1 System of linear differential equations, Part II

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Consider the system of differential equations:

$$\begin{cases} x_1' = a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n \\ x_2' = a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n \\ \vdots \\ x_n' = a_{n1}x_1 + a_{n2}x_2 + ... + a_{nn}x_n, \end{cases} \tag{1}$$

which is written by the following equivalent vector-matrix system:

$$X' = A \cdot X,$$

where the matrix A is **non-diagonalizable**. In this case, the general solution of (1) can be given by:

$$X(t) = e^{tA}c,$$

where $c = \begin{pmatrix} c_1 & c_2 & \dots & c_n \end{pmatrix}^t$ is a constant.

In this program, we only consider certain cases. For example, $A \in \mathcal{M}_n(\mathbb{R})$ but has a unique eigenvalue or when $A \in \mathcal{M}_n(\mathbb{R})$ with $n \leq 4$. The situation is particularly simple whenever $A \in \mathcal{M}_2(\mathbb{R})$.

Corollary 1 *Let $A \in \mathcal{M}_n(\mathbb{R})$ be a square matrix having a unique eigenvalue, say λ. Then*

$$e^{tA} = e^{\lambda t} \sum_{k=0}^{n-1} (A - \lambda I_n)^k \frac{t^k}{k!}.$$

Proof. We first have $p_A(x) = (x - \lambda)^n$ since A has a unique eigenvalue λ. We have

$$\begin{aligned} e^{tA} &= e^{\lambda t I_n + t(A - \lambda I_n)} && (2) \\ &= e^{\lambda t I_n} e^{t(A-\lambda I_n)} \quad \text{(because } \lambda t I_n \text{ and } t(A - \lambda I_n) \text{ commute)} \\ &= e^{\lambda t} e^{t(A-\lambda I_n)} \quad \text{(because } e^{\alpha I_n}B = e^{\alpha}B \text{ for any } B \in \mathcal{M}_n(\mathbb{R}) \text{ and } \alpha \in \mathbb{R}) \\ &= e^{\lambda t} \sum_{k=0}^{+\infty} (A - \lambda I_n)^k \frac{t^k}{k!} && (3) \\ &= e^{\lambda t} \sum_{k=0}^{n-1} (A - \lambda I_n)^k \frac{t^k}{k!}, \end{aligned}$$

where $\sum_{k=n}^{+\infty} (A - \lambda I_n)^k = 0$; this is obtained by Cayley-Hamilton theorem since $p_A(A) = (A - \lambda I_n)^n = 0$. ■

Remark 2 *In particular, by Corollary 1, if* $A \in \mathcal{M}_2(\mathbb{R})$ *with* $Sp(A) = \{\lambda\}$ *then*

$$e^{tA} = e^{\lambda t} \{I_2 + (A - \lambda I_2) t\}. \tag{4}$$

If $A \in \mathcal{M}_3(\mathbb{R})$ *with* $Sp(A) = \{\lambda\}$ *then*

$$e^{tA} = e^{\lambda t} \left\{ I_3 + (A - \lambda I_3) t + \frac{1}{2} (A - \lambda I_3)^2 t^2 \right\}. \tag{5}$$

Example 3 *Solve the system of différentiel equations*

$$\begin{cases} x' = 2x + y \\ y' = 2y \end{cases} \tag{6}$$

Let A *be the matrix of* (6), *i.e.,*

$$A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}.$$

From (4), *we have*

$$\begin{aligned} e^{tA} &= e^{2t} \{I_2 + (A - 2I_2) t\} \\ &= e^{2t} \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \left(\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} - 2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) t \right\} \\ &= \begin{pmatrix} e^{2t} & te^{2t} \\ 0 & e^{2t} \end{pmatrix}. \end{aligned}$$

Thus, the solution of (6) *is given by*

$$X(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} e^{2t} & te^{2t} \\ 0 & e^{2t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} c_1 e^{2t} + t c_2 e^{2t} \\ c_2 e^{2t} \end{pmatrix},$$

where c_1, c_2 *are constants.*

Example 4 *Solve the system of differential equations:*

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \underbrace{\begin{pmatrix} -4 & 1 & 1 \\ 1 & -1 & -2 \\ -2 & 1 & -1 \end{pmatrix}}_{A} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Solution: The characteristic polynomil of A is given by

$$p_A(x) = (x + 2)^3.$$

This means that A has a unique eigenvalu, $\lambda = -2$. From (5), we obtain

$$e^{tA} = e^{-2t} \left\{ I_3 + (A + 2I_3) t + \frac{1}{2} (A + 2I_3)^2 t^2 \right\},$$

where

$$A + 2I_3 = \begin{pmatrix} -2 & 1 & 1 \\ 1 & 1 & -2 \\ -2 & 1 & 1 \end{pmatrix} \text{ and } A + 2I_3 = \begin{pmatrix} 3 & 0 & -3 \\ 3 & 0 & -3 \\ 3 & 0 & -3 \end{pmatrix}.$$

