University 08 Mai 45 Guelma, 2019/2020
Department of Mathematics, 2^{nd} Year-Maths
Module: Algebra 3
Exams 2020
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1 Sub-test of Algebra III, January 15th-2020

Exercise1. Consider the matrix

$$A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}.$$

1. Find eigenvalues of the matrix A.
2. Find eigenvectors for each eigenvalue of A.
3. Diagonalize the matrix A. That is, find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$.
4. Diagonalize the matrix $A^3 - 5A^2 + 3A + I$, where I is the 2×2 identity matrix.
5. Calculate A^n, for $n \geq 0$.
6. Calculate $(A^3 - 5A^2 + 3A + I)^{2020}$.

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Exercise 2.

a. Let $A \in \mathcal{M}_n(\mathbb{R})$ be an orthogonal matrix (i.e., $A^tA = AA^t = I$). Prove that $\det(A) = \pm 1$.

b. Let $A \in \mathcal{M}_n(\mathbb{R})$. Show that if A is diagonalizable by an orthogonal matrix P, then A is a symmetric matrix.

c. For which values of constants a, b and c is the matrix

$$A = \begin{pmatrix} 1 & a & b \\ 0 & 2 & c \\ 0 & 0 & 3 \end{pmatrix}$$

diagonalizable?

d. Let $A \in \mathcal{M}_n(\mathbb{R})$. Prove that

$$\det\left(e^A\right) = e^{tr(A)}.$$

e. Let $A \in \mathcal{M}_n(\mathbb{R})$. Explain how a system of differentail equations $X' = AX$ can be solved whenever A is diagonalizable or not.

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Good Luck.

2 Solution of Sub-test of Algebra III, 2020

Exercise1. Consider the matrix

$$A=\begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}.$$

1. We find eigenvalues of the matrix A:

$$\begin{aligned} p_A(x) &= \begin{vmatrix} 1-x & 2 \\ 4 & 3-x \end{vmatrix} = \begin{vmatrix} 5-x & 5-x \\ 4 & 3-x \end{vmatrix} = (5-x)\begin{vmatrix} 1 & 1 \\ 4 & 3-x \end{vmatrix} \\ &= (5-x)(3-x-4) = (x-5)(x+1). \end{aligned}$$

   Therefore, $\lambda_1=-1$ and $\lambda_2=5$.

2. We find eigenvectors for each eigenvalue of A:

$$E_{\lambda_1}=\left\{(x,y)\in\mathbb{R}^2;\ \begin{matrix} x+2y=-x \\ 4x+3y=-y \end{matrix}\right\}=Vect\{(1,-1)\},$$

   also, we have

$$E_{\lambda_2}=\left\{(x,y)\in\mathbb{R}^2;\ \begin{matrix} x+2y=5x \\ 4x+3y=5y \end{matrix}\right\}=Vect\{(1,2)\}.$$

3. Next, we diagonalize the matrix A. That is, we will find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP=D$. In fact, we see taht

$$P^{-1}AP=\begin{pmatrix} \frac{2}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}=\begin{pmatrix} -1 & 0 \\ 0 & 5 \end{pmatrix}=D.$$

4. We diagonalize the matrix A^3-5A^2+3A+I, where I is the 2×2 identity matrix:

$$A^3-5A^2+3A+I=P\left(D^3-5D^2+3D+I\right)P^{-1}, \qquad (*)$$

   where by few computation we find:

$$D^3-5D^2+3D+I=\begin{pmatrix} -8 & 0 \\ 0 & 16 \end{pmatrix}. \qquad (**)$$

5. We calculate A^n, for $n\geq 0$. Since $A=PDP^{-1}$, then

$$A^n=\begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} (-1)^n & 0 \\ 0 & 5^n \end{pmatrix}\begin{pmatrix} \frac{2}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} \end{pmatrix}=\begin{pmatrix} \dfrac{2(-1)^n+5^n}{3} & \dfrac{5^n-(-1)^n}{3} \\ \dfrac{2.5^n-2(-1)^n}{3} & \dfrac{(-1)^n+2.5^n}{3} \end{pmatrix}.$$

