# Algebra III

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# 2 Characteristic polynomial

In this section we consider only the characteristic polynomial of an $n$ by $n$ matrix which is a polynomial of degree $n$, from which we give a practical way to find the eigenvalues of a given square matrix $A$.

**Definition 1** *Let $A \in \mathcal{M}_n(\mathbb{R})$ be a square matrrix. The characteristic polynomial of $A$ is the polynomial of degree $n$ given by $p_A(x) = \det(A - xI_n)$, where $I_n$ is the identity $n$-by-$n$ matrix*[1]*.*

**Proposition 2** *Let $A \in \mathcal{M}_n(\mathbb{R})$. The characteristic polynomial $p_A(x)$ is given by*

$$p_A(x) = (-1)^n x^n + \sum_{i=0}^{n-1} c_i x^i \quad \text{with } c_{n-1} = (-1)^{n-1} tr(A) \text{ and } c_0 = \det(A).$$

*The leading coefficient of $p_A(x)$ is $\pm 1$ (i.e. $p_A(x)$ is monic.*

For example, if $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ then $tr(A) = 5$ and $\det(A) = -2$. Moreover, by definition we have

$$\begin{aligned} p_A(x) &= \det(A - xI_2) = \begin{vmatrix} 1-x & 2 \\ 3 & 4-x \end{vmatrix} = x^2 - 5x - 2 \\ &= (-1)^2 x^2 + -tr(A)x + \det(A). \end{aligned}$$

**Remark 3** *Recall that the roots of $p_A(x)$ are called **eigenvalues** of $A$. Also, we have the notation:*

$$Sp(A) = \{\lambda \in \mathbb{K} \; ; \lambda \text{ is an eigenvalue of } A\},$$

*which is called the **spectral set** of $A$. Thus, $\lambda \in Sp(A) \Leftrightarrow p_A(\lambda) = 0$.*

[1] In some references the characteristic polynomial of $A$ is the polynomial of degree $n$ given by $p_A(x) = \det(xI_n - A)$.

**Example 4** *Calculate the characteristic polynomial of the following matrix:*

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

*From definition, we obtain*

$$\begin{aligned} p_A(x) &= \begin{vmatrix} 2-x & 1 \\ 1 & 2-x \end{vmatrix} \quad \begin{matrix} c_1 \\ \downarrow \\ c_1 + c_2 \end{matrix} \quad \text{(the first column } c_1 \text{ becomes } c_1 + c_2\text{)} \\ &= \begin{vmatrix} (3-x) & 1 \\ (3-x) & 2-x \end{vmatrix} = (3-x) \begin{vmatrix} 1 & 1 \\ 1 & 2-x \end{vmatrix} = (3-x)(2-x-1) \\ &= (3-x)(1-x). \end{aligned}$$

*Thus,* $p_A(x) = (1-x)(3-x)$, *and so* $Sp(A) = \{1, 3\}$.

**Example 5** *Consider the matrix*

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

*In the same manner, we get*

$$\begin{aligned} p_A(x) &= \begin{vmatrix} 1-x & 1 & 1 \\ 1 & 1-x & 1 \\ 1 & 1 & 1-x \end{vmatrix} = \begin{vmatrix} -x & 0 & 1 \\ x & -x & 1 \\ 0 & x & 1-x \end{vmatrix} \\ &= x^2 \begin{vmatrix} \overset{+}{-1} & \overset{-}{0} & \overset{+}{1} \\ 1 & -1 & 1 \\ 0 & 1 & 1-x \end{vmatrix} \\ &= x^2 \left[ -(x-1-1) + (1-0) \right] \\ &= x^2 (3-x). \end{aligned}$$

*Hence,* $p_A(x) = x^2(3-x)$, *and so* $Sp(A) = \{0, 3\}$.

**Example 6** *Calculate the characteristic polynomial of each of the following:*

$$\begin{aligned} A_1 &= \begin{pmatrix} 4 & 2 & -1 \\ 2 & 7 & -2 \\ -1 & -2 & 4 \end{pmatrix}, \; A_2 = \begin{pmatrix} 13 & -12 & -6 \\ 6 & -5 & -3 \\ 18 & -18 & -8 \end{pmatrix} \\ A_3 &= \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}, \; A_4 = \begin{pmatrix} 4 & 1 & -1 \\ 2 & 5 & -2 \\ 1 & 1 & 2 \end{pmatrix} \end{aligned}$$

*(i) From the definition of the characteristic polynomial, we get*

$$\begin{aligned}
p_{A_1}(x) &= \det(A_1 - xI_3) \\
&= \begin{vmatrix} \mathbf{4-x} & 2 & -1 \\ 2 & \mathbf{7-x} & -2 \\ -1 & -2 & \mathbf{4-x} \end{vmatrix} \quad \begin{matrix} 1^{st}\,column \\ \downarrow \\ 1^{st}+3^{rd} \end{matrix} \\
&= \begin{vmatrix} (3-x) & 2 & -1 \\ 0 & 7-x & -2 \\ (3-x) & -2 & 4-x \end{vmatrix} = (3-x)\begin{vmatrix} 1 & 2 & -1 \\ 0 & 7-x & -2 \\ 1 & -2 & 4-x \end{vmatrix} \quad \begin{matrix} 2^{nd}\,column \\ \downarrow \\ 2\times 3^{rd}+2^{nd} \end{matrix} \\
&= (3-x)\begin{vmatrix} 1 & 0 & -1 \\ 0 & 3-x & -2 \\ 1 & 2(3-x) & 4-x \end{vmatrix} = (3-x)^2\begin{vmatrix} \overset{+}{1} & \overset{-}{0} & \overset{+}{-1} \\ 0 & 1 & -2 \\ 1 & 2 & 4-x \end{vmatrix} \\
&= (3-x)^2\left[4-x+4-(0-1)\right] \\
&= (3-x)^2(9-x).
\end{aligned}$$

*That is,* $p_{A_1}(x) = (3-x)^2(9-x)$.

*(ii) Compute* $p_{A_2}(x)$ *:*

$$\begin{aligned}
p_{A_2}(x) &= \begin{vmatrix} 13-x & -12 & -6 \\ 6 & -5-x & -3 \\ 18 & -18 & -8-x \end{vmatrix} \quad \begin{matrix} 1^{st}\,column \\ 1^{st}+2^{nd} \end{matrix} \\
&= \begin{vmatrix} (1-x) & -12 & -6 \\ (1-x) & -5-x & -3 \\ 0 & -18 & -8-x \end{vmatrix} \quad \begin{matrix} 2^{nd}\,column \\ \downarrow \\ (-2)\times 3^{rd}+2^{nd} \end{matrix} \\
&= \begin{vmatrix} (1-x) & 0 & -6 \\ (1-x) & (1-x) & -3 \\ 0 & (-2)(1-x) & -8-x \end{vmatrix} \\
&= (1-x)^2\begin{vmatrix} \overset{+}{1} & \overset{-}{0} & \overset{+}{-6} \\ 1 & 1 & -3 \\ 0 & -2 & -8-x \end{vmatrix} \\
&= (1-x)^2(-8-x-6-6(-2)) \\
&= (1-x)^2(-2-x).
\end{aligned}$$

*(iii) Computre* $p_{A_3}(x)$ *:*

$$\begin{aligned}
p_{A_3}(x) &= \begin{vmatrix} \mathbf{1-x} & -1 & -1 \\ -1 & \mathbf{1-x} & -1 \\ -1 & -1 & \mathbf{1-x} \end{vmatrix} \quad \begin{matrix} c_1 & c_2 \\ \downarrow & \downarrow \\ c_1-c_2 & c_2-c_3 \end{matrix} \\
&= \begin{vmatrix} (2-x) & 0 & -1 \\ -(2-x) & 2-x & -1 \\ 0 & -(2-x) & 1-x \end{vmatrix} = (2-x)^2\begin{vmatrix} \overset{+}{1} & \overset{-}{0} & \overset{+}{-1} \\ -1 & 1 & -1 \\ 0 & -1 & 1-x \end{vmatrix} \\
&= (2-x)^2\left[1-x-1-1\right] \\
&= -(1+x)(2-x)^2.
\end{aligned}$$

*Thus,* $p_{A_3}(x) = -(1+x)(2-x)^2$.

*(iiii) Compute* $p_{A_4}(x)$ :

$$p_{A_4}(x) = \begin{vmatrix} 4-x & 1 & -1 \\ 2 & 5-x & -2 \\ 1 & 1 & 2-x \end{vmatrix} \quad \begin{matrix} 1^{st} column \\ \downarrow \\ 1^{st} + 3^{rd} \end{matrix} \quad \begin{matrix} 2^{nd} column \\ \\ 2^{nd} + 3^{rd} \end{matrix}$$

$$= \begin{vmatrix} (3-x) & 0 & -1 \\ 0 & 3-x & -2 \\ (3-x) & 3-x & 2-x \end{vmatrix}$$

$$= (3-x)^2 \begin{vmatrix} \overset{+}{1} & \overset{-}{0} & \overset{+}{-1} \\ 0 & 1 & -2 \\ 1 & 1 & 2-x \end{vmatrix}$$

$$= (3-x)^2 (2-x+2+1)$$

$$= (3-x)^2 (5-x).$$

**Example 7** *(a) Calculate the characteristic polynomial of the following matrix:*

$$A_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

*(b) Deduce the characteristic polynomial of the* $n \times n$ *matrix*

$$A_n = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 1 \end{pmatrix} \in \mathcal{M}_n(\mathbb{R}).$$

*For the matrix* $A_4$*, we see that*

$$p_{A_4}(x) = \begin{vmatrix} \mathbf{1-x} & 1 & 1 & 1 \\ 1 & \mathbf{1-x} & 1 & 1 \\ 1 & 1 & \mathbf{1-x} & 1 \\ 1 & 1 & 1 & \mathbf{1-x} \end{vmatrix}$$

$$= \begin{vmatrix} -x & 0 & 0 & 1 \\ x & -x & 0 & 1 \\ 0 & x & -x & 1 \\ 0 & 0 & x & 1-x \end{vmatrix} = x^3 \begin{vmatrix} \overset{+}{-1} & \overset{-}{0} & \overset{+}{0} & \overset{-}{1} \\ 1 & -1 & 0 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 1-x \end{vmatrix}$$

$$= x^3(-1) \begin{vmatrix} -1 & 0 & 1 \\ 1 & -1 & 1 \\ 0 & 1 & 1-x \end{vmatrix} + x^3(-1) \begin{vmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= x^3(x-4).$$

**Remark 8** *For the matrix $A_n$, we can easily prove that*

$$p_{A_n}(x) = \begin{cases} x^{n-1}(x-n), \text{ if } n \text{ is even} \\ x^{n-1}(n-x), \text{ if } n \text{ is odd.} \end{cases}$$

**Example 9** *Calculate the characteristic polynomial of the following matrix:*

$$A = \begin{pmatrix} 7 & -6 & -2 \\ 2 & 0 & -1 \\ 2 & -3 & 2 \end{pmatrix}.$$

*It is clear that*

$$\begin{aligned} p_A(x) &= \begin{vmatrix} 7-x & -6 & -2 \\ 2 & -x & -1 \\ 2 & -3 & 2-x \end{vmatrix} \quad \begin{matrix} c_1 \\ \downarrow \\ 2\times c_3 + c_1 \end{matrix} \\ &= \begin{vmatrix} (3-x) & -6 & -2 \\ 0 & -x & -1 \\ 2(3-x) & -3 & 2-x \end{vmatrix} \\ &= (3-x)\begin{vmatrix} 1 & -6 & -2 \\ 0 & -x & -1 \\ 2 & -3 & 2-x \end{vmatrix} \quad \begin{matrix} c_2 \\ \downarrow \\ 3\times c_3 - c_2 \end{matrix} \\ &= (3-x)\begin{vmatrix} 1 & 0 & -2 \\ 0 & -(3-x) & -1 \\ 2 & 3(3-x) & 2-x \end{vmatrix} \\ &= (3-x)^2 \begin{vmatrix} \overset{+}{1} & \overset{-}{0} & \overset{+}{-2} \\ 0 & -1 & -1 \\ 2 & 3 & 2-x \end{vmatrix} \\ &= (3-x)^2(-2+x+3-2(2)) \\ &= (x-3)^3. \end{aligned}$$

**Example 10** *Consider the matrix*

$$A = \begin{pmatrix} 3 & 2 & -2 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

*From definition, we obtain*

$$\begin{aligned}
p_A(x) &= \begin{vmatrix} 3-x & 2 & -2 \\ -1 & -x & 1 \\ 1 & 1 & -x \end{vmatrix} \qquad \begin{matrix} c_2 \\ \downarrow \\ c_2+c_3 \end{matrix} \\
&= \begin{vmatrix} 3-x & 0 & -2 \\ -1 & 1-x & 1 \\ 1 & 1-x & -x \end{vmatrix} \\
&= (1-x)\begin{vmatrix} 3-x & 0 & -2 \\ -1 & 1 & 1 \\ 1 & 1 & -x \end{vmatrix} \qquad \begin{matrix} c_1 \\ \downarrow \\ c_1+c_3 \end{matrix} \\
&= (1-x)\begin{vmatrix} 1-x & 0 & -2 \\ 0 & 1 & 1 \\ 1-x & 1 & -x \end{vmatrix} \\
&= (1-x)^2\begin{vmatrix} \overset{+}{1} & \overset{-}{0} & \overset{+}{-2} \\ 0 & 1 & 1 \\ 1 & 1 & -x \end{vmatrix} \\
&= (1-x)^2\left[(-x-1)-2(0-1)\right] \\
&= (1-x)^3.
\end{aligned}$$

*Thus,* $p_A(x) = (1-x)^3$.

**Example 11** *Let $A$ be the matrix given by*

$$A = \begin{pmatrix} -3 & 1 & -1 \\ -7 & 5 & -1 \\ -6 & 6 & -2 \end{pmatrix}.$$

*We have*

$$\begin{aligned}
p_A(x) &= \begin{vmatrix} -3-x & 1 & -1 \\ -7 & 5-x & -1 \\ -6 & 6 & -2-x \end{vmatrix} = \begin{vmatrix} -2-x & 0 & -1 \\ -2-x & 4-x & -1 \\ 0 & 4-x & -2-x \end{vmatrix} \\
&= -(2+x)(4-x)\begin{vmatrix} 1 & 0 & -1 \\ 1 & 1 & -1 \\ 0 & 1 & -2-x \end{vmatrix} \\
&= -(2+x)(4-x)(-2-x+1-1) \\
&= (2+x)^2(4-x).
\end{aligned}$$

*Hence,* $p_A(x) = (2+x)^2(4-x)$.

**Example 12** *Calculate the determinant*

$$\Delta_n = \begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \mathbf{1+x} & 1 & \dots & 1 \\ 1 & 1 & \mathbf{1+x} & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \dots & \mathbf{1+x} \end{vmatrix}$$

***Solution.*** *We compute* $\Delta_n$ *:*

- $1^{st}$*column* $\longrightarrow$ $1^{st}$*column*
- $2^{nd}$*column* $\longrightarrow$ $2^{nd}$*column -* $1^{st}$*column*
- $3^{rd}$*column* $\longrightarrow$ $3^{rd}$*column -* $1^{st}$*column, .... and so on. We obtain*

$$\Delta_n = \begin{vmatrix} 1 & 0 & 0 & \dots & 0 \\ 1 & \mathbf{x} & 0 & \dots & 0 \\ 1 & 0 & \mathbf{x} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \dots & \mathbf{x} \end{vmatrix} = x^{n-1}.$$

*Therefore,* $\Delta_n = x^{n-1}$.

**Proposition 13** *Let* $A \in \mathcal{M}_n(\mathbb{R})$ *and* $r \in \mathbb{R}^*$. *We have*

$$p_{rA}(x) = r^n p_A\left(\frac{x}{r}\right).$$

**Proof.** Indeed, we see that

$$\begin{aligned} p_{rA}(x) &= \begin{vmatrix} ra_{11} - x & ra_{12} & \dots & ra_{1n} \\ ra_{21} & ra_{22} - x & \dots & ra_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ ra_{n1} & ra_{n2} & \dots & ra_{nn} - x \end{vmatrix} \\ &= \begin{vmatrix} r\left(a_{11} - \frac{x}{r}\right) & ra_{12} & \dots & ra_{1n} \\ ra_{21} & r\left(a_{22} - \frac{x}{r}\right) & \dots & ra_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ ra_{n1} & ra_{n2} & \dots & r\left(a_{nn} - \frac{x}{r}\right) \end{vmatrix} \\ &= r^n \begin{vmatrix} a_{11} - \frac{x}{r} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} - \frac{x}{r} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} - \frac{x}{r} \end{vmatrix} \\ &= r^n p_A\left(\frac{x}{r}\right). \end{aligned}$$

This completes the proof. ■

**Exercise 14** *Consider the vendermonde's determinant* [2]*:*

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}.$$

[2] In linear algebra, a Vandermonde matrix is a matrix with a geometric progression in each row. It takes its name from the French mathematician Alexandre-Théophile Vandermonde. It is, in particular, used in numerical analysis for solving a system formed by polynomial interpolation.

*Prove that $\Delta = (b-a)(c-a)(c-b)$, and give a generalization formula.*

***Solution 15*** *We have*

$$\begin{aligned}
\Delta &= \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} \quad \begin{matrix} c_1 & & c_2 \\ \downarrow & & \downarrow \\ c_2 - c_1 & & c_3 - c_2 \end{matrix} \\
&= \begin{vmatrix} 0 & 0 & 1 \\ b-a & c-b & c \\ b^2-a^2 & c^2-b^2 & c^2 \end{vmatrix} = (b-a)(c-b) \begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & c \\ b+a & c+b & c^2 \end{vmatrix} \\
&= (b-a)(c-b)(c-a).
\end{aligned}$$

*In the general case, the vendermonde's determinant is given by*

$$\Delta_n = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ x_0 & x_1 & \cdots & x_n \\ x_0^2 & x_1^2 & \cdots & x_n^2 \\ \vdots & \vdots & \cdots & \vdots \\ x_0^n & x_1^n & \cdots & x_n^n \end{vmatrix} = \prod_{i>j} (x_i - x_j).$$

## 2.1 Problems.

**Ex 01.** Consider the following two matrices:

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \ B = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ -3 & 2 & 4 \end{pmatrix}.$$

Calculate $p_A(x)$ and $p_B(x)$. **Ans.**

$$p_A(x) = (1+x)^2 (2-x) \text{ and } p_B(x) = -(x-2)^3.$$

**Ex 02.** Let $A$ be the matrix given by

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & -3 \\ 1 & -1 & 0 \end{pmatrix}.$$

Verify that $p_A(x) = (x+1)(x-1)(x-3)$.

**Ex 03.** Let

$$A = \begin{pmatrix} -3 & 1 & -1 \\ -7 & 5 & -1 \\ -6 & 6 & -2 \end{pmatrix}.$$

Verify that $p_A(x) = (2+x)^2 (4-x)$.

**Ex 04.** Let $A \in \mathcal{M}_n(\mathbb{R})$ be the tridiagonal matrix given by

$$A = \begin{pmatrix} a & b & & \\ c & a & \ddots & \\ & \ddots & \ddots & b \\ & & c & a \end{pmatrix}, \; a, b, c \in \mathbb{R}.$$

Calculate $p_A(x)$.

**Ex 05.** Consider the matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{M}_2(\mathbb{R}).$$

Show that the characteristic polynomial $p_A(x)$ satisfying the following formula:

$$p_A(x) = x^2 - tr(A)\, x + \det(A).$$

Note that $tr(A)$ is the trace of $A$.

**Ex 06.** Let $A$ be the matrix

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Verify that $p_A(x) = x^4 - 1$.

