Definition: A matrix of the form

$$U = \begin{bmatrix} u_{11} & u_{12} & u_{13} & \cdots & u_{1n} \\ & u_{22} & u_{23} & \cdots & u_{2n} \\ & & \ddots & & \\ & 0 & & \ddots & \\ & & & & u_{nn} \end{bmatrix}$$

is Called an **upper triangular matrix**.

A matrix of the form

$$L = \begin{bmatrix} \ell_{11} & & & 0 \\ \ell_{21} & \ell_{22} & & \\ & & \ddots & \\ \ell_{n1} & \ell_{n2} & \cdots & \ell_{nn} \end{bmatrix}$$

is Called a **lower triangular matrix**.

Notation: An upper triangular matrix is typically denoted with U and a lower triangular matrix is typically denoted with L

• properties:

① $U^t = L$

That is, $L^t = U$.

If you transpose an upper (lower) triangular matrix, you get a lower (upper) triangular matrix.

② $\begin{cases} L_1 \cdot L_2 = L \\ U_1 \, U_2 = U \end{cases}$

that is, the product of two lower (upper) triangular matrices is lower (upper) triangular

③ A triangular matrix is **invertible** if and only if all diagonal entries are nonzero.

Example: the matrix

$$A = \begin{bmatrix} 1 & -5 & 3 & 4 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$ is not invertible,

and the matrix

$$B = \begin{bmatrix} 4 & 0 & 0 \\ 1 & 3 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$ is invertible.

Diagonal Matrices:

Definition: A diagonal matrix is a square matrix with zero entries except possibly on the main diagonal.

- The following are examples of diagonal matrices:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

- In general, a diagonal matrix is given by:

$$D = \begin{bmatrix} d_1 & 0 & 0 & \cdots & 0 \\ 0 & d_2 & 0 & \cdots & 0 \\ 0 & 0 & d_3 & \cdots & 0 \\ & & & & \vdots \\ & & & & d_n \end{bmatrix}$$

Notation: A lot of time, a diagonal matrix is referenced with a capital D.

Finding the power of a diagonal matrix:

If D is a diagonal matrix, then D^n for $n > 0$ is given by

$$D^n = \begin{bmatrix} d_1^n & & & 0 \\ & d_2^n & & \\ & & \ddots & \\ 0 & & & d_n^n \end{bmatrix}$$

- **The inverse:** A diagonal matrix D is invertible if and only if all the diagonal elements are nonzero.

In this case, D^{-1} is given by

$$D^{-1} = \begin{bmatrix} \frac{1}{d_1} & & & 0 \\ & \frac{1}{d_2} & & \\ & & \ddots & \\ 0 & & & \frac{1}{d_n} \end{bmatrix}$$

Example: For $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$

Then $A^{-1} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{bmatrix}$.

Identity matrix :

Definition: the identity matrix I_n of size n is the $n \times n$ matrix in which all the elements on the main diagonal are equal to 1 and all other elements are equal to 0.

Examples :

$$I_1 = \begin{bmatrix} 1 \end{bmatrix}, \quad I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \ \ldots$$

$$I_n = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

Remark :

$A \cdot I_n = I_m \cdot A$ for any m-by-n matrix A. In particular,

$A \cdot I_n = I_n \cdot A = A$ for any $n \times n$ matrix.

Symmetric matrix

A symmetric matrix is a square matrix that is equal to its transpose. Formally,

$$A \text{ is symmetric} \iff A = A^t$$

If we put, $A = (a_{ij})$ $1 \leq i, j \leq n$

Then,

$$A \text{ is symmetric} \iff a_{ij} = a_{ji}$$

for every i, j.

Example : Zero matrix and identity matrix are symmetric (any diagonal matrix is symmetric).

- the matrix $A = \begin{bmatrix} 6 & 8 & 5 \\ 8 & 2 & -1 \\ 5 & -1 & 0 \end{bmatrix}$

is symmetric.

Theorem : Let A be a square matrix Then AA^t and A^tA are symmetric.

Proof : It is clear that

$$(A^tA)^t = A^t(A^t)^t = A^tA$$

i.e A^tA is symmetric.

Theorem: If A and B are symmetric matrices with the same size, and if k is any scalar, then

1. A^t is symmetric
2. $A+B$ and $A-B$ are symmetric
3. $k.A$ is symmetric
4. The product of two symmetric matrices is symmetric if and only if the matrices commute, i.e $AB = BA$.
5. If A is invertible symmetric matrix, then A^{-1} is symmetric.

Definition: An $n \times n$ matrix B is called skew-symmetric if $B = -B^t$.

Example: the matrix $B = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{bmatrix}$ is symmetric.

The matrix $B = \begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & 1 \\ -2 & -1 & 0 \end{bmatrix}$ is skew-symmetric.

Eigenspace

Definition: The eigenspace associated with an eigenvalue λ of an $n \times n$ matrix is the Kernel of the matrix $A - \lambda I_n$ and is denoted by E_λ.
So E_λ Consists of all solutions v of the equation $Av = \lambda v$. In other words, E_λ Consists of all eigenvectors with eigenvalue λ, together with the zero vector.

Definition: The geometric multiplicity for a given eigenvalue is the dimension of the eigenspace E_λ. That is,

$$G_m(\lambda) = \dim E_\lambda.$$

- The algebraic multiplicity for a given eigenvalue λ is the number of times the eigenvalue is repeated. For example, if the characteristic polynomial is $(\lambda-1)^2(\lambda-2)^3$ then for $\lambda = 1$ the algebraic multiplicity is 2 and for $\lambda = 2$ the algebraic multiplicity is 3.
- The algebraic multiplicity is greater than or equal to the geometric multiplicity.