6. We calculate $(A^3-5A^2+3A+I)^{2020}$. From (*) and (**), we see that

$$(A^3-5A^2+3A+I)^{2020}=\begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} (-8)^{2020} & 0 \\ 0 & 16^{2020} \end{pmatrix}\begin{pmatrix} \frac{2}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} \end{pmatrix}.$$

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Exercise 2.

a. Let $A \in \mathcal{M}_n(\mathbb{R})$ be an orthogonal matrix (i.e., $A^tA = AA^t = I$). We prove that $\det(A) = \pm 1$. Indeed, we see that

$$\det\left(A^tA\right) = \det(I) = 1,$$

But

$$\det\left(A^tA\right) = \det\left(A^t\right)\det(A) = (\det(A))^2 = 1,$$

then $\det(A) = \pm 1$.

b. Let $A \in \mathcal{M}_n(\mathbb{R})$. We show that if A is diagonalizable by an orthogonal matrix P, then A is a symmetric matrix. In fact, since $A = PDP^{-1}$ with P is orthogonal, then $P^{-1} = P^t$ and $A = PDP^t$. It follows that

$$A^t = \left(PDP^t\right)^t = \left(P^t\right)^t D^t P^t = PDP^t = A.$$

c. For which values of constants a, b and c is the matrix

$$A = \begin{pmatrix} 1 & a & b \\ 0 & 2 & c \\ 0 & 0 & 3 \end{pmatrix}$$

diagonalizable?

In fact, since A is an upper triangular matrix, then $1, 2, 3$ are the eigenvalues of A, which are all simple. That is, $A_m(\lambda) = G_m(\lambda) = 1$ for $\lambda = 1, 2, 3$. Then A is diagonalizable for each values of constants a, b and c.

d. Let $A \in \mathcal{M}_n(\mathbb{R})$. Note that if λ is an eigenvalue of A, then e^λ is also an eigenvalue of e^A. Then, we have

$$\det\left(e^A\right) = \prod e^{\lambda_i} = e^{\lambda_1+\lambda_2+\dots} = e^{tr(A)}, \text{ because } tr(A) = \sum \lambda_i.$$

e. Let $A \in \mathcal{M}_n(\mathbb{R})$. Consider a system of differentail equations $X' = AX$. In general, the solution is given by the following formula:

$$X(t) = e^{tA}c, \text{ where } c \in \mathbb{R}^n.$$

But, if A is diagonalizable (that is, $A_m(\lambda_i) = G_m(\lambda_i)$ for $i = 1, 2, ..., n$), then

$$X(t) = c_1e^{\lambda_1 t}V_1 + c_2e^{\lambda_2 t}V_2 + ... + c_ne^{\lambda_n t}V_n,$$

where $c_1, c_2, ..., c_n$ are constants, and $V_1, V_2, ..., V_n$ are the corresponding eigenvectors.

== **End**

3 Exam of Algebra III, January 19th-2020

University 08 Mai 45 Guelma, 2019/2020
Department of Mathematics, 2^{nd} Year-Maths
Module: Algebra 3 January 19th-2020
Final Exam 2 Hours

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Exercise 1. Let $A_4 = \begin{pmatrix} \mathbf{0} & 0 & 0 & 0 \\ 1 & \mathbf{1} & 1 & 1 \\ 0 & 0 & \mathbf{0} & 0 \\ 1 & 1 & 1 & \mathbf{1} \end{pmatrix}$. Find the eigenvalues of the matrix A_4. Also, give the algebraic multiplicity of each eigenvalue[1]. (*2 pts*).

Exercise 2.

