

1 Linear Systems of differential equations, Part I

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Define the **linear system of differential equations** $(x'_1(t), x'_2(t), \dots, x'_n(t))$ by

$$\begin{cases} x'_1(t) = a_{11}x_1(t) + a_{12}x_2(t) + \dots + a_{1n}x_n(t) + f_1(t) \\ x'_2(t) = a_{21}x_1(t) + a_{22}x_2(t) + \dots + a_{2n}x_n(t) + f_2(t) \\ \vdots \\ x'_n(t) = a_{n1}x_1(t) + a_{n2}x_2(t) + \dots + a_{nn}x_n(t) + f_n(t), \end{cases} \quad (1)$$

where $a_{ij} \in \mathbb{R}$. The unknowns are the functions $x_1(t), x_2(t), \dots, x_n(t)$ which are derivable and $f_i(t)$ are some given functions.

The system is called **homogeneous** if all $f_i = 0$, otherwise it is called **non-homogeneous**.

Matrix Notation

A non-homogeneous system of linear equations (1) is written as the equivalent vector-matrix system

$$X'(t) = A \cdot X(t) + f(t),$$

where

$$X(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}, \quad A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}, \quad f = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix}$$

In this section, we consider only homogeneous systems: We wish to solve the system

$$X' = AX. \quad (2)$$

There are two cases:

Case 1. Assume that A is diagonalizable. There exists an invertible matrix P such that $A = PDP^{-1}$, where D is diagonal. Thus,

$$\begin{cases} X' = PDP^{-1}X = PY' \\ Y' = DY \\ Y = P^{-1}X. \end{cases}$$

The system (2) becomes

$$Y' = DY,$$

which is easier to solve since D is diagonal. Then after, we solve the equation $Y = P^{-1}X$, that is, $X = PY$.

Example 1 Solve the system of differential equations:

$$X' = AX, \quad A = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}, \quad \text{where } X(0) = \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$

Solution. At first, the eigenvalues of A are $\lambda_1 = -1$ and $\lambda_2 = 4$. The corresponding eigenvectors are $v_1 = (1, -1)$ and $v_2 = (2, 3)$. Thus, we have

$$D = \begin{pmatrix} -1 & 0 \\ 0 & 4 \end{pmatrix}, P = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}.$$

We put $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $Y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$. It follows that

$$Y' = DY \Leftrightarrow \begin{cases} y_1' = -y_1 \\ y_2' = 4y_2 \end{cases} \Leftrightarrow \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} c_1 e^{-t} \\ c_2 e^{4t} \end{pmatrix},$$

and hence

$$X = PY = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} c_1 e^{-t} \\ c_2 e^{4t} \end{pmatrix} = \begin{pmatrix} c_1 e^{-t} + 2c_2 e^{4t} \\ -c_1 e^{-t} + 3c_2 e^{4t} \end{pmatrix}.$$

Since $X(0) = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$, then

$$\begin{cases} c_1 + 2c_2 = 3 \\ -c_1 + 3c_2 = 2 \end{cases} \Rightarrow c_1 = c_2 = 1.$$

Thus is,

$$\begin{cases} x_1 = e^{-t} + 2e^{4t} \\ x_2 = -e^{-t} + 3e^{4t} \end{cases}.$$

We present another method to solve the system $X' = AX$, where A is diagonalizable.

Proposition 2 Let $A \in \mathcal{M}_n(\mathbb{R})$ be diagonalizable matrix and let

$$P = [X_1 \ X_2 \ \dots \ X_n]$$

be the invertible matrix formed by n linearly eigenvectors X_1, X_2, \dots, X_n of A . Then the system $X' = AX$ has a unique solution given by

$$X(t) = c_1 e^{\lambda_1 t} X_1 + c_2 e^{\lambda_2 t} X_2 + \dots + c_n e^{\lambda_n t} X_n, \quad (3)$$

where $c_1, c_2, \dots, c_n \in \mathbb{R}^n$ and $\lambda_1, \lambda_2, \dots, \lambda_n$ are the eigenvalues of A .

