Matrix Norms

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Definition

Let E be a vector space over $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$). The norm over E, denoted by $\|.\|$, is a mapping

$$\begin{array}{rcl} \|.\| & : & E \to \mathbb{R}_+ \\ x & \mapsto & \|x\| \quad \text{(we say: the norm of } x\text{)} \end{array}$$

satisfying the following properties:

1. For all $x \in E$: $\|x\| \geq 0$ and $\|x\| = 0 \Leftrightarrow x = 0_E$;
2. For all $x \in E$ and scalar $\alpha \in \mathbb{K}$: $\|\alpha x\| = |\alpha| \,.\, \|x\|$;
3. For all $x, y \in E$: $\|x + y\| \leq \|x\| + \|y\|$.

In this case, the couple $(E, \|.\|)$ is called **normed vector space** or **normed space**. So, a normed space E is a vector space with a norm defined on it.

Example

In this lesson, we only use the two vector spaces, $\mathbb{K}^n$ and $\mathcal{M}_n(\mathbb{K})$ with $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

1. Define over $\mathbb{K}^n$ the following norms:

$$\begin{aligned} \|x\|_1 &= \sum_{i=1}^{n} |x_i|, \ \|x\|_2 = \left(\sum_{i=1}^{n} |x_i|^2 \right)^{\frac{1}{2}}, \\ \|x\|_\infty &= \max_{1 \le i \le n} (|x_i|). \end{aligned}$$

Example

2. Define over $\mathcal{M}_n(\mathbb{K})$ the following norms:

$$\begin{aligned} \|A\|_1 &= \max_j \sum_{i=1}^{n} |a_{ij}| \text{ and } \|A\|_\infty = \max_i \sum_{j=1}^{n} |a_{ij}| \\ \|A\|_2 &= \left(\sum_{i,j}^{n} |a_{ij}|^2 \right)^{\frac{1}{2}}. \end{aligned}$$

As an application, for $x = \begin{pmatrix} -1 & 1 & -2 \end{pmatrix}^t$, we have

$$\|x\|_1 = 4, \quad \|x\|_2 = \sqrt{6} \text{ and } \|x\|_\infty = 2.$$

and for $A = \begin{pmatrix} -1 & -2 \\ 7 & 3 \end{pmatrix} \in \mathcal{M}_n(\mathbb{R})$, we also have

$$\|A\|_1 = \max(8,5) = 8, \|A\|_2 = 3\sqrt{7} \text{ and } \|A\|_\infty = \max(3,10) = 10.$$

Lemma

For each matrix $A \in \mathcal{M}_n(\mathbb{K})$ and for each $x \in \mathbb{K}^n$, we have the following inequality:

$$\|Ax\| \leq \|A\| \, \|x\|.$$

Definition

Let E be real vectot space. The inner product of E (over E) is a function $\langle .,. \rangle$ defined by

$$\begin{array}{rcl} \langle .,. \rangle & : & E \times E \longrightarrow \mathbb{R} \\ (x,y) & \longmapsto & \langle x,y \rangle \end{array}$$

satisfying the following properties:

1. For all $x \in E$: $\langle x,x \rangle \geq 0$ and $\langle x,x \rangle = 0 \Leftrightarrow x = 0$.
2. For all $x,y \in E$: $\langle x,y \rangle = \langle y,x \rangle$.
3. For all $x \in E$ and scalar $\alpha \in \mathbb{R}$: $\langle \lambda x,y \rangle = \lambda \langle x,y \rangle$
4. For all $x,y,z \in E$: $\langle x+y,z \rangle = \langle x,z \rangle + \langle y,z \rangle$.

Define on the vector space $\mathbb{R}^n$ the inner product $\langle .,. \rangle$ by

$$\forall\ x = \begin{pmatrix} x_1 & x_2 & \dots & x_n \end{pmatrix}^t, y = \begin{pmatrix} y_1 & y_2 & \dots & y_n \end{pmatrix}^t \in \mathbb{R}^n$$

we have

$$\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i.$$

Remark

For each $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$, we have

$$\langle x, y \rangle = x^t y.$$

Also, the inner product over $\mathbb{C}^n$ is given by

$$\langle x, y \rangle = x^t \overline{y}, \tag{1}$$

where $\overline{y}$ is the conjugate of y.

Example

Let $A \in \mathcal{M}_n(\mathbb{R})$. Find a symmetric matrix $B \in \mathcal{S}_n(\mathbb{R})$ such that

$$x^t Ax = x^t Bx \text{ for every } x \in \mathbb{R}^n.$$

In fact, for every $x \in \mathbb{R}^n$, we have

$$\begin{aligned} x^t Ax &= \left(x^t Ax\right)^t \quad (\text{since } x^t Ax = a \in \mathbb{R}) \\ &= x^t A^t x, \end{aligned}$$

It follows that

$$x^t Ax = \frac{1}{2}x^t Ax + \frac{1}{2}x^t A^t x = x^t \left(\frac{A + A^t}{2}\right) x.$$

Note that the matrix $B = \dfrac{A + A^t}{2}$ is symmetric.

Scalar Product

Examples

Also, define over the vector space $C([a,b])$ the inner product

$$\forall\ f,g \in C([a,b]) : \langle f,g \rangle = \int_a^b f(x)\,g(x)\,dx.$$

Proposition

Let A be a symmetric matrix and let $(\alpha, x), (\beta, y)$ be two eigenpairs of A with $\alpha \neq \beta$. Then x and y are orthogonal, i.e., $x \perp y$. Or, equivalently, $\langle x, y \rangle = 0$.

Proof.

Indeed, we have

$$\alpha \langle x, y \rangle = \langle \alpha x, y \rangle = \langle Ax, y \rangle = \langle x, A^t y \rangle = \langle x, Ay \rangle = \langle x, \beta y \rangle = \beta \langle x, y \rangle,$$

and since $\alpha \neq \beta$, it follows that $\langle x, y \rangle = 0$. □

Ex 01. Consider the equation

$$ax^2 + 2hxy + by^2 = 0. \quad (2)$$

Write (2) in the form $X^t AX = 0$, where $A \in \mathcal{M}_2(\mathbb{R})$ and $X = \begin{pmatrix} x \\ y \end{pmatrix}$.

Ans. $A = \begin{pmatrix} a & h \\ h & b \end{pmatrix}$.

Ex 02. Write the equation $\lambda_1 x_1^2 + \lambda_2 x_2^2 = 0$ in the form $X^t A X = 0$, where $A \in \mathcal{M}_2(\mathbb{R})$ and $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.

Ex 03. Let $A \in \mathcal{M}_n(\mathbb{R})$. We ask if $x^t A x = 0;\ \forall\, x \in \mathbb{R}^n \overset{\text{implies}}{\Rightarrow} A = 0$?

Ans. No, take the matrix $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.