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Algebra 3 **Test Number 01** 2020 - 2021

Characteristic Polynomial, Eigenvalues and Eigenvectors of a square matrix

Ex 01 : ***a)*** Compute the characteristic polynomial of the matrix

$$A_3 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

Ans: $x^2(3-x)$

b) Deduce the characteristic polynomial of the general matrix:

$$A_n = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix} \in M_n(\mathbb{R})$$

i.e., extend to *n*-dimensions.

Ex 02 : Find the characteristic polynomial of the matrix:

$$A = \begin{pmatrix} 7 & -6 & -2 \\ 2 & 0 & -1 \\ 2 & -3 & 2 \end{pmatrix}, \quad p_A(x) = (x-3)^3$$

Ex 03 : Let $A \in \mathbb{M}_n(\mathbb{R})$ be a square matrix. Prove the following equality

$$p_{rA}(x) = r^n p_A\left(\frac{x}{r}\right) \quad ; r \neq 0$$

Ex 04 : (*i*) Let *A* and *B* be two matrices such that

$$A^2 = B^2 = (AB)^2 = I_n$$

Prove that *AB* = *BA*.

(*ii*) Let $A,\ B \in M_n(\mathbb{R})$. Assume that *A* is invertible. Show that

$$p_{AB}(\lambda) = p_{BA}(\lambda)$$

Ex 05 : Consider the following matrix

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$$

Find all eigenvalues and corresponding eigenvectors for the matrix A.

<u>Ex 06</u> : Determine the eigenvalues and eigenvectors of the following matrices

$$1)\ A = \begin{pmatrix} 1 & 2 \\ 0 & 5 \end{pmatrix}$$

Ans:

eigenvalues	eigenvectors
1	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$
5	$\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$$2)\ A = \begin{pmatrix} 2 & 6 \\ 0 & 2 \end{pmatrix}$$

Ans:

eigenvalues	eigenvector
2 (double)	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$3)\ A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & -5 \end{pmatrix}$$

Ans:

eigenvalues	eigenvectors
1	$(1, 0, 0)$
2	$(2, 1, 0)$
-5	$(5, 6, -14)$

$$4)\ \ A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}$$

Ans :

$$\text{eigenvectors: } \left\{ \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \right\} \leftrightarrow 1, \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\} \leftrightarrow 2 \text{ (double)}$$

$$5)\ \ A = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Ans: $\lambda = 0$ is the only eigenvalue of A (the algebraic multiplicity is 3) $(1,0,0)\,,(0,1,-1)$ are corresponding eigenvectors of λ.

$$6)\quad A = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 3 & 2 \end{pmatrix}$$

Ans: $\lambda = 2$ and $E_\lambda = Vect\{(0,0,1)\}$

$$7)\quad A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

Ans:

eigenvectors: $\left\{\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}\right\} \leftrightarrow 0, \left\{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}\right\} \leftrightarrow 2$

$$8)\quad A = \begin{pmatrix} a & 2 & 3 \\ 0 & 2a & 8 \\ 0 & 0 & 3a \end{pmatrix}, a \in \mathbb{R}$$

Ans :

eigenvalues	eigenvectors
a	$(1,0,0)$
$2a$	$(2,a,0)$
$3a$	$(3a+16, 16a, 2a^2)$

<u>Ex 07</u> : Determine all eigenvalues and their algebraic multiplicities of the matrix

$$A = \begin{pmatrix} 1 & a & 1 \\ a & 1 & a \\ 1 & a & 1 \end{pmatrix},$$

where *a* is a real number.

Similar Matrices, Diagonalizable Matrices

Ex 01 : *(a)* State the definition of two similar matrices $A,\ B \in M_n(\mathbb{R})$ (in this case, we denote $A \sim B$).
(b) Prove that : $A - \lambda I_n \sim B \Rightarrow A \sim \lambda I_n + B$.
(c) Let A and B be two similar matrices, *i.e* ,

$$A = PBP^{-1}$$

for some invertible matrix P, and let (λ, x) be an eigenpair of A. Show that $(\lambda, P^{-1}x)$ is an eigenpair of B.

(d) Let $f_r(x) = a_0 + a_1 x + ... + a_r x^r$ be a polynomial of degree r. If $B = P^{-1}AP$, then prove that

$$f_r(B) = P^{-1} f_r(A) P$$

Ex 02 : Using two methods, prove that similar matrices have the same eigenvalues.