1. Let $A \in \mathcal{M}_n(\mathbb{R})$. Explain how a system of differentail equations $X' = AX$ can be solved whenever A is diagonalizable or not. (*2 pts*)

2. Trigonalize the matrix $A = \begin{pmatrix} 2 & -1 \\ 1 & 4 \end{pmatrix}$. Compute A^n for any $n \in \mathbb{N}$. (*4 pts*)

3. Let $A \in \mathcal{M}_n(\mathbb{R})$. Assume that A has a unique eigenvalue, say λ. Prove that

$$e^{tA} = e^{\lambda t}\left[I + (A - \lambda I)\frac{t}{1!} + (A - \lambda I)^2 \frac{t^2}{2!} + \dots + \frac{t^{n-1}}{(n-1)!}(A - \lambda I)^{n-1}\right]. \quad (1.5 \text{ pts})$$

4. Solve the system of differential equations: $\begin{cases} x' = 2x - y \\ y' = x + 4y. \end{cases}$ (*1.5 pts*)

5. Solve the following system of linear recurrence sequences:

$$\begin{cases} a_{n+1} = 2a_n - b_n \\ b_{n+1} = a_n + 4b_n \end{cases}, \text{ where } a_0 = 1 \text{ and } b_0 = 0. \quad (2 \text{ pts})$$

Exercise 3.

1. Let A, B be two square matrices. Prove that if A or B is invertible, then AB is similar to BA. (*1 pt*)

2. Let A be an $n \times n$ matrix and let $\lambda_1, \lambda_2, \dots, \lambda_n$ be its eigenvalues. Show that

$$\det(A) = \prod_{i=1}^{n} \lambda_i. \quad (2 \text{ pts})$$

3. Let A be an $n \times n$ matrix. We say that A is idempotent if $A^2 = A$.

[1] **Optional Question:** Determine all eigenvalues and their algebraic multiplicities of the matrix A_{2n}, for $n \geq 2$. (+ *2 pts*)

3.a) Find a nonzero, nonidentity idempotent matrix. (*1 pt*)

3.b) Show that any eigenvalue of an idempotent matrix A is either 0 or 1. (*1 pt*)

3.c) Let A be an $n \times n$ idempotent complex matrix. Then prove that A is diagonalizable. (*2 pts*)

Good Luck.

4 Solution

Exercise 1. Let $A_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}$. We find the eigenvalues and their algebraic multiplicities of the matrix A_4. In fact, we have

$$p_{A_4}(x) = \begin{vmatrix} -x & 0 & 0 & 0 \\ 1 & 1-x & 1 & 1 \\ 0 & 0 & -x & 0 \\ 1 & 1 & 1 & 1-x \end{vmatrix} = -x \begin{vmatrix} 1-x & 1 & 1 \\ 0 & -x & 0 \\ 1 & 1 & 1-x \end{vmatrix} =$$

$$-x \begin{vmatrix} -x & 0 & x \\ 0 & -x & 0 \\ 1 & 1 & 1-x \end{vmatrix} = -x^3 \begin{vmatrix} -1 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 1 & 1-x \end{vmatrix} = -x^3 \left[-(-1+x) + 1 \right] = x^3 (x-2).$$

Then $\lambda_1 = 0$ with $A_m(\lambda_1) = 3$ and $\lambda_2 = 2$ with $A_m(\lambda_2) = 1$.

Exercise 2.

1. Let $A \in \mathcal{M}_n(\mathbb{R})$. Consider a system of differentail equations $X' = AX$. In general, the solution is given by the following formula:

$$X(t) = e^{tA}c, \text{ where } c \in \mathbb{R}^n.$$

But, if A is diagonalizable (that is, $A_m(\lambda_i) = G_m(\lambda_i)$ for $i = 1, 2, ..., n$), then

$$X(t) = c_1 e^{\lambda_1 t} V_1 + c_2 e^{\lambda_2 t} V_2 + ... + c_n e^{\lambda_n t} V_n,$$

where $c_1, c_2, ..., c_n$ are constants, and $V_1, V_2, ..., V_n$ are the corresponding eigenvectors.

2. We trigonalize the matrix $A = \begin{pmatrix} 2 & -1 \\ 1 & 4 \end{pmatrix}$, and then we compute A^n for any $n \in \mathbb{N}$.