# 3 On the inverse of a square matrix

**Criterion 16** *Let $A \in \mathcal{M}_n(\mathbb{R})$. If $\det(A) \neq 0$, then $A^{-1}$ exists. Moreover, the formula of $A^{-1}$ is given by:*

$$A^{-1} = \frac{1}{\det(A)} (Com(A))^t, \tag{1}$$

*where $Com(A)$ denotes the comatrice of $A$. If $A^{-1}$ exists, we say that $A$ is **invertible**. By French "inversible".*

**Example 17** *Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{M}_2(\mathbb{R})$. We have*

$$\det(A) = ad - cb \text{ and } A^{-1} = \frac{1}{ad - cb} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

**Example 18** *Consider the matrix*

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 8 & 8 & 9 \end{pmatrix} \in \mathcal{M}_3(\mathbb{R}).$$

*By definition, we obtain*

$$\begin{aligned}\det(A) &= \begin{vmatrix} \overset{+}{1} & \overset{-}{2} & \overset{+}{3} \\ 4 & 5 & 6 \\ 8 & 8 & 9 \end{vmatrix} = \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2\begin{vmatrix} 4 & 6 \\ 8 & 9 \end{vmatrix} + 3\begin{vmatrix} 4 & 5 \\ 8 & 8 \end{vmatrix} \\ &= -3 + 24 - 24 \\ &= -3 \neq 0.\end{aligned}$$

*From* (1), *we have*

$$\begin{aligned}A^{-1} &= \frac{-1}{3}\begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix}^t \\ &= \frac{-1}{3}\begin{pmatrix} \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} & -\begin{vmatrix} 4 & 6 \\ 8 & 9 \end{vmatrix} & \begin{vmatrix} 4 & 5 \\ 8 & 8 \end{vmatrix} \\ -\begin{vmatrix} 2 & 3 \\ 8 & 9 \end{vmatrix} & \begin{vmatrix} 1 & 3 \\ 8 & 9 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 8 & 8 \end{vmatrix} \\ \begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix} & -\begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix} \end{pmatrix}^t \\ &= \frac{-1}{3}\begin{pmatrix} -3 & 12 & -8 \\ 6 & -15 & 8 \\ -3 & 6 & -3 \end{pmatrix}^t = \frac{-1}{3}\begin{pmatrix} -3 & 6 & -3 \\ 12 & -15 & 6 \\ -8 & 8 & -3 \end{pmatrix}.\end{aligned}$$

*As required.*

## 3.1 Problems

**Ex 01.** Consider the matrix

$$A = \begin{pmatrix} 1 & -\alpha & & & \\ & 1 & -\alpha & & \\ & & \ddots & \ddots & \\ & & & 1 & -\alpha \\ & & & & 1 \end{pmatrix}; \ \alpha \in \mathbb{R}$$

Prove that

$$A^{-1} = \begin{pmatrix} 1 & \alpha & \alpha^2 & \dots & \alpha^{n-1} \\ & 1 & \alpha & \dots & \alpha^{n-2} \\ & & \ddots & \ddots & \vdots \\ & & & 1 & \alpha \\ & & & & 1 \end{pmatrix}.$$

**Ex 02.** Let $A, B \in \mathcal{M}_2(\mathbb{R})$. Assume that one of the matrices $A$ or $B$ is invertible. Show that $AB$ amd $BA$ have the same characteristic polynomial, i.e., $p_{AB}(x) = p_{BA}(x)$.

# 4 Eigenvalues and Eigenvectors

Throughout this chapter $\mathbb{K}$ denotes the field $\mathbb{R}$ or $\mathbb{C}$, and $\mathcal{M}_n(\mathbb{K})$ denotes the vector space of $n$ by $n$ matrices over $\mathbb{K}$.

**Definition.** Let $A$ be an $n \times n$ square matrix. When $Ax = \lambda x$ has a non-zero vector solution $x$, then

- $\lambda$ is called an **eigenvalue** of $A$.
- $x$ is called an **eigenvector** of $A$ corresponding to $\lambda$.
- The couple $(\lambda, x)$ is called an **eigenpair** of $A$.

**Notes:** **(i)** eigenvectors must be non-zero. **(ii)** But, eigenvalue $\lambda$ can be zero, can be non-zero.

**Conclusion 19** *A vector $x \in E$ is an eigenvector of $A$ if*

1. *$x$ is non-zero,*
2. *there exists $\lambda \in \mathbb{K}$, $Ax = \lambda x$.*

The **eigenspace** of $A$ corresponding to $\lambda$ is the subspace:

$$E_\lambda = \{v \in \mathbb{K}^n \; ; \; Av = \lambda v\}.$$

Note that $E_\lambda$ is a vector subspace of $\mathbb{K}^n$. This is the **kernel** of the matrix $A - \lambda I_n$. So $E_\lambda$ consists of all solutions $v$ of the equation $Av = \lambda v$. In other words, $E_\lambda$ consists of all eigenvectors with eigenvalue $\lambda$, together with the zero vector.

**Example 20** *Let $A = I_2$. Then any non-zero vector $v$ of $\mathbb{R}^2$ will be an eigenvector of $A$ corresponding to eigenvalue $\lambda = 1$.*

**Example 21** *Consider the matrix*

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}.$$

*Calculate the eigenvalues and eigenvectors of $A$.*

**Solution.**

1. First, we find the eigenvalues of $A$. We start with calculating the characteristic polynomial of $A$. From definition, we obtain

$$\begin{aligned} p_A(x) &= \begin{vmatrix} 2-x & 1 \\ 1 & 2-x \end{vmatrix} \quad \begin{matrix} c_1 \\ \downarrow \\ c_1 + c_2 \end{matrix} \quad \text{(the first column } c_1 \text{ becomes } c_1 + c_2\text{)} \\ &= \begin{vmatrix} (3-x) & 1 \\ (3-x) & 2-x \end{vmatrix} = (3-x) \begin{vmatrix} 1 & 1 \\ 1 & 2-x \end{vmatrix} = (3-x)(2-x-1) \\ &= (3-x)(1-x). \end{aligned}$$

Hence, $p_A(x) = (1-x)(3-x)$, and so the eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = 3$.

1. Second, we find the eigenvectors. By definition, the eigenspace $E_{\lambda_1}$ is given by

$$\begin{aligned} E_{\lambda_1} &= \left\{ (x,y) \in \mathbb{R}^2 ; \begin{array}{c} x+2y=x \\ 2x+y=y \end{array} \right\} \\ &= \left\{ (x,y) \in \mathbb{R}^2 ; \, y=-x \right\} \\ &= Vect\left\{ (1,-1) \right\}. \end{aligned}$$

Thus, $v_1 = (1,-1)$.

Using the same manner, the eigenspace $E_{\lambda_2}$ is given by

$$\begin{aligned} E_{\lambda_2} &= \left\{ (x,y) \in \mathbb{R}^2 ; \begin{array}{c} x+2y=3x \\ 2x+y=3y \end{array} \right\} \\ &= \left\{ (x,y) \in \mathbb{R}^2 ; \, y=x \right\} \\ &= Vect\left\{ (1,1) \right\}. \end{aligned}$$

That is, $v_2 = (1,1)$.

**Definition 22** *The* ***geometric multiplicity*** *for a given eigenvalue $\lambda$, denoted by $G_m(\lambda)$, is the dimension of the eigenspace $E_\lambda$. That is,*

$$G_m(\lambda) = \dim E_\lambda.$$

*The* ***algebraic multiplicity*** *for a given eigenvalue $\lambda$, denoted by $A_m(\lambda)$, is the number of times the eigenvalue is repeated. For example, if the characteristic polynomial is $(x-1)^2(x-5)^3$ then for $\lambda = 1$ the algebraic multiplicity is 2 and for $\lambda = 5$ the algebraic multiplicity is 3.*

**Remark 23** *The algebraic multiplicity is greater than or equal to the geometric multiplicity. That is, we always have $A_m(\lambda) \geq G_m(\lambda)$.*

**Examples.** Calculate eigenvalues and eigenvectors of the following matrices. Deduce the algebraic multiplicity and the geometric multiplicity of each eigenvalue of $A$.

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}.$$

**Ans.** We have $\lambda_1 = 4,\ v_1 = (2,3)$ and $\lambda_2 = -1,\ v_2 = (1,-1)$.

$$A = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

**Ans.** We have $\lambda_1 = e^{i\theta},\ v_1 = (-i,1)$ and $\lambda_2 = e^{-i\theta},\ v_2 = (i,1)$.

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 5 \end{pmatrix}.$$

**Ans.** We have $\lambda_1 = 1,\ E_1 = Vect\{(1,0)\}$ and $\lambda_2 = 5,\ E_5 = Vect\{(1,2)\}$.

$$A = \begin{pmatrix} 2 & 6 \\ 0 & 2 \end{pmatrix}.$$

**Ans.** We have $\lambda = 2$ (double, i.e., the algebraic multiplicity is 2), $E_\lambda = Vect\{(1,0)\}$.

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & -5 \end{pmatrix}.$$

**Ans.** We have $\lambda_1 = 1$, $E_1 = Vect\{(1,0,0)\}$, $\lambda_2 = 2$, $E_2 = Vect\{(2,1,0)\}$ and $\lambda_3 = -5$, $E_{-5} = Vect\{(5,6,-14)\}$.

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}.$$

**Ans.** We have $\lambda_1 = 1$, $E_{\lambda_1} = Vect\{(-1,1,1)\}$, $\lambda_2 = 2$ (double, , i.e., the algebraic multiplicity is 2), $E_{\lambda_2} = Vect\{(0,1,0),(0,0,1)\}$.

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

**Ans.** We have $\lambda = 0$ (triple eigenvalue, , i.e., the algebraic multiplicity is 3), $E_\lambda = Vect\{(1,0,0),(0,1,-1)\}$. The eigenspace corresponding to $\lambda = 0$ is of dimension 2.

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 3 & 2 \end{pmatrix}.$$

**Ans.** We have $\lambda = 2$ (the algebraic multiplicity is 3), $E_\lambda = Vect\{(0,0,1)\}$. The eigenspace corresponding to $\lambda = 2$ is of dimension 1.

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

**Ans.** We have $\lambda_1 = 0$ (simple eigenvalue), $E_{\lambda_1} = Vect\{(-1,1,0)\}$ and $\lambda_2 = 2$ (double eigenvalue), $E_{\lambda_2} = Vect\{(0,0,1),(1,1,0)\}$. The eigenspace corresponding to $\lambda_1$ is of dimension 1 and the eigenspace corresponding to $\lambda_2 = 2$ is of dimension 2.

$$A = \begin{pmatrix} a & 2 & 3 \\ 0 & 2a & 8 \\ 0 & 0 & 3a \end{pmatrix}; a \in \mathbb{R}.$$

**Ans.** We have $\lambda_1 = a$ and $E_{\lambda_1} = Vect\{(1,0,0)\}$, $\lambda_2 = 2a$ and $E_{\lambda_2} = Vect\left\{\left(\frac{2}{a},1,0\right)\right\}$, $\lambda_3 = 3a$ and $E_{\lambda_3} = Vect\left\{\left(\frac{1}{2a^2}(3a+16), \frac{8}{a}, 1\right)\right\}$.

**Corollary 24** *Let $(\lambda, x)$ be an eigenpair of $A$. Then $(\lambda^k, x)$ is an eigenpair of $A^k$.*

**Proof.** In fact, we see that

$$Ax = \lambda x \Rightarrow A^2 x = A(\lambda x) = \lambda A x = \lambda^2 x.$$

Therefore,

$$Ax = \lambda x \Rightarrow \forall\, k \geq 0 : A^k x = \lambda^k x.$$

The result is proved. ■

**Corollary 25** *Let $A$ be an invertible matrix and let $(\lambda, x)$ be an eigenpair of $A$ with $\lambda \neq 0$. Then $\left(\frac{1}{\lambda}, x\right)$ is an eigenpair of $A^{-1}$.*

**Proof.** By definition, we have

$$\begin{aligned} A^{-1}x &= A^{-1}(1.x) = A^{-1}\left(\frac{\lambda}{\lambda}.x\right) = \frac{1}{\lambda}A^{-1}(\lambda x) \\ &= \frac{1}{\lambda}A^{-1}(Ax) \quad (\text{since } Ax = \lambda x) \\ &= \frac{1}{\lambda}x. \end{aligned}$$

Thus, $A^{-1}x = \frac{1}{\lambda}x$. The proof is finished. ■

## 4.1 Problems

**Ex 01.** Calculate the eigenvalues and eigenvectors of the following matrix:

$$A = \begin{pmatrix} -3 & 1 & -1 \\ -7 & 5 & -1 \\ -6 & 6 & -2 \end{pmatrix}.$$

**Ans.** $\lambda_1 = -2$, $v_1 = (1, 1, 0)$ and $\lambda_2 = 4$, $v_2 = (0, 1, 1)$.

**Ex 02.** Let $P \in \mathbb{GL}_n(\mathbb{R})$ and let $D$ be the following diagonal matrix:

$$D = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}$$

Calculate the eigenpairs of $D$, then deduce the eigenpairs of the matrix $PDP^{-1}$.

**Ex 03.** Let $A \in \mathcal{M}_n(\mathbb{R})$ and $\alpha \in \mathbb{R}^*$. Prove that

$$v \text{ is an eigenvector of } A \Rightarrow \alpha v \text{ is also an eigenvector of } A.$$

**Ex 04.** Let $A \in \mathcal{M}_n(\mathbb{R})$ and $\lambda_1, \lambda_2$ be two eigenvalues of $A$ with $\lambda_1 \neq \lambda_2$. Prove that

$$E_{\lambda_1} \cap E_{\lambda_2} = \{0_{\mathbb{R}^n}\}.$$

Recall that $E_\lambda = \{x \in \mathbb{R}^n \ ;\ Ax = \lambda x\}$.

# 5 Similar Matrices

We will now introduce the notion of similarity.

**Definition 26** *Let $A$ and $B$ be two $n$-by-$n$ matrices. We say that $A$ is* ***similar to*** *$B$ if there exists an invertible matrix $P$ such that*

$$A = PBP^{-1}.$$

In linear algebra, two $n$-by-$n$ matrices $A$ and $B$ are called **similar** if there exists an invertible $n$-by-$n$ matrix $P$ such that $A = PBP^{-1}$. We also write: $A$ and $B$ are similar if $A = PBP^{-1}$ for some invertible matrix $P$.

**Notation 27** *The notation $A \sim B$ means that the matrix $A$ is similar to the matrix $B$.*

Next, we give an example.

**Example 28** *Let $A$ and $B$ be the two matrices given by*

$$A = \begin{pmatrix} -4 & 7 \\ 3 & 0 \end{pmatrix}, \ B = \begin{pmatrix} 13 & -8 \\ 25 & -17 \end{pmatrix}.$$

*Then $A$ is similar to $B$ because for the matrix $P = \begin{pmatrix} 4 & -3 \\ -1 & 1 \end{pmatrix}$, we have after few computation*

$$PBP^{-1} = \begin{pmatrix} 4 & -3 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 13 & -8 \\ 25 & -17 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 1 & 4 \end{pmatrix} = \begin{pmatrix} -4 & 7 \\ 3 & 0 \end{pmatrix} = A.$$

But, the question we ask here: *How to find the invertible matrix $P$ so that $A = PBP^{-1}$?* We have the following properties:

**Theorem 29** *Let $A$ and $B$ be two $n$-by-$n$ similar matrices; i.e., there exists an invertible matrix $P$ such that $A = PBP^{-1}$. Then*

1. *For each positive integer $k$, $A^k = PB^kP^{-1}$.*
2. *$p_A(x) = p_B(x)$, that is $A$ and $B$ have the same characteristic polynomial.*

**Proof.** Let us show the theorem as follows:

1. Assume that $A$ and $B$ are two similar matrices, and let $P$ be an invertible matrix such that $A = PBP^{-1}$. For each integer $k \geq 0$ we have

$$\begin{aligned} A^k &= \underbrace{(PBP^{-1})(PBP^{-1})\ldots(PBP^{-1})}_{k-\text{times}} \\ &= P\underbrace{BB\ldots B}_{k-\text{times}}P^{-1} \\ &= PB^kP^{-1}. \end{aligned}$$

2. We prove the following implication

$$A \sim B \Rightarrow p_A(x) = p_B(x). \tag{2}$$

That is, if the matrices $A$ and $B$ are similar to each other, then $A$ and $B$ have the same characteristic equation, and hence have the same eigenvalues. In fact, we have

$$\begin{aligned} p_A(x) &= \det(A - xI) \\ &= \det\left(PBP^{-1} - xPP^{-1}\right), \text{ since } PP^{-1} = I_n \in \mathbb{R} \\ &= \det\left(P(B - xI)P^{-1}\right), \text{ since } x \in \mathbb{R} \\ &= \det(P)\det(B - xI)\det\left(P^{-1}\right) \quad (3) \\ &= \det(B - xI) \quad (4) \\ &= p_B(x). \end{aligned}$$

Note that the passage from (3) to (4) because $\det(P^{-1}) = \dfrac{1}{\det(P)}$.

The proof is finished. ■

**Remark 30** *The converse of* (2) *is false. For example, for*

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2$$

*We see that* $p_A(x) = p_B(x)$. *Therefore,* $Sp(A) = Sp(B) = \{1\}$ *and* $\det(A) = \det(B)$. *Further, if* $A$ *is similar to* $B$ *then there exists an invertible matrix* $P$ *such that*

$$A = PBP^{-1} = PI_2P^{-1} = I_2.$$

*A contradiction since* $A \neq I_2$. *Thus,* $A$ *is not similar to* $B$ *(we denote* $A \nsim B$*).*

**Conclusion:** We can also write

$$\begin{cases} Sp(A) = Sp(B) \nRightarrow A \sim B, \\ p_A(x) = p_B(x) \nRightarrow A \sim B, \\ \det(A) = \det(B) \nRightarrow A \sim B. \end{cases}$$

**Remark 31** *By applying the following rule:*

$$\det(A) = 0 \Leftrightarrow 0 \in Sp(A). \tag{5}$$

*Let* $A$ *and* $B$ *be two similar matrices, i.e., there exists an invertible matrix* $P$ *such that* $A = PBP^{-1}$. *We can also prove that* $Sp(A) = Sp(B)$. *Let* $\lambda \in Sp(A)$, *there exists a nonzero vector* $x$ *tel que* $Ax = \lambda x$. *That is,*

$$(A - \lambda I)x = 0 = 0.x$$

*Which gives* $0 \in Sp(A - \lambda I)$. *On the other hand, we have*

$$A - \lambda I = P(B - \lambda I)P^{-1}. \tag{6}$$

*Asssume that* $0 \notin Sp(B - \lambda I)$. *By* (5) *and* (6) *we have* $B - \lambda I \in \mathbb{GL}_n(\mathbb{R})$. *Consequently,* $A - \lambda I \in \mathbb{GL}_n(\mathbb{R})$. *From* (5), $0 \notin Sp(A - \lambda I)$. *A contradiction.*

*Finally, we deduce that* $0 \in Sp(B - \lambda I)$, *and hence* $\lambda \in Sp(B)$. *Thus,* $Sp(A) \subset Sp(B)$.

**Corollary 32** *Two similar matrices $A$ and $B$ have the same determinant.*

**Proof.** Let $P$ be an invertible matrix $P$ such that $A = PBP^{-1}$. It follows that

$$\det(A) = \det\left(PBP^{-1}\right) = \det(P)\det(B)\det\left(P^{-1}\right) = \det(B),$$

and so $\det(A) = \det(B)$. This completes the proof. ∎

**Example 33** *Consider the following two matrices:*

$$A = \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 5 & 2 \\ 4 & 1 \end{pmatrix}.$$

*How can we tell (rather quickly) that the matrices $A$ and $B$ are not similar to each other?*

*In fact, $A \nsim B$ because $\det(A) = -1 \neq \det(B) = -3$. Thus, we have the result:*

$$\det(A) \neq \det(B) \Rightarrow A \nsim B.$$

**Theorem 34** *The relation " $\sim$ " similarity is an* ***equivalence relation****.*

**Proof.** This relation is what we call an **equivalence relation**, because we have the following three properties:

1. The relation " $\sim$ " is reflexive, because for each matrix $A \in \mathcal{M}_n(\mathbb{R})$ we have
$$A = I_n A I_n^{-1}.$$
Then $A \sim A$.

2. The relation " $\sim$ " is symmetric, because for all matrices $A, B \in \mathcal{M}_n(\mathbb{R})$ we have
$$A \sim B \Rightarrow \exists\ P \in \mathbb{GL}_n(\mathbb{R}) \text{ such that } A = PBP^{-1}.$$
It follows that
$$B = \underbrace{P^{-1}}_{C} AP = CAC^{-1} \text{ and } C \in \mathbb{GL}_n(\mathbb{R}).$$
Thus, $B \sim A$ (i.e., we can just say that $A$ and $B$ are similar to each other). For the matrices $A$, $B$, and $P$ of Example 28, verify by direct computation that $A = PBP^{-1}$ and that $B = P^{-1}AP$.

3. The relation " $\sim$ " is transitive, because for all matrices $A, B, C \in \mathcal{M}_n(\mathbb{R})$ we have
$$\left.\begin{array}{l} A \sim B \\ B \sim C \end{array}\right\} \Rightarrow \left\{\begin{array}{l} \exists\ P \in \mathbb{GL}_n(\mathbb{R}) \text{ such that } A = PBP^{-1}, \\ \exists\ Q \in \mathbb{GL}_n(\mathbb{R}) \text{ such that } B = QCQ^{-1}. \end{array}\right.$$
Which gives
$$A = P\left(QCQ^{-1}\right)P^{-1} = \underbrace{(PQ)}_{R} C (PQ)^{-1} = RCR^{-1} \text{ with } R \in \mathbb{GL}_n(\mathbb{R}).$$
Hence, $A \sim C$.