Proof. It is clear that $X' = AX$ implies

$$X(t) = e^{At} \cdot \xi, \text{ where } \xi \in \mathcal{M}_{n,1}(\mathbb{R}).$$

Since A is diagonalizable, then

$$X(t) = P e^{Dt} P^{-1} = P \begin{pmatrix} e^{\lambda_1 t} & & & \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ & & & e^{\lambda_n t} \end{pmatrix} P^{-1} \cdot \xi \quad (4)$$

Setting

$$P^{-1} \cdot \xi = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = C.$$

It follows from (4) that

$$\begin{aligned} X(t) &= [X_1 \ X_2 \ \dots \ X_n] \begin{pmatrix} e^{\lambda_1 t} & & & \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ & & & e^{\lambda_n t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} \\ &= [e^{\lambda_1 t} X_1 \ e^{\lambda_2 t} X_2 \ \dots \ e^{\lambda_n t} X_n] \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} \\ &= c_1 e^{\lambda_1 t} X_1 + c_2 e^{\lambda_2 t} X_2 + \dots + c_n e^{\lambda_n t} X_n. \end{aligned}$$

Therefore,

$$X(t) = c_1 e^{\lambda_1 t} X_1 + c_2 e^{\lambda_2 t} X_2 + \dots + c_n e^{\lambda_n t} X_n. \quad (5)$$

This completes the proof. ■

Example 3 Solve the system of differential equations:

$$X' = AX, \quad A = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}, \quad \text{where } X(0) = \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$

Solution. After the computation of the eigenvalues and eigenvectors of the matrix A . It follows from (5) that

$$X(t) = c_1 e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{4t} \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

Hence

$$\begin{cases} x(t) = c_1 e^{-t} + 2c_2 e^{4t}, \\ y(t) = -c_1 e^{-t} + 3c_2 e^{4t}. \end{cases}$$

Since $X(0) = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$, then

$$\begin{cases} x_1 = e^{-t} + 2e^{4t} \\ x_2 = -e^{-t} + 3e^{4t}. \end{cases}$$

Example 4 Solve the system of differential equations:

$$X' = AX \quad \text{with } A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}.$$

Solution. Simple computation we get

$$\begin{cases} \lambda_1 = 1, v_1 = (-1, 1, 1) \\ \lambda_1 = 2, v_2 = (0, 1, 0) \text{ and } v_3 = (0, 0, 1). \end{cases}$$

The matrix A is diagonalizable, and by (3) we obtain

$$X(t) = c_1 e^t \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_3 e^{2t} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

where c_1, c_2, c_3 are constants. That is,

$$\begin{cases} x(t) = -c_1 e^t \\ y(t) = c_1 e^t + c_2 e^{2t} \\ z(t) = c_1 e^t + c_3 e^{2t}. \end{cases}$$

Remark 5 In another way, which is very long and based on the calculation of P and P^{-1} with $A = PDP^{-1}$. From which it follows that

$$e^{At} = P e^{Dt} P^{-1}. \quad (6)$$

Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$. The solution of the differential system $X' = AX$ is $X(t) = e^{At} \cdot C$, where C is an arbitrarily constant. Since $X(0) = \begin{pmatrix} 3 & 2 \end{pmatrix}^t$, then $C = \begin{pmatrix} 3 & 2 \end{pmatrix}^t$. Therefore,

$$X(t) = e^{At} \cdot X(0). \quad (7)$$

Simple computation gives

$$P = \begin{pmatrix} 2 & 1 \\ 3 & -1 \end{pmatrix} \text{ and } P^{-1} = \begin{pmatrix} \frac{1}{5} & \frac{1}{5} \\ \frac{3}{5} & -\frac{2}{5} \end{pmatrix}.$$

Hence

$$\begin{aligned} X(t) &= e^{At} \cdot C_0 = \begin{pmatrix} 2 & 1 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} e^{4t} & 0 \\ 0 & e^{-t} \end{pmatrix} \begin{pmatrix} \frac{1}{5} & \frac{1}{5} \\ \frac{3}{5} & -\frac{2}{5} \end{pmatrix} \begin{pmatrix} 3 \\ 2 \end{pmatrix} \\ &= \begin{pmatrix} 2e^{4t} + e^{-t} \\ 3e^{4t} - e^{-t} \end{pmatrix}. \end{aligned}$$

1.1 Problems

Ex 01. Calculate e^{At} for each $t \in \mathbb{R}$, where

$$A = \begin{pmatrix} 0 & -1 & 1 \\ 0 & 0 & 1 \\ -1 & 0 & 1 \end{pmatrix}.$$

Deduce the general solution of the system of differential equations:

$$\begin{cases} p' = -q + r \\ q' = r \\ r' = -p + r \end{cases}$$

Ex 02. Solve the system of differential equations:

$$\begin{cases} x'(t) = y(t) \\ y'(t) = z(t) \\ z'(t) = w(t) \\ w'(t) = x(t) \end{cases}$$

Ex 03. Solve the system of differential equations $X' = A \cdot X$, where $A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$.