1 The square root of a diagonalizable matrix

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Lemma 1 *Let*

$$D = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}, \text{ where } \lambda_i > 0 \ (1 \leq i \leq n).$$

Then

$$\sqrt{D} = \begin{pmatrix} \sqrt{\lambda_1} & & & \\ & \sqrt{\lambda_2} & & \\ & & \ddots & \\ & & & \sqrt{\lambda_n} \end{pmatrix}.$$

Proof. It is clear by computation that $\sqrt{D}\sqrt{D} = D$. ■

Proposition 2 *Let $A \in \mathcal{M}_n(\mathbb{R})$ be a diagonalizable matrix with $Sp\,(A) \subset \mathbb{R}_+$. Then $\sqrt{A} \in \mathcal{M}_n(\mathbb{R})$.*

Proof. Assume that $A = PDP^{-1}$, where $Sp\,(D) \subset \mathbb{R}_+$. We put

$$H = P\sqrt{D}P^{-1} \in \mathcal{M}_n(\mathbb{R}).$$

Since $\sqrt{D}\sqrt{D} = D$, it follows that

$$H^2 = \left(P\sqrt{D}P^{-1}\right)\left(P\sqrt{D}P^{-1}\right) = PDP^{-1} = A.$$

Thus, $\sqrt{A} = H$. ■

Example 3 *Consider the matrix*

$$A = \begin{pmatrix} 11 & -5 & 5 \\ -5 & 3 & -3 \\ 5 & -3 & 3 \end{pmatrix}.$$

Calculate $\sqrt{A}$.

After simple computation, the eigenpairs of A are:

$$\begin{cases} \lambda_1 = 0, \ E_{\lambda_1} = Vect\left\{(0,1,1)\right\}, \\ \lambda_2 = 1, \ E_{\lambda_2} = Vect\left\{(-1,-1,1)\right\}, \\ \lambda_3 = 16, \ E_{\lambda_3} = Vect\left\{(2,-1,1)\right\}. \end{cases}$$

Further, we see that

$$P = \begin{pmatrix} 0 & -1 & 2 \\ 1 & -1 & -1 \\ 1 & 1 & 1 \end{pmatrix}, \ D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 16 \end{pmatrix} \text{ and } P^{-1} = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{6} & \frac{1}{6} \end{pmatrix}.$$

Which gives

$$\begin{aligned}\sqrt{A} &= P\sqrt{D}P^{-1}\\ &= \begin{pmatrix} 0 & -1 & 2\\ 1 & -1 & -1\\ 1 & 1 & 1\end{pmatrix}\begin{pmatrix}\sqrt{0} & 0 & 0\\ 0 & \sqrt{1} & 0\\ 0 & 0 & \sqrt{16}\end{pmatrix}\begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2}\\ -\frac{1}{3} & -\frac{1}{3} & \frac{1}{3}\\ \frac{1}{3} & -\frac{1}{6} & \frac{1}{6}\end{pmatrix}\\ &= \begin{pmatrix} 3 & -1 & 1\\ -1 & 1 & -1\\ 1 & -1 & 1\end{pmatrix}.\end{aligned}$$

Definition 4 *Let $A = PDP^{-1}$ be a diagonalizable matrix whose eigenvalues are given by the diagonal matrix*

$$D = diag\left\{\lambda_1, \lambda_2, ..., \lambda_n\right\}.$$

For any function $f(x)$ defined at the points $(\lambda_i)_{1\leq i\leq n}$, we have

$$f(A) = Pf(D)P^{-1} = P\begin{pmatrix} f(\lambda_1) & & & \\ & f(\lambda_2) & & \\ & & \ddots & \\ & & & f(\lambda_n)\end{pmatrix}P^{-1}.$$

For example, if $A \in \mathcal{M}_n(\mathbb{R})$ with $A = PDP^{-1}$ then

$$\begin{cases} f(x) = x^k \Rightarrow f(A) = A^k = PD^kP^{-1} \quad \text{for } k \geq 0\\ f(x) = \sqrt{x} \Rightarrow f(A) = \sqrt{A} = P\sqrt{D}P^{-1}\\ f(x) = \cos x \Rightarrow f(A) = \cos A = P(\cos D)P^{-1}\\ f(x) = e^x \Rightarrow f(A) = e^A = Pe^DP^{-1}\\ \qquad\qquad ...\end{cases}$$

1.1 Problems.

Ex 01. Let M be a real n by n matrix. We denote by $\cos M$ the real part of e^{iM} and $\sin M$ its imaginary part.

1. Show that $\cos M$ and $\sin M$ commute and that

$$(\cos M)^2 + (\sin M)^2 = I_n.$$

2. Let θ be a real number. Calculate

$$\cos\begin{pmatrix}\theta & 1\\ 0 & \theta\end{pmatrix} \text{ and } \sin\begin{pmatrix}\theta & 1\\ 0 & \theta\end{pmatrix}.$$

Ex 02. Let

$$A = \begin{pmatrix} 0 & i\\ i & 0\end{pmatrix} \in \mathcal{M}_2(\mathbb{C}).$$

Calculate $\sqrt{A}$.