■

**Proposition 35** *Let $P \in \mathbb{GL}_n(\mathbb{R})$. Define the mapping $T_P$ by:*

$$\begin{array}{rcl} T_P & : & \mathcal{M}_n(\mathbb{R}) \to \mathcal{M}_n(\mathbb{R}) \\ A & \mapsto & T_P(A) = P^{-1}AP. \end{array}$$

*Then the following statements hold:*

1. $T_P(I_n) = I_n$
2. $T_P(A+B) = T_P(A) + T_P(B)$
3. $T_P(AB) = T_P(A)\,T_P(B)$
4. $T_P(rA) = rT_P(A)$
5. $T_P\left(A^k\right) = (T_P(A))^k$
6. $T_P(A^{-1}) = (T_P(A))^{-1}$
7. $T_P\left(e^A\right) = e^{T_P(A)}$
8. $T_Q(T_P(A)) = T_{PQ}(A)$.

**Proof.** We have

1. In fact, $T_P(I_n) = P^{-1}I_nP = P^{-1}P = I_n$.
2. $T_P(A+B) = P^{-1}(A+B)P = P^{-1}AP + P^{-1}BP = T_P(A) + T_P(B)$.
3. $T_P(AB) = P^{-1}ABP = P^{-1}APP^{-1}BP = (P^{-1}AP)(P^{-1}BP) = T_P(A)\,T_P(B)$.
4. $T_P(rA) = P^{-1}(rA)P = r(P^{-1}AP) = rT_P(A)$.
5. $T_P\left(A^k\right) = P^{-1}A^kP = (P^{-1}AP)^k = (T_P(A))^k$.
6. $T_P(A^{-1}) = P^{-1}A^{-1}P = (P^{-1}AP)^{-1} = (T_P(A))^{-1}$.
7. $T_P\left(e^A\right) = P^{-1}e^AP = e^{P^{-1}AP} = e^{T_P(A)}$.
8. It is clear that

$$T_Q(T_P(A)) = Q^{-1}T_P(A)\,Q = Q^{-1}\left(P^{-1}AP\right)Q = (PQ)^{-1}A(PQ) = T_{PQ}(A).$$

This completes the proof.

■

**Remark.** Let $A, B \in \mathcal{M}_n(\mathbb{R})$. If $A \sim B$, then

$$A \in \mathbb{GL}_n(\mathbb{R}) \Leftrightarrow B \in \mathbb{GL}_n(\mathbb{R}).$$

In fact, we have $A = PBP^{-1} \Leftrightarrow B = P^{-1}AP$.

**Conclusion 36** *Let $A \in \mathcal{M}_n(\mathbb{R})$, and let $B = P^{-1}AP \in \mathcal{M}_n(\mathbb{R})$ be a matrix similar to $A$. Then $A$ and $B$ have the same characteristic polynomial. Furthermore, $q(A) = Pq(B)P^{-1}$ for each $q \in \mathbb{K}[X]$, and in particular $A^k = PB^kP^{-1}$for $k \geq 1$.*

**Corollary 37** *Let $A, B \in \mathcal{M}_n(\mathbb{R})$. If $A$ and $B$ are similar, then $Tr(A) = Tr(B)$.*

**Proof.** We know that

$$\forall\ M, N \in \mathcal{M}_n(\mathbb{R}) : Tr\,(MN) = Tr\,(NM)\,.$$

Then

$$Tr\,(A) = Tr\left(PBP^{-1}\right) = Tr\left(BPP^{-1}\right) = Tr\,(B)\,.$$

■

**Corollary 38** *Two similar matrix have the same rank.*

**Proof.** Assume that $A = PBP^{-1}$ for some invertible square matrix $P$. We have $rk\,(B) \geq rk\,(PBP^{-1}) = rk\,(A)$. Now note that $B = P^{-1}AP$, so we similarily get $rk\,(A) \geq rk\,(P^{-1}AP) = rk\,(B)$. ■

**Conclusion 39** *Two similar matrices have the same determinant, same trace, same rank, same characteristic polynomial, same eigenvalues.*

On the other hand, we have the following absolutely remarkable result.

**Theorem 40** *In dimension 2 and 3, two matrices are similar iff they have the same minimal polynomial and the same characteristic polynomial.*

## 5.1 Additional Problems

**Ex 01.** Let $A$ and $B$ be two similar matrices, i.e., there exists an invertible matrix $P$ such that $A = PBP^{-1}$. Prove that

$$(\lambda, x)\ \text{is an eigenpair of } A \Rightarrow \left(\lambda, P^{-1}x\right)\ \text{is an eigenpair of } B.$$

**Ex 02.** Let $A, B\ \mathcal{M}_n(\mathbb{R})$ and $f\,(x) = a_0 + a_1x + ... + a_nx^n \in \mathbb{R}\,[x]$ be a polynomial of degree $n$. Prove that

$$A \sim B \Rightarrow f\,(A) \sim f\,(B)\,.$$

**Ex 03.** Consider the two matrices:

$$A = \begin{pmatrix} 1 & 0 & 4 \\ 1 & 1 & 3 \\ 2 & 1 & 7 \end{pmatrix} \text{ et } B = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 3 & 1 & 2 \end{pmatrix}.$$

Prove that $A \nsim B$ ; i.e., $A$ and $B$ are not similar.

**Ex 04.** Show that

$$A - \lambda I_n \sim B \Rightarrow A \sim B + \lambda_n I.$$

**Ex 05.** Using two methods. Prove that similar matrices have the same eigenvalues.

**Ex 06.** Prove that

$$A \sim B \Rightarrow e^A \sim e^B.$$

**Ex 07.** Without calculating, neither eigenvalues nor eigenvectors, show that

$$\begin{pmatrix} 1 & -1 \\ 3 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 \\ -1 & 1 \end{pmatrix}.$$

**Ex 08.** Show by direct computation that the matrices $A$ and $B$ of Example 28 have the same characteristic equation. What are the eigenvalues of $A$ and $B$?

# 6 Diagonalizable Matrices

**Definition 41** *Let $A = (a_{ij}) \in \mathcal{M}_n(\mathbb{R})$ be a square matrix. $A$ is said to be **diagonal**, if and only if*

$$a_{ij} = 0, \ \forall \ i \neq j.$$

*Or, equivalently*

$$A = \begin{pmatrix} a_{11} & & & \\ & a_{22} & & \\ & & \ddots & \\ & & & a_{nn} \end{pmatrix}.$$

*In this case, $A$ is denoted by $D$. We also write $D = diag\{a_{11}, a_{22}, ..., a_{nn}\}$.*

**Definition 42** *Let $A$ be a square matrix. We say that $A$ is **diagonalizable** if $A$ is similar to a diagonal matrix $D$. That is, there exists an invertible matrix $P$ such that $P^{-1}AP$ is diagonal, say $D$. That is,*

$$A \text{ is diagonalizable} \Leftrightarrow \exists \ P \in \mathbb{GL}_n(\mathbb{R}) \ \text{ such that } A = PDP^{-1},$$

*where $D = diag\{\lambda_1, \lambda_2, ..., \lambda_n\}$ and $\lambda_1, \lambda_2, ..., \lambda_n$ are the eigenvalues of $A$.*

**Example 43** *Consider the following matrices*

$$A = \begin{pmatrix} 5 & -4 \\ 2 & -1 \end{pmatrix}, \ D = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \text{ and } P = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}.$$

*Compute $PDP^{-1}$. What can we conclude?*

*By computation, we obtain*

$$\begin{aligned} PDP^{-1} &= \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 6 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 5 & -4 \\ 2 & -1 \end{pmatrix} = A. \end{aligned}$$

*Thus, $A = PDP^{-1}$ and so $A$ is diagonalizable.*

But the question posed is how to determine $P$ and $D$ if they exist? How to diagonalize a matrix?. Here is the following theorem.

**Theorem 44 (Necessary and sufficient condition for diagonalization)** *Let $A \in \mathcal{M}_n(\mathbb{R})$ be a square matrix. $A$ is diagonalizable, if and only if, there exists a basis $B$ of $\mathbb{R}^n$ formed by $n$ eigenvectors of $A$.*

**Proof.** Assume that $A$ is diagonalizable. That is, there exists an invertible matrix $P$ such that

$$A = PDP^{-1}.$$

Or, equivalently

$$P^{-1}AP = D.$$

Setting

$$P = \begin{bmatrix} y_1 & y_2 & \ldots & y_n \end{bmatrix} = \begin{bmatrix} Pe_1 & Pe_2 & \ldots & Pe_n \end{bmatrix},$$

where $(e_i)_{1 \le i \le n}$ is the canonical basis of $\mathbb{R}^n$ and let

$$\begin{aligned} D &= \begin{pmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{pmatrix} = diag\{d_1, d_2, ..., d_n\} \\ &= \begin{bmatrix} d_1e_1 & d_2e_2 & \ldots & d_ne_n \end{bmatrix}. \end{aligned}$$

It follows that

$$\begin{aligned} \begin{bmatrix} Ay_1 & Ay_2 & \ldots & Ay_n \end{bmatrix} &= AP = I_nAP = PP^{-1}AP = PD \\ &= P\begin{bmatrix} d_1e_1 & d_2e_2 & \ldots & d_ne_n \end{bmatrix} \\ &= \begin{bmatrix} d_1Pe_1 & d_2Pe_2 & \ldots & d_nPe_n \end{bmatrix} \\ &= \begin{bmatrix} d_1y_1 & d_2y_2 & \ldots & d_ny_n \end{bmatrix}. \end{aligned}$$

We deduce that for each $i \in \overline{1, n}$, $Ay_i = d_iy_i$. Then $y_i$ is an eigenvector of $A$ corresponding to $d_i$. Since $P$ is invertible, then the familly $B = \{y_1, y_2, ..., y_n\}$ is a basis of $\mathbb{R}^n$.

Conversely, assume that $\mathbb{R}^n$ has a basis $B = \{x_1, x_2, ..., x_n\}$ formed by $n$ eigenvectors of $A$. In this case, we put

$$P = \begin{bmatrix} x_1 & x_2 & \ldots & x_n \end{bmatrix}.$$

It follows that

$$\begin{aligned} AP &= \begin{bmatrix} Ax_1 & Ax_2 & \ldots & Ax_n \end{bmatrix} \\ &= \begin{bmatrix} \lambda_1x_1 & \lambda_2x_2 & \ldots & \lambda_nx_n \end{bmatrix}, \end{aligned}$$

where $(\lambda_i)_{1 \le i \le n}$ are the eigenvalues of $A$ associated with $(x_i)_{1 \le i \le n}$, respectively. Therefore,

$$\begin{aligned} AP &= \begin{pmatrix} \lambda_1x_{11} & \lambda_2x_{21} & \ldots & \lambda_nx_{n1} \\ \lambda_1x_{12} & \lambda_2x_{22} & \ldots & \lambda_nx_{n2} \\ \vdots & \vdots & \ldots & \vdots \\ \lambda_1x_{1N} & \lambda_2x_{2N} & \ldots & \lambda_nx_{nn} \end{pmatrix} = \begin{pmatrix} x_{11} & x_{21} & \ldots & x_{n1} \\ x_{12} & x_{22} & \ldots & x_{n2} \\ \vdots & \vdots & \ldots & \vdots \\ x_{1N} & x_{2N} & \ldots & x_{nn} \end{pmatrix} \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix} \\ &= PD. \end{aligned}$$

Hence $A = PDP^{-1}$, where $D$ is diagonale and $P$ is invertible. The proof is finished. ■

**Corollary 45** *Let $A \in \mathcal{M}_n(\mathbb{R})$ be a diagonalizable matrix. There exists a basis $B = \{x_1, x_2, ..., x_n\}$ of $\mathbb{R}^n$ formed by $n$ eigenvectors $A$.*

**Proof.** Assume that $A = PDP^{-1}$. We know that $\{e_1, e_2, ..., e_n\}$ are eigenvectors of $D$ associated with $diag(D)$, i.e.,

$$De_i = P^{-1}APe_i = \lambda_i e_i, \text{ for } i = 1, 2, ..., n.$$

Hence

$$APe_i = \lambda_i Pe_i, \text{ for } i = 1, 2, ..., n.$$

That is, $\{Pe_i\}_{1 \leq i \leq n}$ are eigenvectors of $A$. Since $P$ is invertible, then $\{Pe_i\}_{1 \leq i \leq n}$ is a basis of $\mathbb{R}^n$. ■

**Conclusion.** Let $A \in \mathcal{M}_n(\mathbb{R})$ be a square matrix and let $\lambda_1, \lambda_2, ..., \lambda_k$ be its eigenvalues. Let $A_m(\lambda_i)$ and $G_m(\lambda_i)$ denote the algebraic multiplicity and the geometric multiplicity of $\lambda_i$, respectively. Then $A$ is diagonalizable if and oly if

$$A_m(\lambda_i) = G_m(\lambda_i), \text{ for } i = 1, 2, ..., k.$$

**Corollary 46** *Let $A \in \mathcal{M}_n(\mathbb{R})$ be a square matrix. Assume that*

$$p_A(x) = (x - \lambda_1)^{\alpha_1} (x - \lambda_2)^{\alpha_2} ... (x - \lambda_k)^{\alpha_k}, \text{ where } k \leq n.$$

*Then $A$ is diagonalizable if and only if $\dim E_{\lambda_i} = \alpha_i$, for $i = 1, 2, ..., k$.*

**Example 47** *For the following matrices, by calculating the eigenpairs one has:*

| *Matrix* | $p_A(x)$ | $Sp(A)$ | $A_m(\lambda)$ | $G_m(\lambda)$ |
|---|---|---|---|---|
| $A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ | $x(x-2)^2$ | $0$<br>$2$ | $1$<br>$2$ | $1$<br>$2$ |
| $B = \begin{pmatrix} 2 & 1 & 1 \\ 2 & 1 & -2 \\ -1 & 0 & -2 \end{pmatrix}$ | $(x+1)^2(x-3)$ | $-1$<br>$3$ | $2$<br>$1$ | $1$<br>$1$ |
| $C = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & -3 \\ 1 & -1 & 0 \end{pmatrix}$ | $(x+1)(x-1)(x-3)$ | $-1$<br>$1$<br>$3$ | $1$<br>$1$<br>$1$ | $1$<br>$1$<br>$1$ |

*We deduce that $A$ and $C$ are diagonalizable, but $B$ is not.*

We see also the following example:

**Example 48** *Show that the following matrix is diagonalizable.*

$$A = \begin{pmatrix} 4 & 1 & 1 & 1 \\ 1 & 4 & 1 & 1 \\ 1 & 1 & 4 & 1 \\ 1 & 1 & 1 & 4 \end{pmatrix}$$

**Solution.** The characteristic polynomial is $p_A(x) = (x-7)(x-3)^3$. The eigenvalues of $A$ are $\lambda_1 = 7$ (simple), and $\lambda_2 = 3$ (triple). The associated eigenvectors are $v_1 = (1,1,1,1)$ for $\lambda_1$, $v_2 = (-1,1,0,0)$, $v_3 = (-1,0,1,0)$ and $v_4 = (-1,0,0,1)$ for $\lambda_2$. The matrix $A$ is therefore diagonalizable since $\dim E_{\lambda_i} = A_m(\lambda_i)$, for $i = 1, 2$.

From Theorem 44, we have the following corollary:

**Corollary 49** *Let $A \in \mathcal{M}_n(\mathbb{R})$ be a square matrix. If $A$ has $n$ distinct eigenvalues, then $A$ is diagonalizable.*

**Proof.** Since $A \in \mathcal{M}_n(\mathbb{R})$ and $A$ has $n$ distinct eigenvalues, then $\dim E_{\lambda_i} = 1 = A_m(\lambda_i)$, for

$i = 1, 2, ..., n$. Then $A$ is diagonalizable. ■

**Proposition 50** *Let $A$ and $B$ be two diagonalizable matrices with $P^{-1}AP = D_1$ and $P^{-1}BP = D_2$ for some invertible matrix $P$. Then $AB = BA$.*

**Proof.** We can easily verify that if $P^{-1}AP = D_1$ and $P^{-1}BP = D_2$, it follows that

$$\begin{cases} A = PD_1P^{-1} \\ B = PD_2P^{-1}. \end{cases}$$

Note that $D_1D_2 = D_2D_1$, and therefore

$$AB = PD_1D_2P^{-1} = PD_2D_1P^{-1} = PD_2P^{-1}PD_1P^{-1} = BA.$$

Hence the result. ■

**Corollary 51** *Let $A \in \mathcal{M}_n(\mathbb{R})$ be a square matrix, and assume that $A$ has a unique eigenvalue $\lambda$. Then $A$ is diagonalizable if and only if $A = \lambda I_n$.*

**Proof.** It is clear that if $A = \lambda I_n$, then $A$ is diagonalizable. Conversely, assume that $A \in \mathcal{M}_n(\mathbb{R})$ is diagonalizable and has a unique eigenvalue $\lambda$, there is therefore an invertible matrix $P$ such $P^{-1}AP$ is diagonal. We put $P^{-1}AP = D$, where $diag(D) = Sp(A) = \{\lambda\}$. It follows that

$$A = P \begin{pmatrix} \lambda & & & \\ & \lambda & & \\ & & \ddots & \\ & & & \lambda \end{pmatrix} P^{-1} = \lambda P \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} P^{-1} = \lambda P I_n P^{-1} = \lambda I_n.$$

This completes the proof. ■

**Proposition 52** *Let $A$ be a diagonalizable matrix* [3] *with $Sp(A) = \{\lambda_1, \lambda_2, ...., \lambda_n\}$. Then*

$$\det(A) = \lambda_1\lambda_2....\lambda_n. \tag{7}$$

[3] Note that the result of Equation (7) is always true for any matrix $A \in \mathcal{M}_n(\mathbb{C})$ which may or may not be diagonalizable.

**Proof.** Assume that $A = PDP^{-1}$, where $D = diag\{\lambda_1, \lambda_2, ...., \lambda_n\}$. Then

$$\begin{aligned}
\det(A) &= \det\left(PDP^{-1}\right) \\
&= \det(P)\det(D)\det\left(P^{-1}\right) \\
&= \det(D) \\
&= \lambda_1\lambda_2....\lambda_n.
\end{aligned}$$

This commpletes the proof. ■

**Definition 53** *$\lambda \in \mathbb{R}$ is called the eigenvalue of multiplicity $m$ if and only if*

$$p_A(x) = (x-\lambda)^m q(x) \ \ \text{with } q(\lambda) \neq 0.$$

**Example 54** *Let*

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 2 & 1 & -2 \\ -1 & 0 & -2 \end{pmatrix}$$

*Then $p_A(x) = (x-3)(x+1)^2$ and $A$ cannot be diagonalizable on either $\mathbb{R}$ or $\mathbb{C}$. Indeed, we have*

$$E_{-1} = Vect\{(1,-2,-1)\}$$

*In $\mathbb{R}^3$ or $\mathbb{C}^3$, $E_{-1}$ is a vector space of dimension $1$ equipped by $(1,-2,-1)$. Since $-1$ is an eigenvalue of $A$ of multiplicity $2$, $A$ is not diagonalizable.*

## 6.1 Applications of diagonalization

### 6.1.1 Computing of the power of a matrix

A classical application is the computing of the powers of a matrix $A$. Assume that $A$ is given to be diagonalizable. That is, there exist $P$ and $D$ such that

$$D = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ & & \ddots & \\ 0 & 0 & \dots & \lambda_n \end{pmatrix}$$

and $D = P^{-1}AP$. For each $k \geq 0$ we have

$$A^k = PD^kP^{-1}.$$

The preceding formula then generalizes to $k \in \mathbb{Z}$. The matrix $A$ is then invertible if, and only if, $D$ is invertible and

$$A^{-1} = PD^{-1}P^{-1}.$$

**Exercise 55** *Consider the matrix*

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}.$$

*Calculate $A^n$ for every $n \geq 0$.*

**Solution 56** *We start by computing the characteristic polynomial of A*

$$\begin{aligned} p_A(x) &= \begin{vmatrix} 2-x & -1 \\ -1 & 2-x \end{vmatrix} = \begin{vmatrix} 1-x & -1 \\ 1-x & 2-x \end{vmatrix} \\ &= (1-x)\begin{vmatrix} 1 & -1 \\ 1 & 2-x \end{vmatrix} = (1-x)(3-x). \end{aligned}$$

*Then* $Sp(A) = \{1, 3\}$.

