1 Linear recurrence sequences of order k

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Let $(a_0, a_1, ..., a_{k-1})$ be a system of k real numbers not all zero. A **linear recurrence sequence of order** k is defined as follows:

$$\begin{cases} x_{n+k} = a_0 x_n + a_1 x_{n+1} + ... + a_{k-1} x_{n+k-1}, \\ x_0, x_1, ..., x_{k-1} \in \mathbb{R} \text{ are given.} \end{cases}$$

Thus, a sequence defined by a **linear recurrence relation** is uniquely determined by its first k terms: $x_0, x_1, ..., x_{k-1}$. As an example, for $k = 2$:

$$\begin{cases} x_{n+2} = a_0 x_n + a_1 x_{n+1}, \\ x_0, x_1 \in \mathbb{R} \text{ are given.} \end{cases} \tag{S}$$

In the equivalent vector-matrix system, we obtain

$$\begin{pmatrix} x_{n+2} \\ x_{n+1} \end{pmatrix} = \begin{pmatrix} a_1 & a_0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_{n+1} \\ x_n \end{pmatrix},$$

or equivalently

$$\underbrace{\begin{pmatrix} x_{n+1} \\ x_{n+2} \end{pmatrix}}_{X_{n+2}} = \underbrace{\begin{pmatrix} 0 & 1 \\ a_0 & a_1 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x_n \\ x_{n+1} \end{pmatrix}}_{X_{n+1}}, \tag{S$_1$}$$

from which it follows that

$$X_n = AX_{n-1} = A^2 X_{n-2} = ... = A^{n-1} X_1, \tag{1}$$

where $X_1 = \begin{pmatrix} x_0 \\ x_1 \end{pmatrix}$. Thus, we must compute A^n for $n \geq 0$.

Application. Consider the following example:

Example 1 *Let (x_n) be the sequence given by*

$$x_{n+2} = \frac{2}{\frac{1}{x_n} + \frac{1}{x_{n+1}}} \text{ ; } x_0, x_1 \in \mathbb{R}_+^*. \tag{2}$$

Find the formula of x_n in terms of n, then calculate $\lim_{n \to +\infty} x_n$.

Solution. In fact, we write (2) in the form

$$\frac{2}{x_n} = \frac{1}{x_{n-2}} + \frac{1}{x_{n-1}}.$$

Setting $\frac{2}{x_n} = y_n$, we get

$$2y_n = y_{n-1} + y_{n-2}, \text{ that is, } y_n = \frac{1}{2} y_{n-1} + \frac{1}{2} y_{n-2}.$$

In the equivalent vector-matrix system, we have

$$\begin{pmatrix} y_n \\ y_{n-1} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} y_{n-1} \\ y_{n-2} \end{pmatrix}; \begin{cases} y_0 = \dfrac{1}{x_0} \\ y_1 = \dfrac{1}{x_1} \end{cases}$$

Therefore,

$$\begin{pmatrix} y_n \\ y_{n-1} \end{pmatrix} = A^{n-1} \begin{pmatrix} y_1 \\ y_0 \end{pmatrix}, \text{ where } A = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & 0 \end{pmatrix}.$$

From the computation (the matrix diagonalizable), we obtain

$$A^{n-1} = \begin{pmatrix} \dfrac{1}{3}\left[2 + \left(\frac{-1}{2}\right)^{n-1}\right] & \dfrac{1}{3}\left[1 - \left(\frac{-1}{2}\right)^{n-1}\right] \\ \dfrac{1}{3}\left[2 - 2\left(\frac{-1}{2}\right)^{n-1}\right] & \dfrac{1}{3}\left[1 + 2\left(\frac{-1}{2}\right)^{n-1}\right] \end{pmatrix},$$

and so

$$y_n = \frac{1}{3}\left[2 + \left(\frac{-1}{2}\right)^{n-1}\right] y_1 + \frac{1}{3}\left[1 - \left(\frac{-1}{2}\right)^{n-1}\right] y_0.$$

Since $x_n = \dfrac{1}{y_n}$, it follows that

$$x_n = \frac{3}{\left[2 + \left(\frac{-1}{2}\right)^{n-1}\right] \frac{1}{x_1} + \left[1 - \left(\frac{-1}{2}\right)^{n-1}\right] \frac{1}{x_0}}.$$

Passing to the limit as n tends to infinity, we get

$$\lim_{n \to +\infty} x_n = \frac{3}{\frac{2}{x_1} + \frac{1}{x_0}}.$$