1 Nilpotent Matrices

Definition 1 *A **nilpotent** matrix is a square matrix N such that $N^k = 0$ for some positive integer k.*

In other words, a square matrix N is said to be **nilpotent** if there exists a positive integer k such that $N^k = 0$. The smallest such k is called the **index** of N.

Example 2 *The matrix*

$$N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

is nilpotent with index 2, since $N^2 = 0$.

Proposition 3 *Let N be a nilpotent matrix. Then*

- *$Sp(N) = \{0\}$,*
- *$I - N$ is invertible.*

****

Proof. Assume that $N^k = 0$ and $N^{k-1} \neq 0$ for some $k \geq 1$.

- Let (λ, x) be an eigenpair of N, that is, $Nx = \lambda x$ and $x \neq 0$. It follows that $\lambda^k x = N^k x = 0$, and hence $\lambda = 0$.
- Let $x \in \mathbb{R}^n$ such that $(I - N) x = 0$. Therefore, $Nx = x$, form which it follows that $N^k x = N^{k-1} x = 0$. Since $N^{k-1} \neq 0$, then $x = 0$. Thus, $I - N$ is invertible.

The proof is finished. ■

****

Theorem 4 *Let A be a nonzero nilpotent matrix. Then A is nondiagonalizable.*

Proof. Assume, by the way of contradiction that A is diagonalizable, that is, $A = PDP^{-1}$ for some invertible matrix $P = 0$. Since A is nilpotent, there exists a positive integer k such that $A^k = 0$. It follows that $D = P^{-1}AP$, and so

$$D^k = P^{-1}A^k P = 0.$$

Since D is diagonal, then $D = 0$. This means that $A = 0$, a contradiction. ■

Theorem 5 *Any strictely triangular matrix is nilpotent.*

Proof. Setting

$$A=\left(\begin{array}{cccc}\mathbf{0} & 0 & \cdots & 0\\ a_{21} & \mathbf{0} & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ a_{n1} & a_{n2} & \cdots & \mathbf{0}\end{array}\right).$$

Since $p_A(x)=x^n$. By Cayley-Hamilton theorem, $A^n=$. That is, $\exists\, k\leq n$ such that $A^k=0$, and hence A is nilpotent. ■

Example 6 *Determine the index of the following matrix:*

$$N=\left(\begin{array}{ccc}0 & 1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{array}\right).$$

It is clear that

$$N^2=\left(\begin{array}{ccc}0 & 0 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{array}\right) \text{ and } N^3=\left(\begin{array}{ccc}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{array}\right).$$

Since $N^3=0$ and $N^2\neq 0$, then N is nilpotent of index $k=3$.

Remark 7 *The product of two non-zero matrices can be zero. Indeed, for a matrix $A\in\mathcal{M}_n(\mathbb{R})$, we have*

$$A^2=0\nRightarrow A=0.$$

For example, if $A=\left(\begin{array}{cc}1 & -1\\ 1 & -1\end{array}\right)\neq 0$ we see that

$$A^2=\left(\begin{array}{cc}1 & -1\\ 1 & -1\end{array}\right)\left(\begin{array}{cc}1 & -1\\ 1 & -1\end{array}\right)=\left(\begin{array}{cc}0 & 0\\ 0 & 0\end{array}\right),$$

But, $A\neq 0$.

Example 8 *Consider the matrix*

$$A=\left(\begin{array}{ccc}3 & 9 & -9\\ 2 & 0 & 0\\ 3 & 3 & -3\end{array}\right)$$

Show that A is nilpotent.

Solution. First, we determine the characteristic polynomial of A.

$$\begin{aligned}p_A(x) &= \left|\begin{array}{ccc}3-x & 9 & -9\\ 2 & -x & 0\\ 3 & 3 & -3-x\end{array}\right| = \left|\begin{array}{ccc}3-x & 0 & -9\\ 2 & -x & 0\\ 3 & -x & -3-x\end{array}\right|\\ &= -x\left|\begin{array}{ccc}3-x & 0 & -9\\ 2 & 1 & 0\\ 3 & 1 & -3-x\end{array}\right|\\ &= -x^3.\end{aligned}$$

By Cayley-Hamilton theorem, $A^3=0$. Since $A^2\neq 0$, then A is nilpotente of index 3.

Theorem 9 *Let N be a nilpotent matrix of index k and let $x \in \mathbb{R}^n$ be a nonzero vector such that $N^{k-1}x \neq 0$. The family*

$$\{Ix, Nx, N^2x, ..., N^{k-1}x\}$$

is free.

Proof. Let $(\alpha_i)_{0 \leq i \leq k-1} \in \mathbb{R}$ such that

$$\sum_{i=0}^{k-1} \alpha_i N^i x = 0,$$

from which it follows that

$$\left\{\begin{array}{c} \alpha_0 N^{k-1}x + \alpha_1 N^k x + ... + \alpha_{k-1} N^{2k-2} x = 0 \\ \alpha_0 N^{k-2}x + \alpha_1 N^{k-1} x + ... + \alpha_{k-1} N^{2k-3} x = 0 \\ \vdots \\ \alpha_0 N x + \alpha_1 N^2 x + ... + \alpha_{k-1} N^k x = 0 \\ \alpha_0 I x + \alpha_1 N x + ... + \alpha_{k-1} N^{k-1} x = 0 \end{array}\right. \Rightarrow \left\{\begin{array}{c} \alpha_0 N^{k-1}x = 0 \\ \alpha_1 N^{k-1}x \\ \vdots \\ \alpha_{k-2} N^{k-1}x = 0 \\ \alpha_{k-1} N^{k-1}x = 0 \end{array}\right.$$

Since $N^{k-1}x \neq 0$, then $\alpha_0 = \alpha_1 = ... = \alpha_{k-1} = 0$. This completes the proof. ■

1.1 Problems

Ex 01. Let $A \in \mathcal{M}_n(\mathbb{R})$ be a nilpotent matrix. Prove that

$$\det(A + I_n) = 1.$$

Ex 02. We ask if $A^2 = 0 \Rightarrow A = 0$?

Ex 03. Verify that

$$A = \begin{pmatrix} 1 & -3 & -4 \\ -1 & 3 & 4 \\ 1 & -3 & -4 \end{pmatrix}$$

is nilpotent.

Ex 04. Let

$$A = \begin{pmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{pmatrix}$$

Calculate A^3. What do you say ?

Ex 05. Prove the result: If N is nilpotent, then $I + N$ and $I - N$ are invertible, where I is the identity matrix.

Ex 06. Prove that

$$A \sim 2A \Rightarrow A \text{ is nilpotent over } \mathbb{R}.$$