   (a) We compute the characteristic polynomial of A :

$$\begin{aligned} p_A(x) &= \begin{vmatrix} 2-x & -1 \\ 1 & 4-x \end{vmatrix} = \begin{vmatrix} 3-x & -1 \\ -(3-x) & 4-x \end{vmatrix} = (3-x) \begin{vmatrix} 1 & -1 \\ -1 & 4-x \end{vmatrix} \\ &= (3-x)(4-x-1) = (3-x)^2. \end{aligned}$$

(b) We compute the corresponding eigenvector:

$$E_3 = \left\{ (x,y) \in \mathbb{R}^2; \quad \begin{array}{c} 2x - y = 3x \\ x + 4y = 3y \end{array} \right\} = Vect\left\{(1,-1)\right\}.$$

(c) Setting $v_2 = (1,0)$, and let

$$P = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}.$$

Then,

$$P^{-1}AP = \begin{pmatrix} 3 & -1 \\ 0 & 3 \end{pmatrix},$$

and so, $A = PTP^{-1}$.

(d) We compute A^n : In fact, we have $A^n = PT^nP^{-1}$.

(e) Next, it suffices to compute T^n. Since

$$T = \begin{pmatrix} 3 & -1 \\ 0 & 3 \end{pmatrix} = \underbrace{\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}}_{D} + \underbrace{\begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}}_{N},$$

where N is nilpotent (with $N^2 = 0$) and $DN + ND$, it follows that

$$\begin{aligned} T^n &= (D+N)^n = D^n + nD^{n-1}N = \begin{pmatrix} 3^n & 0 \\ 0 & 3^n \end{pmatrix} + n \begin{pmatrix} 3^{n-1} & 0 \\ 0 & 3^{n-1} \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 3^n & -3^{n-1}.n \\ 0 & 3^n \end{pmatrix}. \end{aligned}$$

(f) Finally, we deduce that

$$A^n = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 3^n & -3^{n-1}n \\ 0 & 3^n \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 3^n - 3^{n-1}.n & -3^{n-1}.n \\ 3^{n-1}.n & 3^{n-1}.n + 3^n \end{pmatrix}.$$

3. Let $A \in \mathcal{M}_n(\mathbb{R})$ be a square matrix which has a unique eigenvalue, say λ. We prove that

$$e^{tA} = e^{\lambda t}\left[I + (A - \lambda I)\frac{t}{1!} + (A-\lambda I)^2 \frac{t^2}{2!} + \ldots + \frac{t^{n-1}}{(n-1)!}(A - \lambda I)^{n-1} \right].$$

Firstly, note that $p_A(x) = (x-\lambda)^n$ since A has a unique eigenvalue, say λ. By definition, we have

$$\begin{aligned} e^{tA} &= e^{\lambda t I_n + t(A - \lambda I_n)} \qquad (1) \\ &= e^{\lambda t I_n} e^{t(A-\lambda I_n)} \quad \text{(since } \lambda t I_n \text{ and } t(A - \lambda I_n) \text{ commute)} \\ &= e^{\lambda t} e^{t(A-\lambda I_n)} \quad \text{(since } e^{\alpha I_n}B = e^{\alpha}B \text{ for any } B \in \mathcal{M}_n(\mathbb{R}) \text{ and } \alpha \in \mathbb{R}) \\ &= e^{\lambda t}\sum_{k=0}^{+\infty} (A - \lambda I_n)^k \frac{t^k}{k!} \\ &= e^{\lambda t}\sum_{k=0}^{n-1} (A - \lambda I_n)^k \frac{t^k}{k!}, \end{aligned}$$

where $\sum_{k=n}^{+\infty}(A-\lambda I_n)^k=0$; since by Cayley-Hamilton's Theorem $p_A(A)=(A-\lambda I_n)^n=0$.

4. *Application:* We solve the system of differential equations $\begin{cases} x'=2x-y \\ y'=x+4y. \end{cases}$

$$\begin{aligned} X(t) &= e^{3t}\left[I+(A-\lambda I)t\right].C \\ &= e^{3t}\left\{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}+t\begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}\right\}.C=\begin{pmatrix} e^{3t} & -te^{3t} \\ te^{3t} & (2t+1)e^{3t} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}. \end{aligned}$$

Therefore $x=c_1e^{3t}-c_2te^{3t}$ and $y=c_1te^{3t}+c_2(2t+1)e^{3t}$, c_1,c_2 are constants.

