1 System of linear recurrence sequences

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1.1 Form I (without initial values)

Let (x_n) and (y_n) be two sequences given by the following relation:

$$\begin{cases} x_{n+1} = a_{11}x_n + a_{12}y_n \\ y_{n+1} = a_{21}x_n + a_{22}y_n \end{cases} ; \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}. \tag{1}$$

In the matrix form, we get

$$\underbrace{\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix}}_{X_{n+1}} = \underbrace{\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x_n \\ y_n \end{pmatrix}}_{X_n}.$$

Or, equivalently, we write (1) in the form

$$X_{n+1} = AX_n \text{ , where } X_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}.$$

Consequently,

$$X_n = AX_{n-1} = A(AX_{n-2}) = A^2X_{n-2} = ... = A^nX_0. \tag{2}$$

Remark 1 *If it is given to us X_1, we have only $X_n = A^{n-1}X_1$.*

In the general case, a system of k linear recurrence sequences $x_n^{(i)}$, $i = 1, 2, ..., k$ is given by

$$\begin{cases} x_{n+1}^{(1)} = a_{11}x_n^{(1)} + a_{12}x_n^{(2)} + ... + a_{1k}x_n^{(k)} \\ x_{n+1}^{(2)} = a_{21}x_n^{(1)} + a_{22}x_n^{(2)} + ... + a_{2k}x_n^{(k)} \\ \vdots \\ x_{n+1}^{(k)} = a_{k1}x_n^{(1)} + a_{k2}x_n^{(2)} + ... + a_{kk}x_n^{(k)} \end{cases} ; \; x_0^{(i)} \in \mathbb{R}, \text{ for } i = 1, 2, ..., k. \tag{3}$$

In the matrix form

$$\underbrace{\begin{pmatrix} x_{n+1}^{(1)} \\ x_{n+1}^{(2)} \\ \vdots \\ x_{n+1}^{(k)} \end{pmatrix}}_{X_{n+1}} = \underbrace{\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1k} \\ a_{21} & a_{22} & \dots & a_{2k} \\ \vdots & \vdots & \dots & \vdots \\ a_{k1} & a_{k2} & \dots & a_{kk} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x_n^{(1)} \\ x_n^{(2)} \\ \vdots \\ x_n^{(k)} \end{pmatrix}}_{X_n},$$

where $X_0 = \begin{pmatrix} x_0^{(1)} \\ x_0^{(2)} \\ \vdots \\ x_0^{(k)} \end{pmatrix}$. As in (2), we get

$$X_n = A^nX_0.$$

These problems (the solution of (1) or (3)) reduce to the computation of A^n.

Consider the following example:

Example 2 *Solve the system of linear recurrence sequences*

$$\left\{ \begin{array}{l} x_{n+1} = 2x_n - y_n \\ y_{n+1} = -x_n + 2y_n \end{array} \right. ; \ (x_0, y_0) = (0, -1). \tag{4}$$

Solution. First, we write the system (4) according to the equivalent matrix form

$$\underbrace{\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix}}_{X_{n+1}} = \underbrace{\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x_n \\ y_n \end{pmatrix}}_{X_n} ; X_0 = \begin{pmatrix} 0 \\ -1 \end{pmatrix}.$$

From (2), we have $X_n = A^n X_0$. Moreover, from the previous computation, an explicit formula if A^n in terms of n is given by

$$A^n = \begin{pmatrix} \dfrac{1+3^n}{2} & \dfrac{1-3^n}{2} \\ \dfrac{1-3^n}{2} & \dfrac{1+3^n}{2} \end{pmatrix} ; n \geq 0. \tag{5}$$

It follows that

$$X_n = A^n X_0 = \begin{pmatrix} \dfrac{1+3^n}{2} & \dfrac{1-3^n}{2} \\ \dfrac{1-3^n}{2} & \dfrac{1+3^n}{2} \end{pmatrix} \begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} \dfrac{3^n-1}{2} \\ \dfrac{-3^n-1}{2} \end{pmatrix}. \tag{6}$$