*Next, we find the eigenvectors of* $A$:

$$\begin{aligned} E_1 &= \left\{(x,y) \in \mathbb{R}^2; \begin{array}{c} 2x - y = x \\ -x + 2y = y \end{array} \right\} \\ &= Vect\{(1,1)\}. \end{aligned}$$

*and also we have*

$$\begin{aligned} E_3 &= \left\{(x,y) \in \mathbb{R}^2; \begin{array}{c} 2x - y = 3x \\ -x + 2y = 3y \end{array} \right\} \\ &= Vect\{(1,-1)\}. \end{aligned}$$

*We put*

$$P = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, D = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$$

*It follows that*

$$\begin{aligned} A^n &= \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1^n & 0 \\ 0 & 3^n \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 3^n \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \\ &= \begin{pmatrix} \dfrac{1+3^n}{2} & \dfrac{1-3^n}{2} \\ \dfrac{1-3^n}{2} & \dfrac{1+3^n}{2} \end{pmatrix}. \end{aligned} \tag{8}$$

**Example 57** *Consider the matrix*

$$A = \begin{pmatrix} \dfrac{1}{2} & \dfrac{1}{2} \\ \dfrac{1}{4} & \dfrac{3}{4} \end{pmatrix}.$$

*Calculate* $\lim\limits_{n \to +\infty} A^n$.

*First, let us calculate the eigenvalues and eigenvectors of* $A$. *From computation, we find*

$$\begin{cases} \lambda_1 = 1, \ v_1 = (1,1), \\ \lambda_2 = \dfrac{1}{4}, \ v_2 = (-2,1). \end{cases}$$

*Since* $A = PDP^{-1}$*, then* $A^k = PD^kP^{-1}$*, where* $P = \begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix}$ *and* $D = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{4} \end{pmatrix}$*. It follows that*

$$\begin{aligned}
\lim_{n\to+\infty} A^n &= \lim_{n\to+\infty} \begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1^n & 0 \\ 0 & \left(\frac{1}{4}\right)^n \end{pmatrix} \begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ -\frac{1}{3} & \frac{1}{3} \end{pmatrix} \\
&= \begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \lim_{n\to+\infty} \left(\frac{1}{4}\right)^n \end{pmatrix} \begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ -\frac{1}{3} & \frac{1}{3} \end{pmatrix} \\
&= \begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ -\frac{1}{3} & \frac{1}{3} \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix}.
\end{aligned}$$

**Example 58** *Consider the mapping*

$$\begin{aligned}
f &: \mathbb{R}_3[X] \longrightarrow \mathbb{R}_3[X] \\
p &\mapsto f(p) = 3xp - \left(x^2 - 1\right) p'
\end{aligned}$$

*and let* $\mathcal{B} = \{1, x, x^2, x^3\}$ *be the canonical basis of* $\mathbb{R}_3[X]$*.*

1. *Calculate* $M_f(\mathcal{B})$*.*
2. *Is* $f$ *diagonalizable? if so, give the diagonalization.*

**Solution.** There are two steps:

▷ The calculation of $M_f(\mathcal{B})$. We see that

$$\begin{cases}
f(1) = 3x = 0 + 3x + 0x^2 + 0x^3 \\
f(x) = 1 + 2x^2 = 1 + 0x + 2x^2 + 0x^3 \\
f(x^2) = 2x + x^3 = 0 + 2x + 0x^2 + 1x^3 \\
f(x^3) = 3x^2 = 0 + 0x + 3x^2 + 0x^3
\end{cases}$$

Which gives

$$M_f(\mathcal{B}) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 3 & 0 & 2 & 0 \\ 0 & 2 & 0 & 3 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

▷ Let us calculate the characteristic polynomial of $M_f(\mathcal{B})$. Indeed, we have

$$p_{M_f(\mathcal{B})}(x) = \begin{vmatrix} -x & 1 & 0 & 0 \\ 3 & -x & 2 & 0 \\ 0 & 2 & -x & 3 \\ 0 & 0 & 1 & -x \end{vmatrix} = x^4 - 10x^2 + 9.$$

The eigenvalues of $A$ are $\{-1,1,-3,3\}$. From Corollary **??**, $M_f(\mathcal{B})$ is diagonalizable.

▷ Diagonalization of $M_f(\mathcal{B})$ : First, let us calculate the eigenvectors of $M_f(\mathcal{B})$, we obtain

$$M_f(\mathcal{B}) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -3 & -1 & 1 & 3 \\ 3 & -1 & -1 & 3 \\ -1 & 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} -3 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} \frac{1}{8} & -\frac{1}{8} & \frac{1}{8} & -\frac{1}{8} \\ \frac{3}{8} & -\frac{1}{8} & -\frac{1}{8} & \frac{3}{8} \\ \frac{3}{8} & \frac{1}{8} & -\frac{1}{8} & -\frac{3}{8} \\ \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} \end{pmatrix}.$$

## 6.2 Problems

**Ex 01.** Let $A \in \mathcal{M}_3(\mathbb{R})$ be a square matrix such that

$$p_A(x) = (x-1)(x-2)^2.$$

Is it diagonalizable ?

**Ex 02.** Let $f$ be a diagonalizable endomorphism over a vector space $E$. Prove that

$$E = \ker f \oplus \operatorname{Im} f.$$

**Ex 03.** Let $f$ be a diagonalizable endomorphism over a vector space satisfying $f^k = id_E$ for some natural integer $k$. Show that $f^2 = id_E$.

**Ex 04.** Let $A$ be a 3-by-3 matrix given by

$$A = \begin{pmatrix} 0 & 1 & 0 \\ -4 & 4 & 0 \\ -2 & 1 & 2 \end{pmatrix}.$$

1. Is the matrix $A$ diagonalizable?
2. Calculate $(A - 2I_3)$ and $(A - 2I_3)^n$ for every $n \in \mathbb{N}$. Deduce an explicit formula of $A^n$.

**Ex 05.** Let $M$ be a complex square matrix satisfying $M^k = I$ for some positive integer $k$. Prove that $M$ is diagonalizable.

**Ex 06.** Study the diagonalization of the matrix

$$A = \begin{pmatrix} 3 & 0 & 0 \\ 4 & 1 & 2 \\ a & 0 & 3 \end{pmatrix}; \ a \in \mathbb{R}$$

**Ans.** $A$ is diagonalizable $\Leftrightarrow a = 0$.

**Ex 07.** Verify that the matrix

$$A = \begin{pmatrix} 2 & -2 & 2 \\ 0 & 1 & 1 \\ -4 & 8 & 3 \end{pmatrix}$$

is diagonalizable. **Ans :** $Sp(A) = \{1,2,3\}$.

**Ex 08.** Study the diagonalization of the matrix

$$A = \begin{pmatrix} a & 1 & -1 \\ 0 & a & 2 \\ 0 & 0 & b \end{pmatrix}; \; a, b \in \mathbb{R}.$$

**Ex 09.** Check that the matrices of the form

$$A = \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}; \; c \neq 0$$

are not diagonalizable.

**Ex 10.** Consider the two matrices

$$A = \begin{pmatrix} 2 & 1 & -1 \\ 0 & 2 & -1 \\ -3 & -2 & 3 \end{pmatrix} \text{ and } B = \begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix}.$$

- Check that $A$ and $B$ have the same eigenvalues.
- Prove that $A \not\sim B$.

**Ex 11.** Find a matrix $A \in \mathcal{M}_2(\mathbb{R})$ which is not diagonalizable.

**Ex 12.** Let

$$A = S \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} S^{-1}; \; S \in \mathbb{GL}_2(\mathbb{R}) \text{ and } \lambda_1, \lambda_2 \in \mathbb{R}.$$

Calculate the determinant of $A$ and $A^{-1}$.

**Ex 13.** Calculate the eigenvalues and the eigenvectors of the following matrices. Are they diagonalizable? If so, determine a basis of eigenvectors.

$$\begin{pmatrix} 4 & 1 \\ 0 & 3 \end{pmatrix}, \begin{pmatrix} 2 & 4 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & -3 \\ 1 & -3 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 1 & -2 & -1 \\ 2 & 1 & -2 \\ 2 & 2 & -3 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$, \begin{pmatrix} -7 & -2 & 1 \\ 28 & 8 & -4 \\ 31 & 10 & -5 \end{pmatrix}, \begin{pmatrix} 7 & 4 & 0 & 0 \\ -12 & -7 & 0 & 0 \\ 20 & 11 & -6 & -7 \\ -12 & -6 & 6 & 6 \end{pmatrix}$$

**Ex 14.** Let $A \in \mathcal{M}_n(\mathbb{R})$. Prove that $A$ is diagonalozable $\Leftrightarrow$ $A^t$ is diagonalizable.

**Ex 15.** Study the diagonalization of the following matrix

$$A = \begin{pmatrix} 1 & a & b & c \\ 0 & 1 & d & e \\ 0 & 0 & 2 & f \\ 0 & 0 & 0 & 3 \end{pmatrix} ; a \neq 0 \text{ and } b, c, d, e, f \in \mathbb{R}.$$

**Ex 16.** Study the diagonalization of the following matrices

$$A_1 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \text{ and } A_2 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

**Ans.** $A_1$ : yes, $A_2$ : no

**Ex 17.** Discuss the diagonalization, according to $a, b \in \mathbb{R}$ of the matrix

$$A = \begin{pmatrix} a & b & a-b \\ b & 2b & -b \\ a-b & -b & a \end{pmatrix} ; \; ab \neq 0$$

and find $\alpha, \beta$ and $\gamma$ for which

$$A^3 = \alpha A^2 + \beta A + \gamma I_3.$$

**Ans.** $p_A(x) = x(x - 3b)(x - 2a + b)$.

**Ex 18.** Determine the real number $a$ for which the matrix

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & a \\ 0 & 0 & 1 & -a \end{pmatrix}$$

is diagonalizable.

**Ex 19.** Let $A \in \mathcal{M}_n(\mathbb{R})$ be a diagonalizable matrix with $Sp(A) = \{-1, 1\}$. Prove that $A = A^{-1}$.

**Ex 20.** Let

$$A = \begin{pmatrix} 9 & 0 & 0 \\ -5 & 4 & 0 \\ -8 & 0 & 1 \end{pmatrix}.$$

*i*) Prove that $A$ is diagonalizable and find a matrix $P \in \mathbb{GL}_3(\mathbb{R})$ for which $P^{-1}AP$ is diagonal.

*ii*) Calculate $A^n$, $n \in \mathbb{N}$ and deduce an explicit formula of $e^A$.

**Ex 21.** Let $A \in \mathcal{M}_n(\mathbb{R})$ such that $A^2 = A$. Prove that $A$ is diagonalizable.

**Ex 22.** Calculate $p(A) = 2A^8 - 3A^5 + A^4 + A^2 - 4I_3$, where $A$ is given by

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

**Ex 23.** Consider the matrix

$$A_\alpha(n) = \begin{pmatrix} 1 & \dfrac{\alpha}{n} \\ \dfrac{-\alpha}{n} & 1 \end{pmatrix}$$

Prove that

$$\lim_{n\to+\infty} A_\alpha(n) = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}.$$

**Ex 24.** Let $A$ be the matrix given by

$$A = \begin{pmatrix} 0.6 & 0.8 \\ 0.4 & 0.2 \end{pmatrix}$$

Verify that

$$\lim_{n\to+\infty} A^n = \begin{pmatrix} \dfrac{2}{3} & \dfrac{2}{3} \\ \dfrac{1}{3} & \dfrac{1}{3} \end{pmatrix}$$

**Ex 25.** Consider the matrix

$$A = \begin{pmatrix} 9 & 0 & 0 \\ -5 & 4 & 0 \\ -8 & 0 & 1 \end{pmatrix}$$

Calculate $A^n$, for $n \in \mathbb{N}$. **Ans.**

$$A^n = \begin{pmatrix} 9^n & 0 & 0 \\ 4^n - 9^n & 4^n & 0 \\ 1 - 9^n & 0 & 1 \end{pmatrix}.$$

**Ex 26.** Let

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \; B = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

1. Diagonalize the matrix $B$.
2. Is matrix $A$ similar to $B$?

**Ex 27.** Let $n \geq 2$. Let $A$ be the real $n \times n$ matrix of coefficients $a_{ij} = 0$ if $i = j$ and $a_{ij} = 1$; otherwise. We put $B = A + I_n$.

1. What is the rank of the matrix $B$? Deduce that $-1$ is an eigenvalue of $A$ and determe the dimension of the associated eigenspace.

2. Calculate

$$A \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix},$$

and deduce a new eigenvalue of $A$.

3. Justify that $A$ is diagonalizable, and give its characteristic polynomial.

4. Give an invertible matrix $P$ and a matrix $D$ such that $A = PDP^{-1}$ (one does not ask to calculate $P^{-1}$).

# 7 The Matrix Exponential

Note that the exponential of a matrix deals in particular in solving systems of linear differential equations. In the following section, we present some remarkable definitions and properties on the exponential of a square matrix which may or may not be diagonalizable.

**Definition 59** *For each $n \times n$ complex matrix $A$, define the exponential of $A$ to be the matrix*

$$e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!} = I_n + \frac{A}{1!} + \frac{A^2}{2!} + ... + \frac{A^k}{k!} + ....$$

*This is the matrix exponential of $A$.*

Note that if $A = 0$ (the zero matrix); we have $e^0 = I_n$. Indeed, we see that

$$e^0 = I_n + \frac{0}{1!} + \frac{0}{2!} + ... + \frac{0}{k!} + ... = I_n.$$

We also have for every $k \in \mathbb{Z}$, $e^{kA} = \left(e^A\right)^k$.

**Example 60** *Consider the matrix*

$$A = \begin{pmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{pmatrix}.$$

*Calculate $A^2$ and $A^3$. Deduce $e^A$.*

*Indeed, according computation, we have*

$$A^2 = \begin{pmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{pmatrix} \begin{pmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 3 & 3 & 9 \\ -1 & -1 & -3 \end{pmatrix}$$

*Moreover,*

$$A^3 = \begin{pmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 3 & 3 & 9 \\ -1 & -1 & -3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

*Using Definition 59, we obtain*

$$\begin{aligned} e^A &= I_3 + \frac{A}{1!} + \frac{A^2}{2!} \\ &= I_3 + A + \frac{A^2}{2} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 \\ 3 & 3 & 9 \\ -1 & -1 & -3 \end{pmatrix} \\ &= \begin{pmatrix} 2 & 1 & 3 \\ \frac{13}{2} & \frac{9}{2} & \frac{21}{2} \\ \frac{-5}{2} & \frac{-3}{2} & \frac{-7}{2} \end{pmatrix}. \end{aligned}$$

It is easy to calculate the exponential of a diagonal matrix. We have

**Corollary 61** *Let $D$ be a diagonal matrix, i.e.,*

$$D = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix} = diag\left\{\lambda_1, \lambda_2, ..., \lambda_n\right\}.$$

*Then*

$$e^D = \begin{pmatrix} e^{\lambda_1} & & & \\ & e^{\lambda_2} & & \\ & & \ddots & \\ & & & e^{\lambda_n} \end{pmatrix} = diag\left\{e^{\lambda_1}, e^{\lambda_2}, ..., e^{\lambda_n}\right\}. \qquad (9)$$

**Proof.** In fact, for each $k \geq 0$ we have

$$D^k = \begin{pmatrix} \lambda_1^k & & & \\ & \lambda_2^k & & \\ & & \ddots & \\ & & & \lambda_n^k \end{pmatrix}.$$

From definition 59, we get

$$\begin{aligned}
e^D &= \sum_{k=0}^{+\infty} \frac{D^k}{k!} \\
&= \sum_{k=0}^{+\infty} \frac{1}{k!} \begin{pmatrix} \lambda_1^k & & & \\ & \lambda_2^k & & \\ & & \ddots & \\ & & & \lambda_n^k \end{pmatrix} \\
&= \begin{pmatrix} \sum_{k=0}^{+\infty} \frac{\lambda_1^k}{k!} & & & \\ & \sum_{k=0}^{+\infty} \frac{\lambda_2^k}{k!} & & \\ & & \ddots & \\ & & & \sum_{k=0}^{+\infty} \frac{\lambda_n^k}{k!} \end{pmatrix} \\
&= \begin{pmatrix} e^{\lambda_1} & & & \\ & e^{\lambda_2} & & \\ & & \ddots & \\ & & & e^{\lambda_n} \end{pmatrix}.
\end{aligned}$$

This completes the proof. ■

**Example 62** *Let*

$$A = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}.$$

*Calculate* $e^A$.

*In fact, by* (9)*, we have*

$$e^A = \begin{pmatrix} e^{-1} & 0 \\ 0 & e^2 \end{pmatrix}.$$

**Proposition 63** *Let* $A \in \mathcal{M}_n(\mathbb{R})$ *be a diagonalizable matrix. Then* $e^A$ *is also diagonalizable. In addition, we have*

$$A = PDP^{-1} \Rightarrow e^A = Pe^D P^{-1}.$$

**Proof.** Let $A \in \mathcal{M}_n(\mathbb{R})$ be a diagonalizable matrix. Then there exists an invertible matrix $P$ such that $A = PDP^{-1}$ with $D$ is diagonal. Therefore,

$$\begin{aligned}
e^A &= \sum_{k=0}^{+\infty} \frac{A^k}{k!} = \sum_{k=0}^{+\infty} \frac{\left(PDP^{-1}\right)^k}{k!} \\
&= \sum_{k=0}^{+\infty} \frac{PD^k P^{-1}}{k!} \\
&= P \left( \sum_{k=0}^{+\infty} \frac{D^k}{k!} \right) P^{-1} \\
&= Pe^D P^{-1}.
\end{aligned}$$

As required. ■

**Theorem 64** *Let $S \in \mathbb{GL}_n(\mathbb{R})$ be an invertible matrix and let $A \in \mathcal{M}_n(\mathbb{R})$. We have*

$$e^{SAS^{-1}} = Se^{A}S^{-1}.$$

**Proof.** Let $S \in \mathbb{GL}_n(\mathbb{R})$ and let $A \in \mathcal{M}_n(\mathbb{R})$. From Definition 59, we have

$$\begin{aligned} e^{SAS^{-1}} &= I_n + \frac{SAS^{-1}}{1!} + \frac{(SAS^{-1})^2}{2!} + \frac{(SAS^{-1})^3}{3!} + ... \\ &= I_n + \frac{SAS^{-1}}{1!} + \frac{SA^2S^{-1}}{2!} + + \frac{SA^3S^{-1}}{3!} + ... \\ &= SI_nS^{-1} + \frac{SAS^{-1}}{1!} + \frac{SA^2S^{-1}}{2!} + + \frac{SA^3S^{-1}}{3!} + ... \\ &= S\left(I_n + \frac{A}{1!} + \frac{A^2}{2!} + \frac{A^3}{3!} + ...\right)S^{-1} \\ &= Se^{A}S^{-1}. \end{aligned}$$

The proof is finished. ■

**Corollary 65** *Let $A \in \mathcal{M}_n(\mathbb{R})$ and let $(\lambda, x)$ be an eigenpair of $A$. Then $\left(e^{\lambda}, x\right)$ is an eigenpair of $e^{A}$.*

**Proof.** Assume that $(\lambda, x)$ is an eigenpair of $A$. By definition, we have

$$\begin{aligned} e^{A}x &= \left(\sum_{k=0}^{+\infty} \frac{A^k}{k!}\right)x = \sum_{k=0}^{+\infty} \frac{A^k x}{k!} \\ &= \sum_{k=0}^{+\infty} \frac{\lambda^k x}{k!} = \left(\sum_{k=0}^{+\infty} \frac{\lambda^k}{k!}\right)x \\ &= e^{\lambda}x. \end{aligned}$$

This completes the proof. ■

**Lemma 66** *We have the following two properties:*

*(i) For any $A \in \mathcal{M}_n(\mathbb{R})$ and for any $t \in \mathbb{R}$,*

$$Ae^{At} = e^{At}A.$$

*(ii) For any $A \in \mathcal{M}_n(\mathbb{R})$ and for any $t \in \mathbb{R}$,*

$$e^{tI_n} = e^{t}A.$$

**Proof.** By the definition, we have

$$Ae^{At} = A\sum_{i=0}^{+\infty}\frac{A^k t^k}{k!} = \sum_{i=0}^{+\infty}\frac{A^{k+1}t^k}{k!} = \left(\sum_{i=0}^{+\infty}\frac{A^k t^k}{k!}\right)A = e^{At}A.$$

Likewise, we have

$$e^{tI_n} = e^{\begin{pmatrix} t & & \\ & \ddots & \\ & & t \end{pmatrix}} = \begin{pmatrix} e^t & & \\ & \ddots & \\ & & e^t \end{pmatrix} = e^t\begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix} = e^t I_n.$$

The proof is finished. ■

**Remark 67** *According to the previous lemma, we have*

$$e^{tI_n}I_n = e^{tI_n} = e^t I_n.$$

*Note that* $e^{tI_n} \neq e^t$*; because* $e^{tI_n} \in \mathcal{M}_n(\mathbb{R})$ *and* $e^t \in \mathbb{R}$*.*

The integer series which defines the exponential of a real, or complex number, is also convergent for a matrix. In addition, we have

**Theorem 68** *For any matrix* $A \in \mathcal{M}_n(\mathbb{C})$*, the series*

$$\sum_{k=0}^{+\infty}\frac{A^k}{k!}$$

*is absolutely convergent (therefore convergent) in* $\mathcal{M}_n(\mathbb{C})$*.*

**Proof.** For each $k \geq 0$, we have

$$\left\|\frac{A^k}{k!}\right\| \leq \frac{\|A\|^k}{k!}$$

and according to d'Alembert's Rule[4], we obtain

$$\lim_{k\to+\infty}\left|\frac{\frac{\|A\|^{k+1}}{(k+1)!}}{\frac{\|A\|^k}{k!}}\right| = \lim_{k\to+\infty}\frac{\|A\|}{k+1} = 0 < 1.$$

[4]Let $\sum u_n$ be a series with positive terms. If the limit (finite or not)

$$l = \lim\frac{u_{n+1}}{u_n}$$

exists, then

1. The series $\sum u_n$ is convergent if $l < 1$,
2. The series $\sum u_n$ is divergent if $l > 1$.

Thus, $\sum_{k=0}^{+\infty} \frac{A^k}{k!}$ is convergent. Since

$$\left\| \sum_{k=0}^{+\infty} \frac{A^k}{k!} \right\| \leq \sum_{k=0}^{+\infty} \frac{\|A\|^k}{k!},$$

It follows that $\sum_{k=0}^{+\infty} \frac{A^k}{k!}$ is therefore absolutely convergent. ■

Also we have the following proposition.