Ex 03 : Prove the following implication (**see** *The exponential of a Matrix* **)**

$$A \sim B \Longrightarrow e^A \sim e^B$$

Ex 04 : Let A and B be the matrices

$$\begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 2 \\ -2 & 5 \end{pmatrix}$$

Prove that A is similar to B.
Find the matrix C such that

$$A = C.B.C^{-1}$$

Ex 05 : Let A be a diagonalizable matrix which has a unique eigenvalue λ. Prove that $A = \lambda I$, where I is the identity matrix.

Ex 06 : Study the diagonalization of the following matrix:

$$A = \begin{pmatrix} 1 & a & b & c \\ 0 & 1 & d & e \\ 0 & 0 & 2 & f \\ 0 & 0 & 0 & 3 \end{pmatrix} \quad ; a \neq 0, b, c, d, e, f \in \mathbb{R}$$

Ex 07 :
Let $A \in \mathbb{M}_n(\mathbb{R})$ be a diagonalizable matrix with $sp(A) = \{-1, 1\}$. Prove that $A^{-1} = A$.

Ex 08 : Let A and B be two diagonalizable matrices with $P^{-1}AP = D_1$ and $P^{-1}BP = D_2$ for some invertible matrix P. Prove that $AB = BA$.

Ex 09 : ***a)*** Determine whether the matrix

$$A = \begin{pmatrix} 3 & 0 & 0 \\ 4 & 1 & 2 \\ a & 0 & 3 \end{pmatrix} ; a \in \mathbb{R}$$

is diagonalizable or not.

b) Let

$$A_\alpha = \begin{pmatrix} 2 & \alpha & 1 \\ 0 & 2 & 0 \\ 0 & 0 & \alpha \end{pmatrix}$$

Find the values α for which the matrix A_α is diagonalizable. ***Ans:*** A_α is diago iff $\alpha = 0$.

Ex 10 : For which values of c the matrix

$$A = \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} \; ; \; c \neq 0$$

is diagonalisable.

Ex 11 : Consider the matrix

$$A(a) = \begin{pmatrix} 1 & 0 & 0 \\ a & -2 & 3 \\ 1 & -1 & 2 \end{pmatrix} , a \in \mathbb{R}$$

Find all the eigenvalues of $A(a)$.
Find the real parameter a for which the matrix $A(a)$ is diagonalizable, and give in this case the diagonalization formula of $A(a)$.

Ex 12 : Let

$$\begin{array}{llll} f & : & \mathbb{P}_3[x] \longrightarrow \mathbb{P}_3[x] \\ p & \longmapsto & f(p) = 3xp - (x^2 - 1)\, p' \end{array}$$

We denote by $B = \{1, x, x^2, x^3\}$ the canonical basis of $\mathbb{P}_3[x]$. Calculate $M_f(B)$.

We ask whether f is diagonalizable or not. If the answer is positive, then give the diagonalization formula of f.

Exponential of a matrix, *cosine*(*matrix*), *sine*, *log*,…

On the powers of a square matrix. How to find A^k ?

Ex 01 :

1. State the definition of the notion « *Exponential of a square matrix* »

2. Calculate e^O , where *O* is the zero matrix.

Let $A \in \mathbb{M}_n(\mathbb{R})$ and $\lambda \in \mathbb{R}$. Prove that

$$e^{\lambda I_n} A = e^{\lambda} A$$

Ex 02 : Let *A* be the matrix

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$$

Prove that *A* is diagonalizable, and deduce *exp*(*A*) .

Ex 03 : Prove the following theorem : ***Theorem*** : *Let* $A \in \mathbb{M}_n(\mathbb{R})$. *Then the series*

$$\sum_{k=0}^{\infty} \frac{A^k}{k!}$$

is absolutely convergent (*then convergent*).

Ex 04 : Let *A* be a square matrix. Prove that

$$\lim_{x \longrightarrow 0} \frac{e^{xA} - I}{x} = A.$$

Ex 05 : Let

$$A = \begin{pmatrix} 1 & 3 & -3 \\ 1 & 3 & -1 \\ -2 & 2 & 0 \end{pmatrix}$$

1. Find all the eigenvalues and eigenvectors of *A*.
2. Deduce that *A* is diagonalizable.
3. Compute *exp*(*A*).

Ex 06 : Consider the following two matrices

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$$

Verify that

$$e^{A+B} \neq e^A.e^B \neq e^B.e^A$$

<u>Ex 07</u> : Consider the matrix

$$A = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}$$

Calculate *logA*. *i.e*, find a square matrix $B \in M_2(\mathbb{R})$ such that $A = e^B$.

