}}{2} & 0 \\ \frac{e^{2t}}{2} & \frac{e^{2t}}{2} & 0 \\ 0 & 0 & e^{2t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{c_1}{2} e^{2t} + \frac{c_2}{2} e^{2t} \\ \frac{c_1}{2} e^{2t} + \frac{c_2}{2} e^{2t} \\ c_3 e^{2t} \end{bmatrix}.$$

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