**Proposition 69** *Let $A$ be a square matrix. Then*

$$\lim_{x \to 0} \frac{e^{xA} - I}{x} = A.$$

**Proof.** We know that

$$e^{xA} - I - xA = \frac{(xA)^2}{2!} + \frac{(xA)^3}{3!} + ...$$

So we can write

$$\begin{aligned} \left\| e^{xA} - I - xA \right\| &= \left\| \frac{(xA)^2}{2!} + \frac{(xA)^3}{3!} + ... \right\| \\ &\leq \frac{\|xA\|^2}{2!} + \frac{\|xA\|^3}{3!} + ... \\ &= e^{\|xA\|} - 1 - \|xA\| . \end{aligned}$$

For every $x \neq 0$, we obtain

$$\left\| \frac{e^{xA} - I}{x} - A \right\| \leq \frac{e^{\|xA\|} - 1 - \|xA\|}{|x|} = \left( \frac{e^{|x|.\|x\|} - 1}{|x|} - \|A\| \right) \to 0.$$

As required. ■

## 7.1 Problems

**Ex 01.** Are the matrices

$$A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \ B = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}, C = \begin{pmatrix} -1 & 0 \\ 0 & -4 \end{pmatrix}$$

exponentials of matrices?

**Ex 02.** Prove that the matrix

$$J_2 = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$$

is neither the square nor the exponential of any matrix of $\mathcal{M}_2(\mathbb{R})$, but the matrices

$$J_4 = \begin{pmatrix} J_2 & \mathbf{0} \\ \mathbf{0} & J_2 \end{pmatrix} \text{ and } J_3 = \begin{pmatrix} J_2 & I_2 \\ \mathbf{0} & J_2 \end{pmatrix}$$

are the square and the exponential of a matrix of $\mathcal{M}_4(\mathbb{R})$.

**Ex 03.** Let

$$A = \begin{pmatrix} a & b & c \\ 0 & a & b \\ 0 & 0 & a \end{pmatrix}.$$

Calculate $e^A$.

**Ex 04.** Let

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Calculate $e^A e^B$, $e^{A+B}$ and $e^B e^A$.

**Ex 05.** Considère the following matrices

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}.$$

Calculate $C = e^{A+B}$, $D = e^A e^B$ and $F = e^B e^A$. Check that $C \neq D \neq F$.

**Ex 06.** Consider the matrix

$$A = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}.$$

Calculate $\log A$. i.e., find a matrix $B \in \mathcal{M}_2(\mathbb{C})$ such that $A = e^B$.

**Ex 07.** Consider the matrices

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}.$$

Calculate $e^A, e^B$. Deduce the expression of $e^F$, where

$$F = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

# 8 Special Matrices

**Definition 70** *A matrix with all zero entries is called a* ***zero matrix*** *and is denoted by* $0$. *That is,*

$$A = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}.$$

*Also, $A$ is called the* ***null matrix.***

**Definition 71** *A square matrix $A = (a_{ij})$ is* ***diagonal*** *if $a_{ij} = 0$ for $i \neq j$. In this case, we write $D = diag\{\lambda_1, \lambda_2, ..., \lambda_n\}$. So, a* ***diagonal matrix*** *is given by*

$$D = \begin{pmatrix} \boldsymbol{\lambda_1} & 0 & \cdots & 0 \\ 0 & \boldsymbol{\lambda_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \boldsymbol{\lambda_n} \end{pmatrix}.$$

- Every computation on diagonal matrices are quite easy. For example, $\sqrt{D}$, $D^k$, $D^{-1}$, $e^D$, $\cos D$, $\ln D$, ...

**Definition 72** *The* ***unit matrix*** *or the* ***identity matrix****:*

$$I_n = \begin{pmatrix} \mathbf{1} & 0 & \cdots & 0 \\ 0 & \mathbf{1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{1} \end{pmatrix}$$

*This is a diagonal matrix; but, all the diagonal elements are equal to 1.*

For any $A \in \mathcal{M}_n(\mathbb{R})$ we have

$$A \cdot I_n = I_n \cdot A = A.$$

**Definition 73** *A square matrix is* ***upper triangular*** *if all entries below the main diagonal are zero. The general form of an upper triangular matrix is given by*

$$U = \begin{pmatrix} \mathbf{a_{11}} & a_{12} & \cdots & a_{1n} \\ 0 & \mathbf{a_{22}} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{a_{nn}} \end{pmatrix}.$$

*A is called* ***lower triangular*** *if all entries above the main diagonal are 0. The general form of a lower triangular matrix is given by*

$$L = \begin{pmatrix} \mathbf{a_{11}} & 0 & \cdots & 0 \\ a_{21} & \mathbf{a_{22}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & \mathbf{a_{nn}} \end{pmatrix}.$$

**Definition 74** ***Strictly triangular matrices*** *are of the form:*

$$\begin{pmatrix} \mathbf{0} & a_{12} & \cdots & a_{1n} \\ 0 & \mathbf{0} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{0} \end{pmatrix} \quad or \quad \begin{pmatrix} \mathbf{0} & 0 & \cdots & 0 \\ a_{21} & \mathbf{0} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & \mathbf{0} \end{pmatrix}.$$

## 8.1 Symmetric Matrices

**Definition 75** *The* ***transpose*** *of an $m \times n$ matrix $A$, denoted by $A^t$, is the $n \times m$ matrix obtained by interchanging rows and columns of $A$. That is,*

$$\text{if } A = (a_{ij})_{\substack{1 \le i \le m \\ 1 \le j \le n}} \in \mathcal{M}_{m,n}(\mathbb{K}) \overset{then}{\Rightarrow} A^t = (a_{ji})_{\substack{1 \le j \le n \\ 1 \le i \le m}} \in \mathcal{M}_{n,m}(\mathbb{K}).$$

It is cleat that the mapping $A \mapsto A^t$ from $\mathcal{M}_{m,n}(\mathbb{K})$ to $\mathcal{M}_{n,m}(\mathbb{K})$ is linear, and that if $A \in \mathcal{M}_{m,n}(\mathbb{K})$, then

$$\left(A^t\right)^t = A.$$

Further, if $A \in \mathcal{M}_{m,n}(\mathbb{K})$ and $B \in \mathcal{M}_{n,p}(\mathbb{K})$, we have

$$(AB)^t = B^t A^t \in \mathcal{M}_{p,m}(\mathbb{K}).$$

**Properties of transpose:**

- $(A^t)^t = A$.
- $(A + B)^t = A^t + B^t$.
- For scalar $\alpha$, $(\alpha A)^t = \alpha A^t$.
- $(AB)^t = B^t A^t$.

**Example 76** *For the matrix*

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} \in \mathcal{M}_{3,2}(\mathbb{R}),$$

*we have*

$$A^t = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix} \in \mathcal{M}_{2,3}(\mathbb{R}).$$

**Theorem 77** *Let $A \in \mathcal{M}_n(\mathbb{R})$. Then $A$ and $A^t$ have the same eigenvalues.*

**Proof.** Let $x \in \mathbb{R}$. We have

$$\begin{aligned} p_A(x) &= \det(A - xI) = \det\left((A - xI)^t\right) \quad (\text{since } \det B = \det B^t) \\ &= \det\left(A^t - xI\right) \\ &= p_{A^t}(x). \end{aligned}$$

Thus, $A$ and its transpose have the same characteristic polynomial. ■

**Definition 78** *Let $A = (a_{ij})_{1 \le i,j \le n}$ be a square matrix. $A$ is said to be* ***symmetric*** *if $A^t = A$. That is, $a_{ij} = a_{ji}$ for each $i, j \in \overline{1, n}$. So, an $n \times n$ matrix $A$ is called symmetric if it is equal to its transpose.*

**Example 79** *The matrix*

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 0 & 5 \\ 3 & 5 & 1 \end{pmatrix}$$

*is symmetric; since* $A^t = A$.

**Corollary 80** *For every matrix* $A \in \mathcal{M}_n(\mathbb{R})$, $A^t A$ *and* $AA^t$ *are always symmetric.*

**Proof.** It is clear that

$$\left(A^t A\right)^t = A^t \left(A^t\right)^t = A^t A.$$

That is, for each $A \in \mathcal{M}_n(\mathbb{R})$, $A^t A$ is symmetric. ■

**Proposition 81** *The eigenvalues of a real symmetric matrix are real numbers.*

**Proof.** See Theorem 97. ■

**Corollary 82** *Let* $A \in \mathcal{M}_n(\mathbb{R})$ *be a symmetric matrix and let* $\alpha_0, \alpha_1, ..., \alpha_m \in \mathbb{R}$ *with* $m \geq 1$. *The matrix*

$$\alpha_0 I + \alpha_1 A + ... + \alpha_m A^m$$

*is also symmetric.*

**Proof.** (Easy). ■

## 8.2 Skew-symmetric Matrices

**Definition 83** *Let* $A = (a_{ij})_{1\leq i,j\leq n}$ *be a square matrix. A is said to be* ***skew-symmetric*** *if* $A^t = -A$. *That is,* $a_{ij} = -a_{ji}$ *for each* $i, j \in \overline{1, n}$.

For example, the matrix

$$A = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}$$

is skew-symmetric since $A^t = -A$.

**Lemma 84** *Every square matrix* $M \in \mathcal{M}_n(\mathbb{R})$ *can be written as* $A + B$, *where* $A$ *is skew-symmetric and* $B$ *is symmetric.*

**Proof.** It is clear that for each $M \in \mathcal{M}_n(\mathbb{R})$ we have

$$A = \underbrace{\frac{1}{2}\left(M - M^t\right)}_{\text{skew-symmetric}} + \underbrace{\frac{1}{2}\left(M + M^t\right)}_{\text{symmetric}}.$$

■

**Theorem 85** *Let* $B$ *be a skew-symmetric matrix; i.e.,* $B^t = -B$. *Then the matrix* $A = I - B$ *is invertible.*

**Remark 86** *Note that a matrix $A$ is invertible if and only if $(Ax = 0 \Rightarrow x = 0)$.*

**Proof of Theorem 85.**

It suffices to prove that $Ax = 0$ implies $x = 0$. In fact, if $Ax = 0$, it follows that $Bx = x$. Therefore,

$$\langle x, x\rangle = \langle x, Bx\rangle .$$

On the other hand, we have

$$\begin{aligned}
x^t x &= x^t B x \\
&\Rightarrow x^t x = x^t B^t x \quad (\text{since } \left(x^t x\right)^t = x^t x \text{ and } \left(x^t B x\right)^t = x^t B^t x) \\
&\Rightarrow x^t x = x^t\left(-B\right)x \quad (\text{since } B \text{ is skew-symmetric}) \\
&\Rightarrow x^t x = -x^t B x \\
&\Rightarrow x^t x = -x^t x \\
&\Rightarrow x^t x = 0.
\end{aligned}$$

Setting $x = \begin{pmatrix} x_1 & x_2 & \ldots & x_n \end{pmatrix}^t$, we find

$$x^t x = \begin{pmatrix} x_1 & x_2 & \ldots & x_n \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = x_1^2 + x_2^2 + ... + x_n^2 = 0.$$

Thus, $x_i = 0$ for each $i \in \overline{1, n}$, and so $x = 0$. ■

### 8.2.1 Problems.

1. Let

$$A = \begin{pmatrix} 0 & -2 & 3 \\ 2 & 0 & 4 \\ -3 & -4 & 0 \end{pmatrix}$$

Verify that $A$ is skew-symmetric.

2. Prove that $\mathcal{M}_n(\mathbb{R}) = \mathcal{S}_n(\mathbb{R}) \oplus \mathcal{A}_n(\mathbb{R})$, where $\mathcal{S}_n(\mathbb{R})$ is the subspace of all symmetric matrices and $\mathcal{A}_n(\mathbb{R})$ is the subspace of all skew-symmetric matrices.

## 8.3 Orthogonal Matrices

**Definition 87** *A matrix $A \in \mathcal{M}_n(\mathbb{R})$ is called* ***orthogonal*** *if $A^t = A^{-1}$.*

**Example 88** *The matrix*

$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}; \ \theta \in \mathbb{R}$$

*is orthogonal, since*

$$\begin{aligned} A^t A &= AA^t = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2. \end{aligned}$$

An orthogonal matrix has the following properties:

1. its column vectors (rows) are orthonormal,
2. $A^t A = AA^t = I_n$,
3. $A^t = A^{-1}$,
4. For every $x \in \mathbb{R}^n : \|Ax\| = \|x\|$,
5. For every $x, y \in \mathbb{R}^n : \langle Ax, Ay \rangle = \langle x, y \rangle$.

**Corollary 89** *Let $A \in \mathcal{M}_n(\mathbb{R})$ be an orthogonal matrix. Then*

$$\det(A) = \pm 1.$$

**Proof.** Since $A^t = A^{-1}$, then $A^t A = I_n$. It follows that

$$\det\left(A^t A\right) = \det\left(A^t\right)\det(A) = (\det(A))^2 = \det(I_n) = 1.$$

Hence $\det(A) = \pm 1$. ■

**Theorem 90** *Let $A \in \mathcal{M}_n(\mathbb{R})$ be an orthogonal matrix. The following properties are equivalent.*

*1) $A$ is orthogonal.*

*2) For every $x \in \mathbb{R}^n : \|Ax\| = \|x\|$.*

*3) For every $x, y \in \mathbb{R}^n : \langle Ax, Ay \rangle = \langle x, y \rangle$.*

**Proof.** 1)⇒2). Assume that $A$ is orthogonal. Let $x \in \mathbb{R}^n$, we have

$$\begin{aligned} \|Ax\|^2 &= \langle Ax, Ax \rangle = \left\langle x, A^t Ax \right\rangle \\ &= \langle x, I_n x \rangle = \langle x, x \rangle = \|x\|^2. \end{aligned}$$

Therefore, $\|Ax\| = \|x\|$.

2)⇒3). Assume that $\forall\, x \in \mathbb{R}^n : \|Ax\| = \|x\|$. Let $x, y \in \mathbb{R}^n$, we have

$$\|A(x+y)\|^2 = \|x+y\|^2;$$

That is,

$$\langle Ax + Ay, Ax + Ay \rangle = \langle x + y, x + y \rangle,$$

and so

$$\langle Ax, Ax\rangle + \langle Ay, Ay\rangle + 2\langle Ax, Ay\rangle = \langle x, x\rangle + \langle y, y\rangle + 2\langle x, y\rangle$$

Thus, $\langle Ax, Ay\rangle = \langle x, y\rangle$ .

3)$\Rightarrow$1). Assume that $\forall\ x, y \in \mathbb{R}^n : \langle Ax, Ay\rangle = \langle x, y\rangle$ . It follows that

$$\left\langle x, A^t Ay\right\rangle = \langle x, y\rangle$$

i.e.,

$$\left\langle x, A^t Ay - y\right\rangle = 0$$

In particular, for $x = x^t Ay - y$, we obtain

$$\left\|A^t Ay - y\right\|^2 = 0.$$

Hence $A^t Ay = y$, and therefore $A^t A = I_n$. ■

**Exercise 91** *Consider the matrix*

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

*For each $\theta \in \mathbb{R}$, prove that $e^{\theta A}$ is orthogonal*[5].

**Exercise 92** *Let $A$ be an orthogonal matrix. Prove the following properties:*

1. *$A^{-1}$ is orthogonal.*
2. *For every $\lambda \in Sp(A) \Rightarrow |\lambda| = 1$.*
3. *If $A_1$ and $A_2$ are two orthogonal matrices, then $A_1A_2$ is also orthogonal.*

## 8.4 Hermitian Matrices

**Definition 93** *Let $A = (a_{ij})_{1\le i,j\le n} \in \mathcal{M}_n(\mathbb{C})$. That is $a_{ij}$ is a complex number for $1 \le i, j \le n$. The matrix $(\overline{a_{ij}})_{1\le i,j\le n}$ is called **conjugate** of $A$, denoted by $\overline{A}$. The **transpose conjugate** matrix of $A$ is called the **adjoint** of $A$, denoted by $A^*$. Note that $A^* = \overline{A^t} = \left(\overline{A}\right)^t$.*

**Definition 94** *A matrix $A \in \mathcal{M}_n(\mathbb{C})$ is called **Hermitian**[6] if $A^* = A$. Thta is, if $\overline{A^t} = A$.*

**Example 95** *The matrix*

$$A = \begin{pmatrix} 1 & 1+i & 2+3i \\ 1-i & -2 & -i \\ 2-3i & i & 0 \end{pmatrix}$$

*is Hermitian; because $A^* = A$.*

[5] See the chapter of exponential of square matrices.

[6] On the other hand, a matrix $A$ is said to be skew-Hermitian if $A^* = -A$.

**Proposition 96** *The diagonal coefficients of a Hermitian matrix are real.*

**Proof.** From Definition 93, the result is obvious since $a_{ii} = \overline{a_{ii}}$ for $1 \leq i \leq n$. ■

**Theorem 97** *The eigenvalues of a Hermitian matrix are real.*

**Proof. Proof.** Let $(\lambda, x)$ be an eigenpair of a Hermitian matrix $A$ (note that $x \neq 0$). We can write

$$\begin{aligned}
\lambda \langle x, x \rangle &= \langle \lambda x, x \rangle \\
&= \langle Ax, x \rangle \\
&= (Ax)^t \, \overline{x} \\
&= x^t A^t \overline{x} \\
&= x^t \left( \left( \overline{A} \right)^t \right)^t \overline{x} \qquad \text{(since } \left( \overline{A} \right)^t = A) \\
&= x^t \overline{A} \overline{x} \\
&= x^t \overline{Ax} \\
&= \langle x, Ax \rangle \\
&= \langle x, \lambda x \rangle \\
&= \overline{\lambda} \langle x, x \rangle .
\end{aligned}$$

That is, $\lambda = \overline{\lambda}$.

**Remark 98** *Let $A \in \mathcal{M}_n(\mathbb{C})$. We can easily prove that the matrices $A + A^*$, $AA^*$ and $A^*A$ are Hermitian.*

■

## 8.5 Unitary Matrices

**Definition 99** *A matrix $U \in \mathcal{M}_n(\mathbb{C})$ is said to be* ***unitary*** *if $U^{-1} = U^*$. In other words, a square matrix $U$ with complex coefficients is said to be unitary if it satisfies the equalities:*

$$U^*U = UU^* = I_n.$$

- The unitary matrices with real coefficients are the orthogonal matrices.
- Note that a complex square matrix $A$ is **normal** if it commutes with its conjugate transpose $A^*$. That is, $A^*A = AA^*$. Thus, unitary, Hermitian and skew-Hermitian matrices are normal.