<u>Ex 08</u> : Let *A* be a diagonalizable matrix; *i.e* $A = PDP^{-1}$ for some invertible matrix *P*, prove that

$$\lim_{k \longrightarrow \infty} A^k = P. \lim_{k \longrightarrow \infty} D^k.P^{-1}$$

Consider the matrix

$$A = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix}$$

Calculate $\lim_{k \longrightarrow \infty} A^k$

Ans :

$$\begin{aligned} \lim_{k \longrightarrow \infty} A^k &= P. \lim_{k \longrightarrow \infty} D^k.P^{-1} \\ &= \begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix} \lim_{k \longrightarrow \infty} \begin{pmatrix} 1^k & 0 \\ 0 & \left(\frac{1}{4}\right)^k \end{pmatrix} \begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ -\frac{1}{3} & \frac{1}{3} \end{pmatrix} \\ &= \frac{1}{3} \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}. \end{aligned}$$

<u>Ex 09</u> : Determine the eigenvalues and eigenvectors of the following matrix

$$A = \begin{pmatrix} 3 & 1 \\ -2 & 0 \end{pmatrix}$$

For each positive integer *k*, calculate A^k.

Ans :

$$\text{eigenvectors: } (-1,2) \leftrightarrow 1, (-1,1) \leftrightarrow 2$$

$$A^k = \begin{pmatrix} 2^{k+1}-1 & 2^k-1 \\ 2-2^{k+1} & 2-2^k \end{pmatrix}; k \in \mathbb{N}$$

<u>Ex 10</u> : Consider the matrix

$$A = \begin{pmatrix} 9 & 0 & 0 \\ -5 & 4 & 0 \\ -8 & 0 & 1 \end{pmatrix}$$

$$A^n = \begin{pmatrix} 9^n & 0 & 0 \\ 4^n-9^n & 4^n & 0 \\ 1-9^n & 0 & 1 \end{pmatrix}$$

Compute $A^n; n \in \mathbb{N}$.

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System of linear recurrence sequences, System of Differential equations, Cayley-Hamilton Theorem

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Ex 01 : Solve the following system of linear recurrence sequences

$$\begin{cases} x_{n+1} = 3x_n - y_n \\ y_{n+1} = -x_n + 3y_n \end{cases} ;\ (x_0, y_0) = (1, 2)$$

Ans :

$$\begin{cases} x_n = 2^{n-1}(3 - 2^n) \\ y_n = 2^{n-1}(3 + 2^n) \end{cases} ; n \geq 0$$

Ex 02 : Consider the determinant

$$D_n = \det \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{pmatrix}$$

Prove that $D_n = 2D_{n-1} - D_{n-2}$. Calculate the value D_n in terms of *n*.

Ex 03 : Let (x_n) be the sequence given by

$$x_{n+2} = \frac{2}{\frac{1}{x_n} + \frac{1}{x_{n+1}}} ;\ x_0 ,\ x_1 > 0$$

Evaluate the following limit (in terms of the initial values x_0, x_1):

$$\lim_{n \longrightarrow +\infty} x_n$$

Ex 04 : Solve the system of differential equations

$$X' = AX \ ;\ A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}$$

Ans:

$$\begin{cases} \lambda_1 = 1,\ v_1 = (-1, 1, 1) \\ \lambda_2 = 2,\ v_2 = (0, 1, 0) \text{ et } v_3 = (0, 0, 1) \end{cases}$$

Solve the system of differential equations

$$X'(t)=\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}X(t)$$

Ans :

$$\begin{aligned}X(t)&=c_1e^{3t}\begin{pmatrix}1\\1\\1\end{pmatrix}+c_2e^{0t}\begin{pmatrix}-1\\1\\0\end{pmatrix}+c_3e^{0t}\begin{pmatrix}-1\\0\\1\end{pmatrix}\\&=\begin{pmatrix}c_1e^{3t}-c_2-c_3\\c_1e^{3t}+c_2\\c_1e^{3t}+c_3\end{pmatrix},c_i\in\mathbb{R}\end{aligned}$$

Ex 05 : Using Cayley-Hamilton Theorem, calculate the inverse of the matrix

$$M=\begin{pmatrix}1&1&0\\-1&0&0\\2&0&-1\end{pmatrix}$$

Ex 06 : Let $A\in\mathbb{M}_n(\mathbb{R})$ be a square matrix with

$$p_A(x)=c_0+c_1x+\ldots+c_{n-1}x^{n-1}+c_nx^n\ ;\ c_0\neq 0$$

Prove that

$$A^{-1}=\frac{-1}{c_0}\sum_{k=1}^{n}c_kA^{k-1}$$

Ex 07 : Let $A\in\mathbb{M}_n(\mathbb{R})$ be a square matrix which has one eigenvalue, say λ. Prove that

$$e^{tA}=e^{\lambda}\sum_{k=0}^{n-1}(A-\lambda I_n)^k\frac{t^k}{k!}.$$

Solve the system of differential equations

$$X'=\begin{pmatrix}x'\\y'\end{pmatrix}=\begin{pmatrix}2&1\\0&2\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$$

