1 The Matrix Exponential

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Note that the exponential of a matrix deals in particular in solving systems of linear differential equations. In the following section, we present some remarkable definitions and properties on the exponential of a square matrix which may or may not be diagonalizable.

Definition 1 *For each $n \times n$ complex matrix A, define the exponential of A to be the matrix*

$$e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!} = I_n + \frac{A}{1!} + \frac{A^2}{2!} + ... + \frac{A^k}{k!} + ...$$

This is the matrix exponential of A.

Note that if $A = 0$ (the zero matrix); we have $e^0 = I_n$. Indeed, we see that

$$e^0 = I_n + \frac{0}{1!} + \frac{0}{2!} + ... + \frac{0}{k!} + ... = I_n.$$

We also have for every $k \in \mathbb{Z}$, $e^{kA} = \left(e^A\right)^k$.

Example 2 *Consider the matrix*

$$A = \begin{pmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{pmatrix}.$$

Calculate A^2 and A^3. Deduce e^A.

Indeed, according computation, we have

$$A^2 = \begin{pmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{pmatrix} \begin{pmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 3 & 3 & 9 \\ -1 & -1 & -3 \end{pmatrix}$$

Moreover,

$$A^3 = \begin{pmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 3 & 3 & 9 \\ -1 & -1 & -3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Using Definition 1, we obtain

$$\begin{aligned} e^A &= I_3 + \frac{A}{1!} + \frac{A^2}{2!} \\ &= I_3 + A + \frac{A^2}{2} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 \\ 3 & 3 & 9 \\ -1 & -1 & -3 \end{pmatrix} \\ &= \begin{pmatrix} 2 & 1 & 3 \\ \frac{13}{2} & \frac{9}{2} & \frac{21}{2} \\ \frac{-5}{2} & \frac{-3}{2} & \frac{-7}{2} \end{pmatrix}. \end{aligned}$$

It is easy to calculate the exponential of a diagonal matrix. We have

Corollary 3 *Let D be a diagonal matrix, i.e.,*

$$D = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix} = diag\left\{\lambda_1, \lambda_2, ..., \lambda_n\right\}.$$

Then

$$e^D = \begin{pmatrix} e^{\lambda_1} & & & \\ & e^{\lambda_2} & & \\ & & \ddots & \\ & & & e^{\lambda_n} \end{pmatrix} = diag\left\{e^{\lambda_1}, e^{\lambda_2}, ..., e^{\lambda_n}\right\}. \tag{1}$$

Proof. In fact, for each $k \geq 0$ we have

$$D^k = \begin{pmatrix} \lambda_1^k & & & \\ & \lambda_2^k & & \\ & & \ddots & \\ & & & \lambda_n^k \end{pmatrix}.$$

From definition 1, we get

$$\begin{aligned} e^D &= \sum_{k=0}^{+\infty} \frac{D^k}{k!} \\ &= \sum_{k=0}^{+\infty} \frac{1}{k!} \begin{pmatrix} \lambda_1^k & & & \\ & \lambda_2^k & & \\ & & \ddots & \\ & & & \lambda_n^k \end{pmatrix} \\ &= \begin{pmatrix} \sum_{k=0}^{+\infty} \frac{\lambda_1^k}{k!} & & & \\ & \sum_{k=0}^{+\infty} \frac{\lambda_2^k}{k!} & & \\ & & \ddots & \\ & & & \sum_{k=0}^{+\infty} \frac{\lambda_n^k}{k!} \end{pmatrix} \\ &= \begin{pmatrix} e^{\lambda_1} & & & \\ & e^{\lambda_2} & & \\ & & \ddots & \\ & & & e^{\lambda_n} \end{pmatrix}. \end{aligned}$$

This completes the proof. ■

Example 4 *Let*

$$A = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}.$$

Calculate e^A.