**Example 100** *The matrix*

$$A = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

*is unitary; since*

$$AA^* = A^*A = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2.$$

Any unitary matrix $U$ satisfies the following properties:

a. its determinant has modulus 1;

b. its eigenvectors are orthogonal;

c. $U$ is diagonalizable, i.e.,

$$U = VDV^*,$$

where $V$ is a unitary matrix and $D$ is a unitary diagonal matrix.

d. $U$ can be written as an exponential of a matrix:

$$U = e^{iH},$$

where $i$ is the imaginary unit and $H$ is a Hermitian matrix.

**Proposition 101** *Let $U$ be a square matrix of size $n$ with complex coefficients; the following five propositions are equivalent:*

1. *$U$ is unitary;*
2. *$U^*$ is unitary;*
3. *$U$ is invertible and its inverse is $U^*$;*
4. *the columns of $U$ form an orthonormal basis for the canonical Hermitian product over $\mathbb{C}^n$;*
5. *$U$ is normal and its eigenvalues have modulus 1.*

## 8.6 Idempotent matrix

**Definition 102** *Let $A \in \mathcal{M}_n(\mathbb{K})$. Then $A$ is called **idempotent** if $A^2 = A$.*

Examples of $2 \times 2$ idempotent matrices are:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 3 & -6 \\ 1 & -2 \end{pmatrix}$$

**Theorem 103** *If $A$ is idempotent, then $A$ is diagonalizable.*

**Proof.** Since $A^2 = A$, it follows that $m_A(x) = x(x-1)$ which has simple roots, and hence $A$ is diagonalizable. ■

# 9 Matrx norms

**Definition 104** *Let $E$ be a vector space over $\mathbb{K}$ ( $\mathbb{R}$ or $\mathbb{C}$). The norm over $E$, denoted by $\|.\|$, is a mapping*

$$\begin{array}{rcl} \|.\| & : & E \to \mathbb{R}_+ \\ x & \mapsto & \|x\| \quad \textit{(we say: the norm of } x\textit{)} \end{array}$$

*satisfying the following properties:*

1. *For all $x \in E : \|x\| \geq 0$ and $\|x\| = 0 \Leftrightarrow x = 0_E$;*
2. *For all $x \in E$ and scalar $\alpha \in \mathbb{K} : \|\alpha x\| = |\alpha| . \|x\|$;*
3. *For all $x, y \in E : \|x + y\| \leq \|x\| + \|y\|$.*

In this case, the couple $(E, \|.\|)$ is called **normed vector space** or **normed space**. So, a normed space $E$ is a vector space with a norm defined on it.

**Example 105** *In this lesson, we use only the vector spaces, $\mathbb{K}^n$ and $\mathcal{M}_n(\mathbb{K})$ with $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.*

1. *Define over $\mathbb{K}^n$ the following norms:*

$$\begin{aligned} \|x\|_1 &= \sum_{i=1}^{n} |x_i|, \ \|x\|_2 = \left( \sum_{i=1}^{n} |x_i|^2 \right)^{\frac{1}{2}}, \\ \|x\|_\infty &= \max_{1 \leq i \leq n} (|x_i|). \end{aligned}$$

2. *Define over $\mathcal{M}_n(\mathbb{K})$ the following norms:*

$$\begin{aligned} \|A\|_1 &= \max_j \sum_{i=1}^{n} |a_{ij}| \ \textit{ and } \ \|A\|_\infty = \max_i \sum_{j=1}^{n} |a_{ij}| \\ \|A\|_2 &= \left( \sum_{i,j}^{n} |a_{ij}|^2 \right)^{\frac{1}{2}}. \end{aligned}$$

*As an application, for $x = \begin{pmatrix} -1 & 1 & -2 \end{pmatrix}^t$, we have*

$$\|x\|_1 = 4, \quad \|x\|_2 = \sqrt{6} \textit{ and } \|x\|_\infty = 2.$$

*and for $A = \begin{pmatrix} -1 & -2 \\ 7 & 3 \end{pmatrix} \in \mathcal{M}_n(\mathbb{R})$, we also have*

$$\|A\|_1 = \max(8, 5) = 8, \|A\|_2 = 3\sqrt{7} \textit{ and } \|A\|_\infty = \max(3, 10) = 10.$$

**Lemma 106** *For each matrix $A \in \mathcal{M}_n(\mathbb{K})$ and for each $x \in \mathbb{K}^n$, we have the following inequality:*

$$\|Ax\| \leq \|A\| \|x\|.$$

# 10 Scalar Product (Inner product)

**Definition 107** *Let $E$ be real vectot space. The inner product of $E$ (over $E$) is a function $\langle .,. \rangle$ defined by*

$$\begin{array}{rcl} \langle .,. \rangle & : & E \times E \rightarrow \mathbb{R} \\ (x,y) & \mapsto & \langle x,y \rangle \end{array}$$

*satisfying the following properties:*

1. *For all $x \in E : \langle x,x \rangle \geq 0$ and $\langle x,x \rangle = 0 \Leftrightarrow x = 0$.*
2. *For all $x, y \in E : \langle x,y \rangle = \langle y,x \rangle$.*
3. *For all $x \in E$ and scalar $\alpha \in \mathbb{R} : \langle \lambda x,y \rangle = \lambda \langle x,y \rangle$*
4. *For all $x, y, z \in E : \langle x+y,z \rangle = \langle x,z \rangle + \langle y,z \rangle$.*

Define on the vector space $\mathbb{R}^n$ the inner product $\langle .,. \rangle$ by

$$\forall\, x = \begin{pmatrix} x_1 & x_2 & \ldots & x_n \end{pmatrix}^t, y = \begin{pmatrix} y_1 & y_2 & \ldots & y_n \end{pmatrix}^t \in \mathbb{R}^n$$

we have

$$\langle x,y \rangle = \sum_{i=1}^{n} x_i y_i.$$

**Remark 108** *For each $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n$, we have*

$$\langle x,y \rangle = x^t y.$$

*Also, the inner product over $\mathbb{C}^n$ is given by*

$$\langle x,y \rangle = x^t \overline{y}, \tag{10}$$

*where $\overline{y}$ is the conjugate of $y$.*

**Example 109** *Let $A \in \mathcal{M}_n(\mathbb{R})$. Find a symmetric matrix $B \in \mathcal{S}_n(\mathbb{R})$ such that*

$$x^t A x = x^t B x \text{ for every } x \in \mathbb{R}^n.$$

*In fact, for every $x \in \mathbb{R}^n$, we have*

$$\begin{array}{rcl} x^t A x & = & \left(x^t A x\right)^t \quad (\text{since } x^t A x = a \in \mathbb{R}) \\ & = & x^t A^t x, \end{array}$$

*It follows that*

$$x^t A x = \frac{1}{2} x^t A x + \frac{1}{2} x^t A^t x = x^t \left(\frac{A + A^t}{2}\right) x.$$

*Note that the matrix $B = \dfrac{A + A^t}{2}$ is symmetric.*

Also, define over the vector space $C([a,b])$ the inner product

$$\forall\ f,g \in C([a,b]) : \langle f,g\rangle = \int_a^b f(x)\, g(x)\, dx.$$

**Proposition 110** *Let $A$ be a symmetric matrix and let $(\alpha, x), (\beta, y)$ be two eigenpairs of $A$ with $\alpha \neq \beta$. Then $x$ and $y$ are orthogonal, i.e., $x \perp y$. Or, equivalently, $\langle x, y\rangle = 0$.*

**Proof.** Indeed, we have

$$\alpha \langle x,y\rangle = \langle \alpha x, y\rangle = \langle Ax, y\rangle = \left\langle x, A^t y\right\rangle = \langle x, Ay\rangle = \langle x, \beta y\rangle = \beta \langle x, y\rangle,$$

and since $\alpha \neq \beta$, it follows that $\langle x, y\rangle = 0$. ∎

## 10.1 Problems.

**Ex 01.** Consider the equation

$$ax^2 + 2hxy + by^2 = 0. \tag{11}$$

Write (11) in the form $X^t AX = 0$, where $A \in \mathcal{M}_2(\mathbb{R})$ and $X = \begin{pmatrix} x \\ y \end{pmatrix}$.

**Ans.** $A = \begin{pmatrix} a & h \\ h & b \end{pmatrix}$.

**Ex 02.** Write the equation $\lambda_1 x_1^2 + \lambda_2 x_2^2 = 0$ in the form $X^t AX = 0$, where $A \in \mathcal{M}_2(\mathbb{R})$ and $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.

**Ex 03.** Let $A \in \mathcal{M}_n(\mathbb{R})$. We ask if $x^t Ax = 0; \forall\ x \in \mathbb{R}^n \Rightarrow A = 0$ ?

**Ans.** No, take the matrix $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

# 11 System of linear recurrence sequences

## 11.1 Form I (without initial values)

Let $(x_n)$ and $(y_n)$ be two sequences given by the following relation:

$$\begin{cases} x_{n+1} = a_{11}x_n + a_{12}y_n \\ y_{n+1} = a_{21}x_n + a_{22}y_n \end{cases} ; \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}. \tag{12}$$

In the matrix form, we get

$$\underbrace{\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix}}_{X_{n+1}} = \underbrace{\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x_n \\ y_n \end{pmatrix}}_{X_n}.$$

Or, equivalently, we write (12) in the form

$$X_{n+1} = AX_n \text{ , where } X_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}.$$

Consequently,

$$X_n = AX_{n-1} = A\left(AX_{n-2}\right) = A^2 X_{n-2} = ... = A^n X_0. \tag{13}$$

**Remark 111** *If it is given to us $X_1$, we have only $X_n = A^{n-1}X_1$.*

In the general case, a system of $k$ linear recurrence sequences $x_n^{(i)}$, $i = 1, 2, ..., k$ is given by

$$\begin{cases} x_{n+1}^{(1)} = a_{11}x_n^{(1)} + a_{12}x_n^{(2)} + ... + a_{1k}x_n^{(k)} \\ x_{n+1}^{(2)} = a_{21}x_n^{(1)} + a_{22}x_n^{(2)} + ... + a_{2k}x_n^{(k)} \\ \vdots \\ x_{n+1}^{(k)} = a_{k1}x_n^{(1)} + a_{k2}x_n^{(2)} + ... + a_{kk}x_n^{(k)} \end{cases} ; \; x_0^{(i)} \in \mathbb{R}, \text{ for } i = 1, 2, ..., k. \tag{14}$$

In the matrix form

$$\underbrace{\begin{pmatrix} x_{n+1}^{(1)} \\ x_{n+1}^{(2)} \\ \vdots \\ x_{n+1}^{(k)} \end{pmatrix}}_{X_{n+1}} = \underbrace{\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1k} \\ a_{21} & a_{22} & \dots & a_{2k} \\ \vdots & \vdots & \dots & \vdots \\ a_{k1} & a_{k2} & \dots & a_{kk} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x_n^{(1)} \\ x_n^{(2)} \\ \vdots \\ x_n^{(k)} \end{pmatrix}}_{X_n},$$

where $X_0 = \begin{pmatrix} x_0^{(1)} \\ x_0^{(2)} \\ \vdots \\ x_0^{(k)} \end{pmatrix}$. As in (13), we get

$$X_n = A^n X_0.$$

These problems (the solution of (12) or (14)) reduce to the computation of $A^n$.

Consider the following example:

**Example 112** *Solve the system of linear recurrence sequences*

$$\begin{cases} x_{n+1} = 2x_n - y_n \\ y_{n+1} = -x_n + 2y_n \end{cases} ; \; (x_0, y_0) = (0, -1). \tag{15}$$

**Solution.** First, we write the system (15) according to the equivalent matrix form

$$\underbrace{\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix}}_{X_{n+1}} = \underbrace{\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x_n \\ y_n \end{pmatrix}}_{X_n} ; X_0 = \begin{pmatrix} 0 \\ -1 \end{pmatrix}.$$

From (13), we have $X_n = A^n X_0$. Moreover, from the previous computation, an explicit formula if $A^n$ in terms of $n$ is given by

$$A^n = \begin{pmatrix} \dfrac{1+3^n}{2} & \dfrac{1-3^n}{2} \\ \dfrac{1-3^n}{2} & \dfrac{1+3^n}{2} \end{pmatrix} ; n \geq 0. \tag{16}$$

It follows that

$$X_n = A^n X_0 = \begin{pmatrix} \frac{1+3^n}{2} & \frac{1-3^n}{2} \\ \frac{1-3^n}{2} & \frac{1+3^n}{2} \end{pmatrix} \begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} \frac{3^n-1}{2} \\ \frac{-3^n-1}{2} \end{pmatrix}. \tag{17}$$

## 11.2 Form II (with initial values)

Consider the system of linear recurrence sequences $x_n^{(i)}$, for $i = 1, 2, ..., k$:

$$\begin{cases} x_{n+1}^{(1)} = a_{11}x_n^{(1)} + a_{12}x_n^{(2)} + ... + a_{1k}x_n^{(k)} + c_1 \\ x_{n+1}^{(2)} = a_{21}x_n^{(1)} + a_{22}x_n^{(2)} + ... + a_{2k}x_n^{(k)} + c_2 \\ \vdots \\ x_{n+1}^{(k)} = a_{k1}x_n^{(1)} + a_{k2}x_n^{(2)} + ... + a_{kk}x_n^{(k)} + c_k \end{cases} ;\ c_i, x_0^{(i)} \in \mathbb{R}, \text{ for } i = 1, 2, ..., k.$$

In the matrix form

$$\underbrace{\begin{pmatrix} x_{n+1}^{(1)} \\ x_{n+1}^{(2)} \\ \vdots \\ x_{n+1}^{(k)} \end{pmatrix}}_{X_{n+1}} = \underbrace{\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1k} \\ a_{21} & a_{22} & \dots & a_{2k} \\ \vdots & \vdots & \dots & \vdots \\ a_{k1} & a_{k2} & \dots & a_{kk} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x_n^{(1)} \\ x_n^{(2)} \\ \vdots \\ x_n^{(k)} \end{pmatrix}}_{X_n} + \underbrace{\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{pmatrix}}_{C},$$

where $X_0 = \begin{pmatrix} x_0^{(1)} \\ x_0^{(2)} \\ \vdots \\ x_0^{(k)} \end{pmatrix}$. This means that

$$\begin{aligned} X_n &= AX_{n-1} + C = A\left(AX_{n-2} + C\right) + C = A^2 X_{n-2} + (A + I)\, C \\ &= ... \\ &= A^n X_0 + \left(A^{n-1} + A^{n-2} + ... + A + I\right) C. \end{aligned} \tag{18}$$

These problems are reduced to the computation of $A^n$ and $\sum_{i=0}^{n-1} A^i$.

**Example 113** *Solve the system of linear recurrence sequences*

$$\begin{cases} x_{n+1} = 2x_n - y_n - 1 \\ y_{n+1} = -x_n + 2y_n + 2 \end{cases} ;\ (x_0, y_0) = (0, -1). \tag{19}$$

**Solution.** The system (19) can be written in the following matrix form:

$$\underbrace{\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix}}_{X_{n+1}} = \underbrace{\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x_n \\ y_n \end{pmatrix}}_{X_n} + \underbrace{\begin{pmatrix} -1 \\ 2 \end{pmatrix}}_{C}$$

It suffices to compute $A^{n-1} + A^{n-2} + ... + A + I$. Indeed, in view of (16) we can write

$$A^n = \begin{pmatrix} \frac{1+3^n}{2} & \frac{1-3^n}{2} \\ \frac{1-3^n}{2} & \frac{1+3^n}{2} \end{pmatrix} = \frac{1}{2}U + \frac{3^n}{2}V,$$

where

$$U = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \text{ and } V = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

It follows that

$$\begin{aligned} A^{n-1} + A^{n-2} + ... + A + I &= \frac{n}{2}U + \left(\frac{1+3+...+3^{n-1}}{2}\right)V \\ &= \frac{n}{2}U + \left(\frac{3^n-1}{4}\right)V. \end{aligned}$$

Finally, from (18) we have

$$\begin{aligned} X_n &= \left(\frac{1}{2}U + \frac{3^n}{2}V\right)X_0 + \left[\frac{n}{2}U + \left(\frac{3^n-1}{4}\right)V\right]C = \left(\frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \frac{3^n}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\right)\begin{pmatrix} 0 \\ -1 \end{pmatrix} \\ &+ \left[\frac{n}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \left(\frac{3^n-1}{4}\right)\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\right]\begin{pmatrix} -1 \\ 2 \end{pmatrix} \\ &= \begin{pmatrix} \dfrac{2n-3^n+1}{4} \\ \dfrac{2n+3^n-5}{4} \end{pmatrix};\ n \geq 0. \end{aligned}$$

**Exercise 114** *Let $A \in \mathcal{M}_2(\mathbb{R})$. Assume $(A - I_2)^{-1}$ exists, prove that*

$$A^{n-1} + A^{n-2} + ... + A + I = (A^n - I_2)(A - I_2)^{-1}.$$

# 12 Linear Systems of differential equations, Part I

Define the **linear system of differential equations** $(x_1'(t), x_2'(t), ..., x_n'(t))$ by

$$\begin{cases} x_1'(t) = a_{11}x_1(t) + a_{12}x_2(t) + ... + a_{1n}x_n(t) + f_1(t) \\ x_2'(t) = a_{21}x_1(t) + a_{22}x_2(t) + ... + a_{2n}x_n(t) + f_2(t) \\ \vdots \\ x_n'(t) = a_{n1}x_1(t) + a_{n2}x_2(t) + ... + a_{nn}x_n(t) + f_n(t), \end{cases} \tag{20}$$

where $a_{ij} \in \mathbb{R}$. The unknowns are the functions $x_1(t), x_2(t), ..., x_n(t)$ which are derivable and $f_i(t)$ are some given functions.

The system is called **homogeneous** if all $f_i = 0$, otherwise it is called **non-homogeneous.**
**Matrix Notation**

A non-homogeneous system of linear equations (20) is written as the equivalent vector-matrix system

$$X'(t) = A \cdot X(t) + f(t),$$

where

$$X(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_2(t) \end{pmatrix},\ A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}, f = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_2 \end{pmatrix}$$

In this section, we consider only homogeneous systems**:** We wish to solve the system

$$X' = AX. \tag{21}$$

There are two cases:

**Case 1**. Assume that $A$ is diagonalizable. There exists an invertible matrix $P$ such that $A = PDP^{-1}$, where $D$ is diagonal. Thus,

$$\begin{cases} X' = PDP^{-1}X = PY' \\ Y' = DY \\ Y = P^{-1}X. \end{cases}$$

The system (21) becomes

$$Y' = DY,$$

which is easier to solve since $D$ is diagonal. Then after, we solve the equation $Y = P^{-1}X$, that is, $X = PY$.

**Example 115** *Solve the system of differential equations:*

$$X' = AX,\ A = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix},\ \textit{where } X(0) = \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$

**Solution.** At first, the eigenvalues of $A$ are $\lambda_1 = -1$ and $\lambda_2 = 4$. The corresponding eigenvectors are $v_1 = (1, -1)$ and $v_2 = (2, 3)$. Thus, we have

$$D = \begin{pmatrix} -1 & 0 \\ 0 & 4 \end{pmatrix},\ P = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}.$$

We put $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $Y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$. It follows that

$$Y' = DY \Leftrightarrow \begin{cases} y_1' = -y_1 \\ y_2' = 4y_2 \end{cases} \Leftrightarrow \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} c_1e^{-t} \\ c_2e^{4t} \end{pmatrix},$$

and hence

$$X = PY = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} c_1e^{-t} \\ c_2e^{4t} \end{pmatrix} = \begin{pmatrix} c_1e^{-t} + 2c_2e^{4t} \\ -c_1e^{-t} + 3c_2e^{4t} \end{pmatrix}.$$

Since $X(0) = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$, then

$$\begin{cases} c_1 + 2c_2 = 3 \\ -c_1 + 3c_2 = 2 \end{cases} \Rightarrow c_1 = c_2 = 1.$$

Thus is,

$$\begin{cases} x_1 = e^{-t} + 2e^{4t} \\ x_2 = -e^{-t} + 3e^{4t}. \end{cases}$$

We present another method to solve the system $X' = AX$, where $A$ is diagonalizable.