Solve the system of differential equations

$$\begin{pmatrix}x_1'(t)\\x_2'(t)\\x_3'(t)\end{pmatrix}=\begin{pmatrix}-4&1&1\\1&-1&-2\\-2&1&-1\end{pmatrix}\begin{pmatrix}x_1(t)\\x_2(t)\\x_3(t)\end{pmatrix}$$

Let $A\in\mathbb{M}_3(\mathbb{R})$ be a non diagonalizable matrix. Assume that λ, λ, and μ are the eigenvalues of A with $\lambda\neq\mu$.
Prove that

$$e^{tA}=e^{\lambda t}(I+t(A-\lambda I))+\frac{e^{\mu t}-e^{\lambda t}}{(\mu-\lambda)^2}(A-\lambda I)^2-\frac{te^{\lambda t}}{\mu-\lambda}(A-\lambda I)^2.$$

Solve the system of differential equations

$$\begin{pmatrix}x_1'(t)\\x_2'(t)\\x_3'(t)\end{pmatrix}=\begin{pmatrix}2&-1&2\\10&-5&7\\4&-2&2\end{pmatrix}\begin{pmatrix}x_1(t)\\x_2(t)\\x_3(t)\end{pmatrix}$$

Minimal Polynomial, Trigonalizable Matrices

Ex 01 : Calculate the minimal polynomial of the matrix

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

Verify that all the matrices of the form

$$A = \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} \; ; \; c \neq 0$$

are not diagonalisable.

Ex 02 : Determine the minimal polynomial of the following matrices

$$a) \; A = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, b) \; A = \begin{pmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 4 \end{pmatrix}, \; c) \; \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}$$

Ex 03 : If $m_A(x) = (x-a)(x-b)$. Show that A^n can be only written in terms of A and I.

Ex 04 : Calculate the minimal polynomial of the matrix

$$A = \begin{pmatrix} \lambda & & & & \\ 1 & \lambda & & & \\ & \ddots & \ddots & & \\ & & 1 & \lambda & \\ & & & 1 & \lambda \end{pmatrix} \; ; \lambda \in \mathbb{R}$$

Is it diagonalizable ?

Ex 05 : Consider the matrix

$$A = \begin{pmatrix} a & b & b \\ b & a & b \\ b & b & a \end{pmatrix} \; ; \; a, b \in \mathbb{R}$$

Show that A is diagonalizable.

Ex 06 : Prove that the $n \times n$ matrix

$$A=\begin{pmatrix}1&1&\dots&1\\1&1&\dots&1\\\vdots&\vdots&\ddots&\vdots\\1&1&\dots&1\end{pmatrix}\in M_n(\mathbb{R})$$

is diagonalizable.

<u>Ex 07</u> : Trigonalize the matrix

$$A=\begin{pmatrix}2&-1\\1&4\end{pmatrix}\qquad p_A(x)=(x-3)^2$$

Calculate A^n.

<u>Ex 08</u> : Trigonalize the matrix

$$A=\begin{pmatrix}2&-1&2\\10&-5&7\\4&-2&2\end{pmatrix}\qquad p_A(x)=\begin{vmatrix}x-2&-1&2\\10&x+5&7\\4&-2&x-2\end{vmatrix}=x^2(x+1)$$

Calculate A^n for any positive integer *n*.

<u>Ex 09</u> : Trigonalize the matrix

$$A=\begin{pmatrix}7&-6&-2\\2&0&-1\\2&-3&2\end{pmatrix}\ ;\ p_A(x)=(x-3)^3$$

Then, compute A^n.

<u>Ex 10</u> : Consider the matrix

$$A=\begin{pmatrix}0&1&1\\1&0&1\\1&1&0\end{pmatrix}$$

1) Verify that $A^2 = A + 2I$ and deduce the expression of A^{-1} in terms of *A* and *I*, where *I* is the identity 3 by 3 matrix.
2) Using the proof by induction, prove that

$$A^n=a_n.A+b_n.I\,,$$

where

$$a_n,b_n\in\mathbb{N}\text{ and }\begin{cases}a_{n+1}=a_n+b_n\\b_{n+1}=2a_n\end{cases},n\geq 0\quad(\text{S})$$

3) Write the system (*S*) in the matrix form.
4) Calculate a_n and b_n in terms of *n*. Then, deduce the expression of A^n.
5) Calculate the minimal polynomial of the matrix *A*. What to deduce?
6) Solve the system of differential equations $X'=A.X$