1 Trigonalization

Definition 1 *Let $A \in \mathcal{M}_n(\mathbb{K})$. Then A is called **trigonalizable** if there exists an invertible matrix P, that is, $P \in \mathbb{GL}_n(\mathbb{K})$, such that $A = PTP^{-1}$, where T is an upper triangular matrix having the same eigenvalues of A. Or, equivalently, A is similar to a triangular matrix T.*

Now, we present Schur Theorem decomposition of a square matrix $A \in \mathcal{M}_n(\mathbb{C})$.

Theorem 2 *Any matrix with complex entries is trigonalizable over $\mathcal{M}_n(\mathbb{C})$.*

Proof. Let $A \in \mathcal{M}_n(\mathbb{C})$. We will show that A is trigonalizable over $\mathcal{M}_n(\mathbb{C})$. We use induction on n. Indeed, for $n = 1$ we have

$$A = (a_{11}), \text{ where } a_{11} \in \mathbb{C}.$$

In this case, we write

$$A = I(a_{11})I^{-1} = PTP^{-1} \text{ with } P = I = (1) \text{ and } T = (a_{11}) = A.$$

Assume that every matrix $A_1 \in M_n(\mathbb{C})$ is trigonalizable. Let (λ, x) be an eigenpair of A, and let $\{x, u_2, ..., u_n\}$ be a basis of $\mathbb{C}^n$. We put $U = (x, u_2, ..., u_n)$, it follows that

$$AU = \begin{pmatrix} Ax & Au_2 & \dots & Au_n \end{pmatrix} = \begin{pmatrix} \lambda x & Au_2 & \dots & Au_n \end{pmatrix}.$$

Now, calculate $U^{-1}AU$. In fact, we have

$$U^{-1} = U^{-1}Ue_1 = e_1,$$

where $e_1 = (1, 0, ..., 0)$. Therefore,

$$U^{-1}AU = U^{-1}\begin{pmatrix} \lambda x & Au_2 & \dots & Au_n \end{pmatrix} = \begin{pmatrix} \lambda e_1 & U^{-1}Au_2 & \dots & U^{-1}Au_n \end{pmatrix}.$$

Also we obtain

$$U^{-1}AU = \begin{pmatrix} \lambda & \times & ... & \times \\ 0 & * & ... & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & * & ... & * \end{pmatrix} = \begin{pmatrix} \lambda & C \\ 0 & A_1 \end{pmatrix} = T_1,$$

where $C \in M_{1,n-1}(\mathbb{C})$ and $A_1 \in M_{n-1}(\mathbb{C})$. From the hypothesis, there exists an invertible matrix W such that

$$\begin{pmatrix} 1 & C \\ 0 & W^{-1} \end{pmatrix}\begin{pmatrix} \lambda & C \\ 0 & A_1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & W \end{pmatrix} = \begin{pmatrix} \lambda & CW \\ 0 & W^{-1}A_1W \end{pmatrix} = \begin{pmatrix} \lambda & CW \\ 0 & T' \end{pmatrix}.$$

Hence

$$A \sim T_1 \sim \begin{pmatrix} \lambda & CW \\ 0 & T' \end{pmatrix} = T,$$

where T is upper triangular. That is, $A \sim T$. ■

Exercise 3 *Trigonalize the following matrix:*

$$A = \begin{pmatrix} 2 & -1 \\ 1 & 4 \end{pmatrix}.$$

Then, calculate A^n, for $n \geq 0$.

1. *From simple computation, we haev*

$$p_A(x) = (x-3)^2.$$

This means that $\lambda = 3$ is an eigenvalue of A with multiplicity 2, and hence A is not diagonalizable since $A \neq 3I$.

Next, we find the corresponding eigenvectors. In fact, we have

$$\begin{aligned} E_\lambda &= \left\{(x,y) \in \mathbb{R}^2; \begin{array}{c} 2x - y = 3x \\ x + 4y = 3y \end{array}\right\} \\ &= \left\{(x,y) \in \mathbb{R}^2; y = -x\right\} \\ &= Vect\{(1,-1)\} = Vect\{v_1\}. \end{aligned}$$

Let v_2 be a nonzero vector for which $\{v_1, v_2\}$ is a basis of $\mathbb{R}^2$. For example, we put $v_2 = (1,1)$, and let

$$P = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

Therefore,

$$P^{-1}AP = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & -2 \\ 0 & 3 \end{pmatrix} = T$$

That is, $A \sim T$.

Next, we compute A^n : We have

$$A^n = PT^nP^{-1}.$$

It suffices to compute T^n : We write T in the form

$$T = \begin{pmatrix} 3 & -2 \\ 0 & 3 \end{pmatrix} = \underbrace{\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}}_{D} + \underbrace{\begin{pmatrix} 0 & -2 \\ 0 & 0 \end{pmatrix}}_{N}, \text{ where } N^2 = 0.$$

Hence

$$\begin{aligned} T^n &= D^n + nD^{n-1}N \\ &= \begin{pmatrix} 3^n & 0 \\ 0 & 3^n \end{pmatrix} + n\begin{pmatrix} 3^{n-1} & 0 \\ 0 & 3^{n-1} \end{pmatrix}\begin{pmatrix} 0 & -2 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 3^n & -2n \times 3^{n-1} \\ 0 & 3^n \end{pmatrix};\ n \geq 0. \end{aligned}$$

Finally, we deduce that

$$\begin{aligned} A^n &= \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 3^n & -2n \cdot 3^{n-1} \\ 0 & 3^n \end{pmatrix}\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \\ &= \begin{pmatrix} 3^n - n \cdot 3^{n-1} & -n \cdot 3^{n-1} \\ n \cdot 3^{n-1} & n \cdot 3^{n-1} + 3^n \end{pmatrix};\ n \geq 0. \end{aligned}$$

Theorem 4 *For any matrix $A \in \mathcal{M}_n(\mathbb{C})$, we have*

$$\det(A) = \prod_{\lambda \in Sp(A)} \lambda.$$

Recall that $Sp(A)$ consists of all eigenvalues of A.

Proof. We know that A is trigonalizable, and so there exists an invertible matrix $P \in \mathbb{GL}_n(\mathbb{C})$ and an upper triangular matrix T such that

$$A = PTP^{-1} \; (T = (t_{ij}) \text{ with } t_{ii} \in Sp(A)).$$

Therefore,

$$\begin{aligned} \det(A) &= \det\left(PTP^{-1}\right) \\ &= \det(P)\det(T)\det\left(P^{-1}\right) \\ &= \det(T) = t_{11}t_{22}...t_{nn} \\ &= \prod_{\lambda_i \in Sp(A)} \lambda_i. \end{aligned}$$

This completes the proof. ■

Corollary 5 *Let $A \in \mathcal{M}_n(\mathbb{C})$. Then*

$$0 \notin Sp(A) \Rightarrow A \text{ is invertible.}$$

Proof. By Theorem 4, if we have $0 \notin Sp(A)$ then $\det(A) \neq 0$, and so A is invertible. ■