**Proposition 116** *Let $A \in \mathcal{M}_n(\mathbb{R})$ be diagonalizable matrix and let*

$$P = \begin{bmatrix} X_1 & X_n & ... & X_n \end{bmatrix}$$

*be the invertible matrix formed by $n$ linearly eigenvectors $X_1, X_2, ..., X_n$ of $A$. Then the system $X' = AX$ has a unique solution given by*

$$X(t) = c_1 e^{\lambda_1 t} X_1 + c_2 e^{\lambda_2 t} X_2 + ... + c_n e^{\lambda_n t} X_n, \tag{22}$$

*where $c_1, c_2, ..., c_n \in \mathbb{R}^n$ and $\lambda_1, \lambda_2, ..., \lambda_n$ are the eigenvalues of $A$.*

**Proof.** It is clear that $X' = AX$ implies

$$X(t) = e^{At}.\xi, \text{ where } \xi \in \mathcal{M}_{n,1}(\mathbb{R}).$$

Since $A$ is diagonalizable, then

$$X(t) = Pe^{Dt}P^{-1} = P \begin{pmatrix} e^{\lambda_1 t} & & & \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ & & & e^{\lambda_n t} \end{pmatrix} P^{-1}.\xi \tag{23}$$

Setting

$$P^{-1}.\xi = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = C.$$

It follows from (23) that

$$\begin{aligned} X(t) &= \begin{bmatrix} X_1 & X_n & ... & X_n \end{bmatrix} \begin{pmatrix} e^{\lambda_1 t} & & & \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ & & & e^{\lambda_n t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} \\ &= \begin{bmatrix} e^{\lambda_1 t} X_1 & e^{\lambda_2 t} X_n & ... & e^{\lambda_n t} X_n \end{bmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} \\ &= c_1 e^{\lambda_1 t} X_1 + c_2 e^{\lambda_2 t} X_2 + ... + c_n e^{\lambda_n t} X_n. \end{aligned}$$

Therefore,

$$X(t) = c_1 e^{\lambda_1 t} X_1 + c_2 e^{\lambda_2 t} X_2 + ... + c_n e^{\lambda_n t} X_n. \tag{24}$$

This completes the proof. ■

**Example 117** *Solve the system of differential equations:*

$$X' = AX, \ A = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}, \text{ where } X(0) = \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$

**Solution.** After the computation of the eigenvalues and eigenvectors of the matrix $A$. It follows from (24) that

$$X(t) = c_1 e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{4t} \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

Hence

$$\begin{cases} x(t) = c_1 e^{-t} + 2c_2 e^{4t}, \\ y(t) = -c_1 e^{-t} + 3c_2 e^{4t}. \end{cases}$$

Since $X(0) = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$, then

$$\begin{cases} x_1 = e^{-t} + 2e^{4t} \\ x_2 = -e^{-t} + 3e^{4t}. \end{cases}$$

**Example 118** *Solve the system of differential equations:*

$$X' = AX \text{ with } A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}.$$

**Solution.** Simple computation we get

$$\begin{cases} \lambda_1 = 1, \ v_1 = (-1, 1, 1) \\ \lambda_1 = 2, \ v_2 = (0, 1, 0) \text{ and } v_3 = (0, 0, 1). \end{cases}$$

The matrix $A$ is diagonalizable, and by (22) we obtain

$$X(t) = c_1 e^{t} \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_3 e^{2t} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

where $c_1, c_2, c_3$ are constants. That is,

$$\begin{cases} x(t) = -c_1 e^{t} \\ y(t) = c_1 e^{t} + c_2 e^{2t} \\ z(t) = c_1 e^{t} + c_3 e^{2t}. \end{cases}$$

**Remark 119** *In another way, which is very long and based on the calculation of $P$ and $P^{-1}$ with $A = PDP^{-1}$. From which it follows that*

$$e^{At} = P e^{Dt} P^{-1}. \tag{25}$$

*Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$. The solution of the differential system $X' = AX$ is $X(t) = e^{At}.C$, where $C$ is an arbitrarily constant. Since $X(0) = \begin{pmatrix} 3 & 2 \end{pmatrix}^t$, then $C = \begin{pmatrix} 3 & 2 \end{pmatrix}^t$. Therefore,*

$$X(t) = e^{At} \cdot X(0). \tag{26}$$

*Simple computation gives*

$$P = \begin{pmatrix} 2 & 1 \\ 3 & -1 \end{pmatrix} \text{ and } P^{-1} = \begin{pmatrix} \frac{1}{5} & \frac{1}{5} \\ \frac{3}{5} & \frac{-2}{5} \end{pmatrix}.$$

*Hence*

$$\begin{aligned} X(t) &= e^{At} \cdot C_0 = \begin{pmatrix} 2 & 1 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} e^{4t} & 0 \\ 0 & e^{-t} \end{pmatrix} \begin{pmatrix} \frac{1}{5} & \frac{1}{5} \\ \frac{3}{5} & \frac{-2}{5} \end{pmatrix} \begin{pmatrix} 3 \\ 2 \end{pmatrix} \\ &= \begin{pmatrix} 2e^{4t} + e^{-t} \\ 3e^{4t} - e^{-t} \end{pmatrix}. \end{aligned}$$

## 12.1 Problems

**Ex 01.** Calculate $e^{At}$ for each $t \in \mathbb{R}$, where

$$A = \begin{pmatrix} 0 & -1 & 1 \\ 0 & 0 & 1 \\ -1 & 0 & 1 \end{pmatrix}.$$

Deduce the general solution of the system of differential equations:

$$\begin{cases} p' = -q + r \\ q' = r \\ r' = -p + r \end{cases}$$

**Ex 02.** Solve the system of differential equations:

$$\begin{cases} x'(t) = y(t) \\ y'(t) = z(t) \\ z'(t) = w(t) \\ w'(t) = x(t) \end{cases}$$

**Ex 03.** Solve the system of differential equations $X' = A \cdot X$, where $A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$.

# 13 The square root of a diagonalizable matrix

**By Bellaouar D.**

**Lemma 120** *Let*

$$D = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}, \text{ where } \lambda_i > 0 \ (1 \leq i \leq n).$$

*Then*

$$\sqrt{D} = \begin{pmatrix} \sqrt{\lambda_1} & & & \\ & \sqrt{\lambda_2} & & \\ & & \ddots & \\ & & & \sqrt{\lambda_n} \end{pmatrix}.$$

**Proof.** It is clear by computation that $\sqrt{D}\sqrt{D} = D$. ■

**Proposition 121** *Let $A \in \mathcal{M}_n(\mathbb{R})$ be a diagonalizable matrix with $Sp(A) \subset \mathbb{R}_+$. Then $\sqrt{A} \in \mathcal{M}_n(\mathbb{R})$.*

**Proof.** Assume that $A = PDP^{-1}$, where $Sp(D) \subset \mathbb{R}_+$. We put

$$H = P\sqrt{D}P^{-1} \in \mathcal{M}_n(\mathbb{R}).$$

Since $\sqrt{D}\sqrt{D} = D$, it follows that

$$H^2 = \left(P\sqrt{D}P^{-1}\right)\left(P\sqrt{D}P^{-1}\right) = PDP^{-1} = A.$$

Thus, $\sqrt{A} = H$. ■

**Example 122** *Consider the matrix*

$$A = \begin{pmatrix} 11 & -5 & 5 \\ -5 & 3 & -3 \\ 5 & -3 & 3 \end{pmatrix}.$$

*Calculate $\sqrt{A}$.*

*After simple computation, the eigenpairs of $A$ are:*

$$\begin{cases} \lambda_1 = 0, \ E_{\lambda_1} = Vect\{(0,1,1)\}, \\ \lambda_2 = 1, \ E_{\lambda_2} = Vect\{(-1,-1,1)\}, \\ \lambda_3 = 16, \ E_{\lambda_3} = Vect\{(2,-1,1)\}. \end{cases}$$

*Further, we see that*

$$P = \begin{pmatrix} 0 & -1 & 2 \\ 1 & -1 & -1 \\ 1 & 1 & 1 \end{pmatrix}, \ D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 16 \end{pmatrix} \ \textit{and} \ P^{-1} = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{6} & \frac{1}{6} \end{pmatrix}.$$

*Which gives*

$$\begin{aligned} \sqrt{A} &= P\sqrt{D}P^{-1} \\ &= \begin{pmatrix} 0 & -1 & 2 \\ 1 & -1 & -1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{0} & 0 & 0 \\ 0 & \sqrt{1} & 0 \\ 0 & 0 & \sqrt{16} \end{pmatrix} \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{6} & \frac{1}{6} \end{pmatrix} \\ &= \begin{pmatrix} 3 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}. \end{aligned}$$

**Definition 123** *Let $A = PDP^{-1}$ be a diagonalizable matrix whose eigenvalues are given by the diagonal matrix*

$$D = diag\left\{\lambda_1, \lambda_2, ..., \lambda_n\right\}.$$

*For any function $f(x)$ defined at the points $(\lambda_i)_{1\leq i\leq n}$ , we have*

$$f(A) = Pf(D)P^{-1} = P\begin{pmatrix} f(\lambda_1) & & & \\ & f(\lambda_2) & & \\ & & \ddots & \\ & & & f(\lambda_n)\end{pmatrix}P^{-1}.$$

*For example, if $A \in \mathcal{M}_n(\mathbb{R})$ with $A = PDP^{-1}$ then*

$$\begin{cases} f(x) = x^k \Rightarrow f(A) = A^k = PD^kP^{-1} \quad \text{for } k \geq 0 \\ f(x) = \sqrt{x} \Rightarrow f(A) = \sqrt{A} = P\sqrt{D}P^{-1} \\ f(x) = \cos x \Rightarrow f(A) = \cos A = P(\cos D)P^{-1} \\ f(x) = e^x \Rightarrow f(A) = e^A = Pe^DP^{-1} \\ \qquad ... \end{cases}$$

## 13.1 Problems.

**Ex 01.** Let $M$ be a real $n$ by $n$ matrix. We denote by $\cos M$ the real part of $e^{iM}$ and $\sin M$ its imaginary part.

1. Show that $\cos M$ and $\sin M$ commute and that

$$(\cos M)^2 + (\sin M)^2 = I_n.$$

2. Let $\theta$ be a real number. Calculate

$$\cos\begin{pmatrix} \theta & 1 \\ 0 & \theta \end{pmatrix} \text{ and } \sin\begin{pmatrix} \theta & 1 \\ 0 & \theta \end{pmatrix}.$$

**Ex 02.** Let

$$A = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \in \mathcal{M}_2(\mathbb{C}).$$

Calculate $\sqrt{A}$.

# 14 Cayley-Hamilton Theorem

The goal of this section is to prove the famous Cayley-Hamilton Theorem, which asserts that if $p(x)$ is the characteristic polynomial of an $n$ by $n$ matrix $A$, then $p(A) = 0$.

**Definition 124** *Let $p(x) = a_0 + a_1x + ... + a_kx^k \in \mathbb{K}[X]$, and let $A \in \mathcal{M}_n(\mathbb{K})$. Define the matrix $p(A)$ by*

$$p(A) = a_0I_n + a_1A + ... + a_kA^k.$$

*In other words, $p(A)$ is the matrix obtained by replacing $x^i$ by $A^i$, for each $i = 0, 1, ..., k$, in the expression of $p$, with the convention $A^0 = I_n$.*

**Remark 125** *If we replace $x$ by $A$ in the formula of the characteristic polynomial $p_A(x)$, which gives*

$$p_A(A) = \det(A - A) = \det(0) = 0.$$

*This is impossible since $p_A(A) \in \mathcal{M}_n(\mathbb{K})$ and $\det(A - A) = \det(0) \in \mathbb{K}$.*

Let us recall the statement of one of the very classical theorem.

**Theorem 126 (Cayley-Hamilton Theorem)** *Let $A \in \mathcal{M}_n(\mathbb{R})$ and let $p_A(x)$ be its characteristic polynomial. Then $p_A(A) = 0$.*

In the proof, we need to use the following lemma.

**Lemma 127** *For each $A \in \mathcal{M}_n(\mathbb{R})$, we have*

$$A\,(com\,(A))^t = (com\,(A))^t\,A = \det A I_n. \tag{27}$$

*In particular, if $A$ is invertible, its inverse is given by*

$$A^{-1} = \frac{1}{\det(A)}\,(com\,(A))^t\,.$$

For example, if $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{M}_2(\mathbb{R})$, we have

$$\begin{aligned} A.\,(com\,(A))^t &= \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} ad - bc & 0 \\ 0 & ad - bc \end{pmatrix} \\ &= (ad - bc)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \det(A)\,I_2. \end{aligned}$$

**Proof of Cayley-Hamilton Theorem.** Let

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \in M_n(\mathbb{R})\,.$$

Assume further that $p_A(x) = x^n + c_{n-1}x^{n-1} + c_{n-2}x^{n-2} + ... + c_1 x + c_0$. Applying Lemma 127 using the matrix $xI_n - A$, we obtain

$$(xI_n - A)\,com\,(xI - A)^t = \det(xI_n - A)\,I_n,$$

where

$$xI - A = \begin{pmatrix} x - a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & x - a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \dots & \vdots \\ a_{n1} & a_{n2} & \dots & x - a_{nn} \end{pmatrix}.$$

Hence

$$com\,(xI-A)=\begin{pmatrix} p_{n-1}^{(1,1)}(x) & p_{n-1}^{(1,2)}(x) & \dots & p_{n-1}^{(1,n)}(x) \\ p_{n-1}^{(2,1)}(x) & p_{n-1}^{(2,2)}(x) & \dots & p_{n-1}^{(2,n)}(x) \\ \vdots & \vdots & \vdots & \vdots \\ p_{n-1}^{(n,1)}(x) & p_{n-1}^{(n,2)}(x) & \dots & p_{n-1}^{(n,n)}(x) \end{pmatrix},$$

where $p_{n-1}^{(i,j)}$ are polynomials of degree $n-1$. Setting

$$com\,(xI-A)^t = B_0 + xB_1 + x^2B_2 + ... + x^{n-1}B_{n-1}, \text{ where } (B_i)_{i=0,1,\dots,n-1} \in M_n(\mathbb{R}).$$

We deduce that

$$\begin{aligned}(xI-A)\left(B_0 + xB_1 + x^2B_2 + ... + x^{n-1}B_{n-1}\right) &= \det(xI_n - A).I_n \\ &= x^nI_n + c_{n-1}x^{n-1}I_n + ... + c_1xI_n + c_0I_n.\end{aligned}$$

It follows that

$$\begin{aligned}& x^nB_{n-1} + x^{n-1}(B_{n-2} - AB_{n-1}) + ... + x(B_0 - AB_1) - AB_0 \\ = \; & x^nI_n + c_{n-1}x^{n-1}I_n + ... + c_1xI_n + c_0I_n.\end{aligned}$$

Then

$$\begin{cases} B_{n-1} = I_n \\ B_{n-2} - AB_{n-1} = c_{n-1}x^{n-1}I_n \\ \qquad\vdots \\ B_0 - AB_1 = c_1I_n \\ -AB_0 = c_0I_n. \end{cases}$$

Which gives

$$\begin{aligned}p_A(A) &= c_0I_n + c_1A + ... + c_{n-1}A^{n-1} + A^n \\ &= -AB_0 + A(B_0 - AB_1) + ... + A^{n-1}(B_{n-2} - AB_{n-1}) + A^nB_{n-1} \\ &= 0.\end{aligned}$$

This completes the proof. ■

**Example 128** *Let* $A=\begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix}$. *Find a polynomial* $p(x)$ *of degree* 2 *such that* $p(A)=0$.
***Ans.*** $p(x) = x^2 - 3x - 2$.

**Corollary 129** *Let* $A \in \mathcal{M}_n(\mathbb{R})$ *with*

$$p_A(x) = x^n + c_{n-1}x^{n-1} + c_{n-2}x^{n-2} + ... + c_1x + c_0,$$

*where* $c_0 \in \mathbb{R}^*$ *and* $c_1, c_2, ..., c_{n-1} \in \mathbb{R}$. *Then*

$$A^{-1} = \frac{-1}{c_0}\left(\sum_{i=1}^{n-1} c_iA^{i-1} + A^{n-1}\right).$$

**Proof.** Since

$$p_A(A) = c_0 I + c_1 A + c_2 A^2 + ... + c_{n-1} A^{n-1} + A^n = 0,$$

it follows that

$$\left(c_1 I + c_2 A + ... + c_{n-1} A^{n-2} + A^{n-1}\right) A = -c_0 I,$$

and so

$$A^{-1} = \frac{-1}{c_0}\left(c_1 I + c_2 A + ... + c_{n-1} A^{n-2} + A^{n-1}\right).$$

This completes the proof. ■

**Example 130** *Using Cayley-Hamilton Theorem, calculate the inverse of the matrix*

$$A = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & 0 \\ 2 & 0 & -1 \end{pmatrix}.$$

**Solution.** First, let us calculate $p_A(x)$ :

$$\begin{aligned} p_A(x) &= \begin{vmatrix} \overset{+}{x-1} & \overset{-}{1} & \overset{+}{0} \\ -1 & x & 0 \\ 2 & 0 & x+1 \end{vmatrix} \\ &= (x-1)\left[x(x+1)\right] + (x+1) \\ &= (x-1)\left(x^2 - x + 1\right) \\ &= x^3 + 1. \end{aligned}$$

Therefore, $p_A(x) = x^3 + 1$, and hence

$$\begin{aligned} p_A(A) &= 0 \Rightarrow A^3 + I_3 = 0 \\ &\Rightarrow A^{-1} = -A^2. \end{aligned}$$

Finally, we get

$$A^{-1} = -\begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & 0 \\ 2 & 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & 0 \\ 2 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & -2 & -1 \end{pmatrix}.$$

# 15 Minimal Polynomial

We introduce here a second polynomial extracted from the characteristic polynomial of a square matrix.

**Definition 131** *Let $A$ be a square matrix and let $p_A(x)$ be its characteristic polynomial. The* ***minimal polynomial*** *of $A$, denoted by $m_A(x)$, is a polynomial satisfying the following two properties:*

1. *$m_A(x) | p_A(x)$; i.e., $m_A(x)$ divides the characteristic polynomial $p_A(x)$.*

2. *$m_A(A) = p_A(A) = 0$ (the zero matrix). That is, $m_A(x)$ satisfies Cayley-Hamilton Theorem as does $p_A(x)$.*

**Theorem 132** *The eigenvalues of a matrix $A$ are the roots of $m_A(x)$.*

**Proof.** Let $\lambda$ be an eigenvalue of $A$ and let $x$ be its eigenvector. We do the Euclidean division of $m_A(x)$ by $x-\lambda$, we obtain

$$m_A(x) = Q(x)(x-\lambda) + c,\ c \in \mathbb{R} \text{ and } Q \in \mathbb{R}[X].$$

It follows that

$$0 = m_A(A) = Q(A)(A-\lambda I) + cI.$$

If we apply this to the vector $x$, we get

$$0 = Q(A)(Ax - \lambda x) + cx.$$

Hence $cx = 0$. Since $x$ is not zero, we get $c = 0$, and so $m_A(x) = Q(x)(x-\lambda)$. This means that $\lambda$ is a root of $m_A(x)$. ∎

**Remark 133** *The minimal polynomial of $A$ is a polynomial satisfying the following three properties:*

1. *$m_A(x) | p_A(x)$,*
2. *$m_A(A) = p_A(A) = 0$ (the zero matrix),*
3. *For any $\lambda \in Sp(A) : m_A(\lambda) = 0$.*

**Example 134** *Calculate the minimal polynomial of the matrices:*

1. *$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$,*
2. *$B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.*

**Solution.**

1. We can easily prove that $p_A(x) = (1-x)(3-x)$, and so $m_A(x) = p_A(x)$.
2. First, the characteristic polynomial is $p_A(x) = (x-1)^2$. Hence,

$$m_A(x) = (x-1) \text{ or } m_A(x) = (x-1)^2,$$

and since $A - I_2 \neq 0$, then $m_A(x) = p_A(x) = (x-1)^2$.

**Example 135** *Determine the minimal polynomials of the following matrices:*

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ B = \begin{bmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 4 \end{bmatrix},\ C = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}$$

- It is clear that $p_A(x) = x^3$. Then, $m_A(x) = x^3$ or $x^2$ or $x$. On the other hand, we have $m_A(x) = x^2$; since $A \neq 0$ and $A^2 = 0$.

- Note that after computation, $p_B(x) = (x-3)^2(x-6)$. Since $p_B(x)$ and $m_B(x)$ having the same roots and $m_B(x)$ divides $p_B(x)$, then $m_B(x) = (x-3)(x-6)$ or $m_B(x) = (x-3)^2(x-6)$. But,

$$\begin{aligned}(B - 3I_3)(B - 6I_3) &= \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.\end{aligned}$$

  It follows that $m_B(x) = (x-3)(x-6)$.