Then

$$
\begin{aligned}
e^{tA} &= e^{-2t}\left\{I_3 + (A+2I_3)\,t + \frac{1}{2}(A+2I_3)^2\,t^2\right\} \\
&= e^{-2t}\left\{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} -2 & 1 & 1 \\ 1 & 1 & -2 \\ -2 & 1 & 1 \end{pmatrix} t + \frac{1}{2}\begin{pmatrix} 3 & 0 & -3 \\ 3 & 0 & -3 \\ 3 & 0 & -3 \end{pmatrix} t^2\right\} \\
&= e^{-2t}\begin{pmatrix} \frac{3}{2}t^2 - 2t + 1 & t & t - \frac{3}{2}t^2 \\ \frac{3}{2}t^2 + t & t+1 & -\frac{3}{2}t^2 - 2t \\ \frac{3}{2}t^2 - 2t & t & -\frac{3}{2}t^2 + t + 1 \end{pmatrix}.
\end{aligned}
$$

Exercise 5 *Solve the system of differential equations*

$$X' = A \cdot X, \text{ where } A = \begin{pmatrix} 3 & -18 \\ 2 & -9 \end{pmatrix}.$$

Exercise 6 *Solve the system of differential equations*

$$\begin{pmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \\ x_4'(t) \end{pmatrix} = \begin{pmatrix} 1 & 2 & 1 & 3 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{pmatrix}.$$

Theorem 7 *Let $A \in \mathcal{M}_3(\mathbb{R})$. If A has two distinct eigenvalues λ and μ (where λ has multiplicity 2), then*

$$e^{tA} = e^{\lambda t}\left(I + t\,(A - \lambda I)\right) + \frac{e^{\mu t} - e^{\lambda t}}{(\mu - \lambda)^2}(A - \lambda I)^2 - \frac{te^{\lambda t}}{\mu - \lambda}(A - \lambda I)^2. \tag{7}$$

Proof. From (2) and (3), we have

$$
\begin{aligned}
e^{tA} &= e^{\lambda t}\sum_{k=0}^{+\infty}(A - \lambda I)^k\frac{t^k}{k!} \\
&= e^{\lambda t}\left(I + (A - \lambda I)\right) + e^{\lambda t}\sum_{k=2}^{+\infty}(A - \lambda I)^k\frac{t^k}{k!} \\
&= e^{\lambda t}\left(I + (A - \lambda I)\right) + e^{\lambda t}\sum_{r=0}^{+\infty}(A - \lambda I)^{2+r}\frac{t^{2+r}}{(2+r)!} \qquad (8)
\end{aligned}
$$

Now, let $p_A(x) = (x - \lambda)^2(x - \mu)$ be the characteristic polynomial of A. First, we note that

$$A - \mu I = (A - \lambda I_n) - (\mu - \lambda)\,I.$$

By Cayley-Hamilton theorem, we get

$$0 = (A - \lambda I)^2(A - \mu I) = (A - \lambda I)^3 - (\mu - \lambda)(A - \lambda I)^2,$$

from which is follows that

$$(A-\lambda I)^3 = (\mu-\lambda)(A-\lambda I)^2 .$$

By induction, for every $r \geq 1$,

$$(A-\lambda I)^{2+r} = (\mu-\lambda)^r (A-\lambda I)^2 .$$

It follows from (8) that

$$\begin{aligned}
\sum_{r=0}^{+\infty} (A-\lambda I)^{2+r} \frac{t^{2+r}}{(2+r)!} &= \sum_{r=0}^{+\infty} (\mu-\lambda)^r \frac{t^{2+r}}{(2+r)!}^{r} (A-\lambda I)^2 \\
&= \frac{1}{(\mu-\lambda)^2} \sum_{r=0}^{+\infty} \frac{t^k}{k!} (\mu-\lambda)^k (A-\lambda I)^2 .
\end{aligned}$$

Finally, we obtain

$$\begin{aligned}
e^{tA} &= e^{\lambda t}(I+(A-\lambda I)) + \frac{e^{\lambda t}}{(\mu-\lambda)^2}\left\{e^{(\mu-\lambda)t} - 1 - (\mu-\lambda)t\right\}(A-\lambda I)^2 \\
&= e^{\lambda t}(I+t(A-\lambda I)) + \frac{e^{\mu t}-e^{\lambda t}}{(\mu-\lambda)^2}(A-\lambda I)^2 - \frac{te^{\lambda t}}{\mu-\lambda}(A-\lambda I)^2 .
\end{aligned}$$

This completes the proof. ■

Example 8 *Solve the system of differential equations*

$$\begin{pmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{pmatrix} = \underbrace{\begin{pmatrix} 2 & -1 & 2 \\ 10 & -5 & 7 \\ 4 & -2 & 2 \end{pmatrix}}_{A} \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} .$$

We first find the characteristic polynomial of A. By computation, $p_A(x) = x^2(x+1)$. This means that A has two eigenvalues $\lambda = 0$ (with multiplicity 2) and $\mu = -1$ (simple). It follows from (7) that

$$e^{At} = I_3 + tA + \left(t + e^{-t} - 1\right) A^2.$$

Simple computation we obtain

$$e^{At} = \begin{pmatrix} 4t + \frac{2}{e^t} - 1 & 1 - \frac{1}{e^t} - 2t & 3t + \frac{1}{e^t} - 1 \\ 8t - \frac{2}{e^t} + 2 & \frac{1}{e^t} - 4t & 6t - \frac{1}{e^t} + 1 \\ 4 - \frac{4}{e^t} & \frac{2}{e^t} - 2 & 3 - \frac{2}{e^t} \end{pmatrix} .$$