1.2 Form II (with initial values)

Consider the system of linear recurrence sequences $x_n^{(i)}$, for $i = 1, 2, ..., k$:

$$\left\{ \begin{array}{l} x_{n+1}^{(1)} = a_{11}x_n^{(1)} + a_{12}x_n^{(2)} + ... + a_{1k}x_n^{(k)} + c_1 \\ x_{n+1}^{(2)} = a_{21}x_n^{(1)} + a_{22}x_n^{(2)} + ... + a_{2k}x_n^{(k)} + c_2 \\ \vdots \\ x_{n+1}^{(k)} = a_{k1}x_n^{(1)} + a_{k2}x_n^{(2)} + ... + a_{kk}x_n^{(k)} + c_k \end{array} \right. ; \ c_i, x_0^{(i)} \in \mathbb{R}, \text{ for } i = 1, 2, ..., k.$$

In the matrix form

$$\underbrace{\begin{pmatrix} x_{n+1}^{(1)} \\ x_{n+1}^{(2)} \\ \vdots \\ x_{n+1}^{(k)} \end{pmatrix}}_{X_{n+1}} = \underbrace{\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1k} \\ a_{21} & a_{22} & \dots & a_{2k} \\ \vdots & \vdots & \dots & \vdots \\ a_{k1} & a_{k2} & \dots & a_{kk} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x_n^{(1)} \\ x_n^{(2)} \\ \vdots \\ x_n^{(k)} \end{pmatrix}}_{X_n} + \underbrace{\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{pmatrix}}_{C},$$

where $X_0 = \begin{pmatrix} x_0^{(1)} \\ x_0^{(2)} \\ \vdots \\ x_0^{(k)} \end{pmatrix}$. This means that

$$\begin{aligned} X_n &= AX_{n-1} + C = A(AX_{n-2} + C) + C = A^2X_{n-2} + (A + I)C \\ &= ... \\ &= A^nX_0 + \left(A^{n-1} + A^{n-2} + ... + A + I\right)C. \end{aligned} \tag{7}$$

These problems are reduced to the computation of A^n and $\sum\limits_{i=0}^{n-1} A^i$.

Example 3 *Solve the system of linear recurrence sequences*

$$\left\{\begin{array}{l} x_{n+1}=2x_n-y_n-1 \\ y_{n+1}=-x_n+2y_n+2 \end{array}\right. ;\ \ (x_0,y_0)=(0,-1). \tag{8}$$

Solution. The system (8) can be written in the following matrix form:

$$\underbrace{\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix}}_{X_{n+1}} = \underbrace{\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x_n \\ y_n \end{pmatrix}}_{X_n} + \underbrace{\begin{pmatrix} -1 \\ 2 \end{pmatrix}}_{C}$$

It suffices to compute $A^{n-1}+A^{n-2}+...+A+I$. Indeed, in view of (5) we can write

$$A^n = \begin{pmatrix} \dfrac{1+3^n}{2} & \dfrac{1-3^n}{2} \\ \dfrac{1-3^n}{2} & \dfrac{1+3^n}{2} \end{pmatrix} = \frac{1}{2}U + \frac{3^n}{2}V,$$

where

$$U=\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \text{ and } V=\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

It follows that

$$\begin{aligned} A^{n-1}+A^{n-2}+...+A+I &= \frac{n}{2}U+\left(\frac{1+3+...+3^{n-1}}{2}\right)V \\ &= \frac{n}{2}U+\left(\frac{3^n-1}{4}\right)V. \end{aligned}$$

Finally, from (7) we have

$$\begin{aligned} X_n &= \left(\frac{1}{2}U+\frac{3^n}{2}V\right)X_0+\left[\frac{n}{2}U+\left(\frac{3^n-1}{4}\right)V\right]C = \left(\frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}+\frac{3^n}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\right)\begin{pmatrix} 0 \\ -1 \end{pmatrix} \\ &+\left[\frac{n}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}+\left(\frac{3^n-1}{4}\right)\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\right]\begin{pmatrix} -1 \\ 2 \end{pmatrix} \\ &= \begin{pmatrix} \dfrac{2n-3^n+1}{4} \\ \dfrac{2n+3^n-5}{4} \end{pmatrix};\ n\geq 0. \end{aligned}$$

Exercise 4 *Let* $A\in\mathcal{M}_2(\mathbb{R})$. *Assume* $(A-I_2)^{-1}$ *exists, prove that*

$$A^{n-1}+A^{n-2}+...+A+I=(A^n-I_2)(A-I_2)^{-1}.$$