5. In the matrix form, we obtain

$$\begin{pmatrix} a_{n+1} \\ b_{n+1} \end{pmatrix}=\begin{pmatrix} 2 & -1 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} a_n \\ b_n \end{pmatrix}.$$

Setting $U_n=\begin{pmatrix} a_n \\ b_n \end{pmatrix}$ with $U_0=\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Then

$$U_n=A^nU_0=\begin{pmatrix} 3^n-3^{n-1}.n & -3^{n-1}.n \\ 3^{n-1}.n & 3^{n-1}.n+3^n \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

It follows that $a_n=3^n-3^{n-1}.n$ and $b_n=3^{n-1}.n$.

Exercise 3.

1. Let A,B be two square matrices. We prove that if A or B is invertible, then AB is similar to BA. Assume, for example A is invertible. It is clear that

$$A^{-1}(AB)A=I.BA=BA.$$

2. Let A be an $n\times n$ matrix. Since $A=PTP^{-1}$, where T is an upper triangular matrix with its diagonal entries are the eigenvalues of A (say, $\lambda_1,\lambda_2,\ldots,\lambda_n$) and P is invertible, then

$$\det(A)=\det\left(PTP^{-1}\right)=\det(P)\det(T)\det\left(P^{-1}\right)=\det(T)=\prod_{i=1}^{n}\lambda_i.$$

3. Let A be an $n\times n$ matrix. We say that A is idempotent if $A^2=A$.

3.a) We find a nonzero, nonidentity idempotent matrix. For example, we have

$$A=\begin{pmatrix} 3 & -6 \\ 1 & -2 \end{pmatrix}.$$

3.b) Let A be an idempotent matrix and let (λ,x) be an eigenpair of A. Then $\lambda x=Ax=A^2x=\lambda^2x$ and hence $\lambda^2=\lambda$ since $x\neq 0$. Thus, λ is either 0 or 1.

3.c) Let A be an $n\times n$ idempotent complex matrix. We prove that A is diagonalizable. Indeed, from the above question we see that

$$p_A(x)=x^{\alpha}(x-1)^{\beta}\text{, where }\alpha+\beta=n.$$

Since $A(A-I)=0$, then $m_A(x)=x(x-1)$, which has simple roots. Hence, A is diagonalizable. **End.**

5 Passing Exam, November 18th-2020

University 08 Mai 45 Guelma, 2019/2020
Department of Mathematics, 2^{nd} Year-Maths
Algebra 3 November 18th-2020
Passing Exam 1 H 30 mn

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Exercise 1. The following questions are indepentent.

1. Let A be an n by n matrix over $\mathbb{C}$. Prove that
$$\det(A) = \prod_{\lambda_i \in Sp(A)} \lambda_i.$$

2. Consider the matrix
$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$
Find A^n for each $n \geq 0$.

3. Let $A \in \mathcal{M}_n(\mathbb{R})$ be a nilpotent matrix. Show that
$$A \text{ is diagonalisable} \Rightarrow A = 0 \text{ (zero matrix)}.$$
Let $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Is A a diagonalizable matrix?

4. Let $A \in \mathcal{M}_n(\mathbb{R})$. Show that
$$A \sim 2A \Rightarrow A \text{ is nilpotent}.$$

5. Consider the 2×2 matrix
$$A = \begin{pmatrix} \frac{-\pi}{2} & \frac{\pi}{2} \\ \frac{\pi}{2} & \frac{-\pi}{2} \end{pmatrix}$$
Compute $\cos^2 A$ et $\sin^2 A$.

Exercise 2. Let A be the real matrix given by:
$$A = \begin{pmatrix} 0 & -1 & -1 \\ -1 & 0 & -1 \\ -1 & -1 & 0 \end{pmatrix} \in \mathcal{M}_3(\mathbb{R}).$$

a) Determine the minimal polynomial of A.

b) Prove, for each $n \in \mathbb{N}$, that there exist two real numbers α_n and β_n for which

$$A^n = \alpha_n A + \beta_n I_3.$$

c) For each $n \in \mathbb{N}$, compute α_n and β_n in terms of n

d) Assume that $A \in \mathcal{M}_n(\mathbb{R})$ with $a_{ij} = 0$ for $i = j$, -1; otherwise. Prove that A is diagonalizable.

Good Luck.