In fact, by (1), *we have*

$$e^A = \begin{pmatrix} e^{-1} & 0 \\ 0 & e^2 \end{pmatrix}.$$

Proposition 5 *Let* $A \in \mathcal{M}_n(\mathbb{R})$ *be a diagonalizable matrix. Then* e^A *is also diagonalizable. In addition, we have*

$$A = PDP^{-1} \Rightarrow e^A = Pe^D P^{-1}.$$

Proof. Let $A \in \mathcal{M}_n(\mathbb{R})$ be a diagonalizable matrix. Then there exists an invertible matrix P such that $A = PDP^{-1}$ with D is diagonal. Therefore,

$$\begin{aligned} e^A &= \sum_{k=0}^{+\infty} \frac{A^k}{k!} = \sum_{k=0}^{+\infty} \frac{\left(PDP^{-1}\right)^k}{k!} \\ &= \sum_{k=0}^{+\infty} \frac{PD^k P^{-1}}{k!} \\ &= P\left(\sum_{k=0}^{+\infty} \frac{D^k}{k!}\right) P^{-1} \\ &= Pe^D P^{-1}. \end{aligned}$$

As required. ■

Theorem 6 *Let* $S \in \mathbb{GL}_n(\mathbb{R})$ *be an invertible matrix and let* $A \in \mathcal{M}_n(\mathbb{R})$. *We have*

$$e^{SAS^{-1}} = Se^A S^{-1}.$$

Proof. Let $S \in \mathbb{GL}_n(\mathbb{R})$ and let $A \in \mathcal{M}_n(\mathbb{R})$. From Definition 1, we have

$$\begin{aligned} e^{SAS^{-1}} &= I_n + \frac{SAS^{-1}}{1!} + \frac{\left(SAS^{-1}\right)^2}{2!} + \frac{\left(SAS^{-1}\right)^3}{3!} + ... \\ &= I_n + \frac{SAS^{-1}}{1!} + \frac{SA^2S^{-1}}{2!} + + \frac{SA^3S^{-1}}{3!} + ... \\ &= SI_nS^{-1} + \frac{SAS^{-1}}{1!} + \frac{SA^2S^{-1}}{2!} + + \frac{SA^3S^{-1}}{3!} + ... \\ &= S\left(I_n + \frac{A}{1!} + \frac{A^2}{2!} + \frac{A^3}{3!} + ...\right) S^{-1} \\ &= Se^A S^{-1}. \end{aligned}$$

The proof is finished. ■

Corollary 7 *Let $A \in \mathcal{M}_n(\mathbb{R})$ and let (λ, x) be an eigenpair of A. Then $\left(e^\lambda, x\right)$ is an eigenpair of e^A.*

Proof. Assume that (λ, x) is an eigenpair of A. By definition, we have

$$\begin{aligned} e^A x &= \left(\sum_{k=0}^{+\infty} \frac{A^k}{k!}\right) x = \sum_{k=0}^{+\infty} \frac{A^k x}{k!} \\ &= \sum_{k=0}^{+\infty} \frac{\lambda^k x}{k!} = \left(\sum_{k=0}^{+\infty} \frac{\lambda^k}{k!}\right) x \\ &= e^\lambda x. \end{aligned}$$

This completes the proof. ■

Lemma 8 *We have the following two properties:*

(i) For any $A \in \mathcal{M}_n(\mathbb{R})$ and for any $t \in \mathbb{R}$,

$$Ae^{At} = e^{At} A.$$

(ii) For any $A \in \mathcal{M}_n(\mathbb{R})$ and for any $t \in \mathbb{R}$,

$$e^{tI_n} = e^t A.$$

Proof. By the definition, we have

$$Ae^{At} = A \sum_{i=0}^{+\infty} \frac{A^k t^k}{k!} = \sum_{i=0}^{+\infty} \frac{A^{k+1} t^k}{k!} = \left(\sum_{i=0}^{+\infty} \frac{A^k t^k}{k!}\right) A = e^{At} A.$$

Likewise, we have

$$e^{tI_n} = e^{\begin{pmatrix} t & & \\ & \ddots & \\ & & t \end{pmatrix}} = \begin{pmatrix} e^t & & \\ & \ddots & \\ & & e^t \end{pmatrix} = e^t \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix} = e^t I_n.$$

The proof is finished. ■

Remark 9 *According to the previous lemma, we have*

$$e^{tI_n} I_n = e^{tI_n} = e^t I_n.$$

Note that $e^{tI_n} \neq e^t$; because $e^{tI_n} \in \mathcal{M}_n(\mathbb{R})$ and $e^t \in \mathbb{R}$.

The integer series which defines the exponential of a real, or complex number, is also convergent for a matrix. In addition, we have

Theorem 10 *For any matrix $A \in \mathcal{M}_n(\mathbb{C})$, the series*

$$\sum_{k=0}^{+\infty} \frac{A^k}{k!}$$

is absolutely convergent (therefore convergent) in $\mathcal{M}_n(\mathbb{C})$.

Proof. For each $k \geq 0$, we have

$$\left\| \frac{A^k}{k!} \right\| \leq \frac{\|A\|^k}{k!}$$

and according to d'Alembert's Rule[1], we obtain

$$\lim_{k \to +\infty} \left| \frac{\frac{\|A\|^{k+1}}{(k+1)!}}{\frac{\|A\|^k}{k!}} \right| = \lim_{k \to +\infty} \frac{\|A\|}{k+1} = 0 < 1.$$

Thus, $\sum\limits_{k=0}^{+\infty} \dfrac{A^k}{k!}$ is convergent. Since

$$\left\| \sum_{k=0}^{+\infty} \frac{A^k}{k!} \right\| \leq \sum_{k=0}^{+\infty} \frac{\|A\|^k}{k!},$$

It follows that $\sum\limits_{k=0}^{+\infty} \dfrac{A^k}{k!}$ is therefore absolutely convergent. ■

Also we have the following proposition.

Proposition 11 *Let A be a square matrix. Then*

$$\lim_{x \to 0} \frac{e^{xA} - I}{x} = A.$$

Proof. We know that

$$e^{xA} - I - xA = \frac{(xA)^2}{2!} + \frac{(xA)^3}{3!} + ...$$

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[1] Let $\sum u_n$ be a series with positive terms. If the limit (finite or not)

$$l = \lim \frac{u_{n+1}}{u_n}$$

exists, then

1. The series $\sum u_n$ is convergent if $l < 1$,
2. The series $\sum u_n$ is divergent if $l > 1$.

So we can write

$$\begin{aligned}\left\|e^{xA} - I - xA\right\| &= \left\|\frac{(xA)^2}{2!} + \frac{(xA)^3}{3!} + \ldots\right\| \\ &\leq \frac{\|xA\|^2}{2!} + \frac{\|xA\|^3}{3!} + \ldots \\ &= e^{\|xA\|} - 1 - \|xA\|.\end{aligned}$$

For every $x \neq 0$, we obtain

$$\left\|\frac{e^{xA} - I}{x} - A\right\| \leq \frac{e^{\|xA\|} - 1 - \|xA\|}{|x|} = \left(\frac{e^{|x|.\|x\|} - 1}{|x|} - \|A\|\right) \to 0.$$

As required. ■

1.1 Problems

Ex 01. Are the matrices

$$A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},\ B = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}, C = \begin{pmatrix} -1 & 0 \\ 0 & -4 \end{pmatrix}$$

exponentials of matrices?

Ex 02. Prove that the matrix

$$J_2 = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$$

is neither the square nor the exponential of any matrix of $\mathcal{M}_2(\mathbb{R})$, but the matrices

$$J_4 = \begin{pmatrix} J_2 & \mathbf{0} \\ \mathbf{0} & J_2 \end{pmatrix} \text{ and } J_3 = \begin{pmatrix} J_2 & I_2 \\ \mathbf{0} & J_2 \end{pmatrix}$$

are the square and the exponential of a matrix of $\mathcal{M}_4(\mathbb{R})$.

Ex 03. Let

$$A = \begin{pmatrix} a & b & c \\ 0 & a & b \\ 0 & 0 & a \end{pmatrix}.$$

Calculate e^A.

Ex 04. Let

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Calculate $e^A e^B$, e^{A+B} and $e^B e^A$.

Ex 05. Considère the following matrices

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}.$$

Calculate $C = e^{A+B}$, $D = e^A e^B$ and $F = e^B e^A$. Check that $C \neq D \neq F$.

Ex 06. Consider the matrix

$$A = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}.$$

Calculate $\log A$. i.e., find a matrix $B \in \mathcal{M}_2(\mathbb{C})$ such that $A = e^B$.

Ex 07. Consider the matrices

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}.$$

Calculate e^A, e^B. Deduce the expression of e^F, where

$$F = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$