- From simple computation, we get $p_C(x) = (x-1)^2$. Since $A - I_2 \neq 0$, then

$$m_C(x) = (x-1)^2 = p_C(x).$$

**Corollary 136** *Let $A \in \mathcal{M}_n(\mathbb{R})$ with $m_A(x) = (x-a)(x-b)$; $a, b \in \mathbb{R}$. Then $A^n$ can be written in terms of $A$ and $I$.*

**Proof.** The proof is by induction on $n$. Indeed,for $n = 1$, we have

$$A^1 = 1.A + 0.I.$$

Moreover, for $n = 2$, $A^2 = (a+b)A - abI$, since $m_A(A) = 0$. Assume that $A^n$ can be written in terms of $A$ and $I$, i.e.,

$$A^n = a_n A + b_n I.$$

Therefore,

$$\begin{aligned} A^{n+1} &= AA^n = A(a_n A + b_n I) \\ &= a_n A^2 + b_n A \\ &= a_n((a+b)A - abI) + b_n A \\ &= ((a+b)a_n + b_n)A - aba_n I \\ &= f(A, I). \end{aligned}$$

This means that $A^{n+1}$ can be written in terms of $A$ and $I$. ■

**Corollary 137** *The matrix $A$ is diagonalizable if and only if the roots of $m_A(x)$ are simple.*

**Example 138** *Let*

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

*Verify that $A$ is diagonalizable.*

**Solution.** From computation, we get

$$p_A(x) = (1+x)^2 (x-2).$$

This means that $m_A(x) = (1+x)(x-2)$ or $m_A(x) = (1+x)^2(x-2)$. But,

$$\begin{aligned}(I+A)(A-2I) &= \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.\end{aligned}$$

Thus, $m_A(x) = (1+x)(x-2)$. It is clear that the roots of $m_A(x)$ are simple, and hence $A$ is diagonalizable.

**Example 139** *Study the diagonalization of the matrix*

$$A = \begin{pmatrix} a & 0 & 0 \\ 1 & a & 0 \\ 1 & 1 & a \end{pmatrix}, \text{ where } a \in \mathbb{R}.$$

*Since $A$ is a lower triangular matrix, then $p_A(x) = (x-a)^3$. Since $(A-aI) \neq 0$, then $m_A(x)$ can not be $(x-a)$. This means that the roots of $m_A(x)$ are not simple, and so $A$ is not diagonalizable.*

**Example 140** *Consider the matrix*

$$A = \begin{pmatrix} a & b & b \\ b & a & b \\ b & b & a \end{pmatrix}.$$

*Show that $A$ is diagonalizable.*

*In fact, we have*

$$A = a \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + b \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} = aI_3 + bB.$$

*It suffices to prove that $B$ is diagonalizable. After computation we obtain*

$$m_B(x) = (x+1)(x-2),$$

*and hence $B$ is diagonalizable. That is, $B$ can be written in the form $B = PDP^{-1}$, from which it follows that*

$$\begin{aligned} A &= aI_3 + bPDP^{-1} \\ &= P(aI_3 + bD)P^{-1}. \end{aligned}$$

*Since $aI_3 + bD$ is diagonal, then $A$ is diagonalizable.*

**Example 141** *Cnsider the matrix*

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

*By computation,* $m_A(x) = x(x-3)$*. This means that* $A$ *is diagonalizable since the roots of* $m_A(x)$ *are simple.*

## 15.1 Problems

**Ex 01.** Find minimal polynomial of the matrix

$$A = \begin{pmatrix} 2 & 2 & -5 \\ 3 & 7 & -15 \\ 1 & 2 & -4 \end{pmatrix}.$$

Deduce that $A$ is diagonalizable. **Ans.**

$$p_A(x) = (x-3)(x-1)^2 \text{ and } m_A(x) = (x-3)(x-1).$$

**Ex 02.** Consider the matrix

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Calculate the minimal polynomial of $A$. **Ans.** $m_A(x) = x(x-2)$.

**Ex 03.** Calculate the characteristic polynomial of the matrix

$$\begin{pmatrix} 4 & 1 & 1 & 1 \\ 1 & 4 & 1 & 1 \\ 1 & 1 & 4 & 1 \\ 1 & 1 & 1 & 4 \end{pmatrix}.$$

Deduce its minimal polynomial. **Ans.**

$$p_A(x) = (3-x)^3(7-x) \text{ and } m_A(x) = (3-x)(7-x).$$

**Ex 04.** Calculate the minimal polynomial of the following matrices

$$\begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix}, \begin{pmatrix} 3 & 0 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix}, \begin{pmatrix} 3 & 0 & 0 \\ 1 & 3 & 0 \\ 0 & 1 & 4 \end{pmatrix}, \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 1 & 0 & 4 \end{pmatrix}.$$

**Ex 05.** Verify that all matrices of the forn

$$A = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix};\ \alpha \in \mathbb{R}^*$$

are not diagonalizable.

**Ex 06.** Calculate the minimal polynomial of the matrix

$$A = \begin{pmatrix} \lambda & & & & \\ 1 & \lambda & & & \\ & \ddots & \ddots & & \\ & & 1 & \lambda & \\ & & & 1 & \lambda \end{pmatrix},\ \lambda \in \mathbb{R}.$$

Is it diagonalizable ?

**Ex 07.** Let $A \in \mathcal{M}_3(\mathbb{R})$ given by

$$A = \begin{pmatrix} 3 & 2 & -2 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

a) Determine the characteristic polynomial of $A$.
b) Determin the minimal polynomial of $A$.
c) Is the matrix $A$ diagonalizable?

**Ex 08.** Find all the matrices $A \in \mathcal{M}_2(\mathbb{C})$ whose minimal polynomial is $x^2 + 1$.

**Ex 09.** Calculate the minimal polynomial of the matrix:

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

**Ans.** $m_A(x) = x(x-8)$.

**Ex 10.** Calculate the characteristic polynomial and its minimal polynomial of the matrix

$$A = \begin{pmatrix} 2 & 5 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 4 & 2 & 0 \\ 0 & 0 & 3 & 5 & 0 \\ 0 & 0 & 0 & 0 & 7 \end{pmatrix}.$$

**Ans.** $p_A(x) = (x-2)^3 (x-7)^2$ and $m_A(x) = (x-2)^2 (x-7)$.

# 16 Linear recurrence sequences of order $k$

Let $(a_0, a_1, ..., a_{k-1})$ be a system of $k$ real numbers not all zero. A **linear recurrence sequence of order** $k$ is defined as follows:

$$\begin{cases} x_{n+k} = a_0 x_n + a_1 x_{n+1} + ... + a_{k-1} x_{n+k-1}, \\ x_0, x_1, ..., x_{k-1} \in \mathbb{R} \text{ are given.} \end{cases}$$

Thus, a sequence defined by a **linear recurrence relation** is uniquely determined by its first $k$ terms: $x_0, x_1, ..., x_{k-1}$. As an example, for $k = 2$ :

$$\begin{cases} x_{n+2} = a_0 x_n + a_1 x_{n+1}, \\ x_0, x_1 \in \mathbb{R} \text{ are given.} \end{cases} \tag{S}$$

In the equivalent vector-matrix system, we obtain

$$\begin{pmatrix} x_{n+2} \\ x_{n+1} \end{pmatrix} = \begin{pmatrix} a_1 & a_0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_{n+1} \\ x_n \end{pmatrix},$$

or equivalently

$$\underbrace{\begin{pmatrix} x_{n+1} \\ x_{n+2} \end{pmatrix}}_{X_{n+2}} = \underbrace{\begin{pmatrix} 0 & 1 \\ a_0 & a_1 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x_n \\ x_{n+1} \end{pmatrix}}_{X_{n+1}}, \tag{S$_1$}$$

from which it follows that

$$X_n = AX_{n-1} = A^2 X_{n-2} = ... = A^{n-1} X_1, \tag{28}$$

where $X_1 = \begin{pmatrix} x_0 \\ x_1 \end{pmatrix}$. Thus, we must compute $A^n$ for $n \geq 0$.

**Application.** Consider the following example:

**Example 142** *Let $(x_n)$ be the sequence given by*

$$x_{n+2} = \frac{2}{\frac{1}{x_n} + \frac{1}{x_{n+1}}} \; ; x_0, x_1 \in \mathbb{R}_+^*. \tag{29}$$

*Find the formula of $x_n$ in terms of $n$, then calculate $\lim\limits_{n \to +\infty} x_n$.*

**Solution.** In fact, we write (29) in the form

$$\frac{2}{x_n} = \frac{1}{x_{n-2}} + \frac{1}{x_{n-1}}.$$

Setting $\frac{2}{x_n} = y_n$, we get

$$2y_n = y_{n-1} + y_{n-2}, \text{ that is, } y_n = \frac{1}{2} y_{n-1} + \frac{1}{2} y_{n-2}.$$

In the equivalent vector-matrix system, we have

$$\begin{pmatrix} y_n \\ y_{n-1} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} y_{n-1} \\ y_{n-2} \end{pmatrix}; \begin{cases} y_0 = \dfrac{1}{x_0} \\ y_1 = \dfrac{1}{x_1} \end{cases}$$

Therefore,

$$\begin{pmatrix} y_n \\ y_{n-1} \end{pmatrix} = A^{n-1} \begin{pmatrix} y_1 \\ y_0 \end{pmatrix}, \text{ where } A = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & 0 \end{pmatrix}.$$

From the computation (the matrix diagonalizable), we obtain

$$A^{n-1} = \begin{pmatrix} \dfrac{1}{3}\left[2 + \left(\frac{-1}{2}\right)^{n-1}\right] & \dfrac{1}{3}\left[1 - \left(\frac{-1}{2}\right)^{n-1}\right] \\ \dfrac{1}{3}\left[2 - 2\left(\frac{-1}{2}\right)^{n-1}\right] & \dfrac{1}{3}\left[1 + 2\left(\frac{-1}{2}\right)^{n-1}\right] \end{pmatrix},$$

and so

$$y_n = \frac{1}{3}\left[2 + \left(\frac{-1}{2}\right)^{n-1}\right] y_1 + \frac{1}{3}\left[1 - \left(\frac{-1}{2}\right)^{n-1}\right] y_0.$$

Since $x_n = \dfrac{1}{y_n}$, it follows that

$$x_n = \frac{3}{\left[2 + \left(\frac{-1}{2}\right)^{n-1}\right] \dfrac{1}{x_1} + \left[1 - \left(\frac{-1}{2}\right)^{n-1}\right] \dfrac{1}{x_0}}.$$

Passing to the limit as $n$ tends to infinity, we get

$$\lim_{n \to +\infty} x_n = \frac{3}{\dfrac{2}{x_1} + \dfrac{1}{x_0}}.$$

# 17 System of linear differential equations, Part II

Consider the system of differential equations:

$$\begin{cases} x_1' = a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \\ x_2' = a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \\ \vdots \\ x_n' = a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n, \end{cases} \tag{30}$$

which is written by the following equivalent vector-matrix system:

$$X' = A \cdot X,$$

where the matrix $A$ is **non-diagonalizable**. In this case, the general solution of (30) can be given by:

$$X(t) = e^{tA}c,$$

where $c = \begin{pmatrix} c_1 & c_2 & \dots & c_n \end{pmatrix}^t$ is a constant.

In this program, we only consider certain cases. For example, $A \in \mathcal{M}_n(\mathbb{R})$ but has a unique eigenvalue or when $A \in \mathcal{M}_n(\mathbb{R})$ with $n \leq 4$. The situation is particularly simple whenever $A \in \mathcal{M}_2(\mathbb{R})$.

**Corollary 143** *Let $A \in \mathcal{M}_n(\mathbb{R})$ be a square matrix having a unique eigenvalue, say $\lambda$. Then*

$$e^{tA} = e^{\lambda t} \sum_{k=0}^{n-1} (A - \lambda I_n)^k \frac{t^k}{k!}.$$

**Proof.** We first have $p_A(x) = (x-\lambda)^n$ since $A$ has a unique eigenvalue $\lambda$. We have

$$\begin{aligned}
e^{tA} &= e^{\lambda t I_n + t(A - \lambda I_n)} && (31)\\
&= e^{\lambda t I_n} e^{t(A-\lambda I_n)} \quad \text{(because } \lambda t I_n \text{ and } t(A-\lambda I_n) \text{ commute)}\\
&= e^{\lambda t} e^{t(A-\lambda I_n)} \quad \text{(because } e^{\alpha I_n} B = e^{\alpha} B \text{ for any } B \in \mathcal{M}_n(\mathbb{R}) \text{ and } \alpha \in \mathbb{R})\\
&= e^{\lambda t} \sum_{k=0}^{+\infty} (A - \lambda I_n)^k \frac{t^k}{k!} && (32)\\
&= e^{\lambda t} \sum_{k=0}^{n-1} (A - \lambda I_n)^k \frac{t^k}{k!},
\end{aligned}$$

where $\sum_{k=n}^{+\infty} (A - \lambda I_n)^k = 0$; this is obtained by Cayley-Hamilton theorem since $p_A(A) = (A - \lambda I_n)^n = 0$. ■

**Remark 144** *In particular, by Corollary 143, if $A \in \mathcal{M}_2(\mathbb{R})$ with $Sp(A) = \{\lambda\}$ then*

$$e^{tA} = e^{\lambda t} \{I_2 + (A - \lambda I_2) t\}. \tag{33}$$

*If $A \in \mathcal{M}_3(\mathbb{R})$ with $Sp(A) = \{\lambda\}$ then*

$$e^{tA} = e^{\lambda t} \left\{ I_3 + (A - \lambda I_3) t + \frac{1}{2} (A - \lambda I_3)^2 t^2 \right\}. \tag{34}$$

**Example 145** *Solve the system of différentiel equations*

$$\begin{cases} x' = 2x + y \\ y' = 2y \end{cases} \tag{35}$$

*Let $A$ be the matrix of* (35), *i.e.,*

$$A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}.$$

*From* (33), *we have*

$$\begin{aligned}
e^{tA} &= e^{2t} \{I_2 + (A - 2I_2) t\}\\
&= e^{2t} \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \left( \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} - 2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) t \right\}\\
&= \begin{pmatrix} e^{2t} & te^{2t} \\ 0 & e^{2t} \end{pmatrix}.
\end{aligned}$$

*Thus, the solution of* (35) *is given by*

$$X(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} e^{2t} & te^{2t} \\ 0 & e^{2t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} c_1 e^{2t} + tc_2 e^{2t} \\ c_2 e^{2t} \end{pmatrix},$$

*where* $c_1, c_2$ *are constants.*

**Example 146** *Solve the system of differential equations:*

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \underbrace{\begin{pmatrix} -4 & 1 & 1 \\ 1 & -1 & -2 \\ -2 & 1 & -1 \end{pmatrix}}_{A} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

**Solution:** The characteristic polynomil of $A$ is given by

$$p_A(x) = (x+2)^3.$$

This means that $A$ has a unique eigenvalu, $\lambda = -2$. From (34), we obtain

$$e^{tA} = e^{-2t}\left\{ I_3 + (A + 2I_3)\, t + \frac{1}{2}(A + 2I_3)^2\, t^2 \right\},$$

where

$$A + 2I_3 = \begin{pmatrix} -2 & 1 & 1 \\ 1 & 1 & -2 \\ -2 & 1 & 1 \end{pmatrix} \text{ and } A + 2I_3 = \begin{pmatrix} 3 & 0 & -3 \\ 3 & 0 & -3 \\ 3 & 0 & -3 \end{pmatrix}.$$

Then

$$\begin{aligned} e^{tA} &= e^{-2t}\left\{ I_3 + (A + 2I_3)\, t + \frac{1}{2}(A + 2I_3)^2\, t^2 \right\} \\ &= e^{-2t}\left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} -2 & 1 & 1 \\ 1 & 1 & -2 \\ -2 & 1 & 1 \end{pmatrix} t + \frac{1}{2} \begin{pmatrix} 3 & 0 & -3 \\ 3 & 0 & -3 \\ 3 & 0 & -3 \end{pmatrix} t^2 \right\} \\ &= e^{-2t} \begin{pmatrix} \frac{3}{2}t^2 - 2t + 1 & t & t - \frac{3}{2}t^2 \\ \frac{3}{2}t^2 + t & t + 1 & -\frac{3}{2}t^2 - 2t \\ \frac{3}{2}t^2 - 2t & t & -\frac{3}{2}t^2 + t + 1 \end{pmatrix}. \end{aligned}$$

**Exercise 147** *Solve the system of differential equations*

$$X' = A \cdot X, \text{ where } A = \begin{pmatrix} 3 & -18 \\ 2 & -9 \end{pmatrix}.$$

**Exercise 148** *Solve the system of differential equations*

$$\begin{pmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \\ x_4'(t) \end{pmatrix} = \begin{pmatrix} 1 & 2 & 1 & 3 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{pmatrix}.$$

**Theorem 149** *Let $A \in \mathcal{M}_3(\mathbb{R})$. If $A$ has two distinct eigenvalues $\lambda$ and $\mu$ (where $\lambda$ has multiplicity 2), then*

$$e^{tA} = e^{\lambda t}\left(I + t\left(A - \lambda I\right)\right) + \frac{e^{\mu t} - e^{\lambda t}}{\left(\mu - \lambda\right)^2}\left(A - \lambda I\right)^2 - \frac{te^{\lambda t}}{\mu - \lambda}\left(A - \lambda I\right)^2. \tag{36}$$

**Proof.** From (31) and (32), we have

$$\begin{aligned} e^{tA} &= e^{\lambda t}\sum_{k=0}^{+\infty}\left(A - \lambda I\right)^k \frac{t^k}{k!} \\ &= e^{\lambda t}\left(I + \left(A - \lambda I\right)\right) + e^{\lambda t}\sum_{k=2}^{+\infty}\left(A - \lambda I\right)^k \frac{t^k}{k!} \\ &= e^{\lambda t}\left(I + \left(A - \lambda I\right)\right) + e^{\lambda t}\sum_{r=0}^{+\infty}\left(A - \lambda I\right)^{2+r} \frac{t^{2+r}}{(2+r)!} \end{aligned} \tag{37}$$

Now, let $p_A(x) = (x - \lambda)^2 (x - \mu)$ be the characteristic polynomial of $A$. First, we note that

$$A - \mu I = \left(A - \lambda I_n\right) - \left(\mu - \lambda\right) I.$$

By Cayley-Hamilton theorem, we get

$$0 = \left(A - \lambda I\right)^2\left(A - \mu I\right) = \left(A - \lambda I\right)^3 - \left(\mu - \lambda\right)\left(A - \lambda I\right)^2,$$

from which is follows that

$$\left(A - \lambda I\right)^3 = \left(\mu - \lambda\right)\left(A - \lambda I\right)^2.$$

By induction, for every $r \geq 1$,

$$\left(A - \lambda I\right)^{2+r} = \left(\mu - \lambda\right)^r\left(A - \lambda I\right)^2.$$

It follows from (37) that

$$\begin{aligned} \sum_{r=0}^{+\infty}\left(A - \lambda I\right)^{2+r}\frac{t^{2+r}}{(2+r)!} &= \sum_{r=0}^{+\infty}\left(\mu - \lambda\right)^r\frac{t^{2+r}}{(2+r)!}^{r}\left(A - \lambda I\right)^2 \\ &= \frac{1}{\left(\mu - \lambda\right)^2}\sum_{r=0}^{+\infty}\frac{t^k}{k!}\left(\mu - \lambda\right)^k\left(A - \lambda I\right)^2. \end{aligned}$$

Finally, we obtain

$$\begin{aligned} e^{tA} &= e^{\lambda t}\left(I + \left(A - \lambda I\right)\right) + \frac{e^{\lambda t}}{\left(\mu - \lambda\right)^2}\left\{e^{(\mu - \lambda)t} - 1 - \left(\mu - \lambda\right)t\right\}\left(A - \lambda I\right)^2 \\ &= e^{\lambda t}\left(I + t\left(A - \lambda I\right)\right) + \frac{e^{\mu t} - e^{\lambda t}}{\left(\mu - \lambda\right)^2}\left(A - \lambda I\right)^2 - \frac{te^{\lambda t}}{\mu - \lambda}\left(A - \lambda I\right)^2. \end{aligned}$$

This completes the proof. ■

**Example 150** *Solve the system of differential equations*

$$\begin{pmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{pmatrix} = \underbrace{\begin{pmatrix} 2 & -1 & 2 \\ 10 & -5 & 7 \\ 4 & -2 & 2 \end{pmatrix}}_{A} \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix}.$$

*We first find the characteristic polynomial of $A$. By computation, $p_A(x) = x^2(x+1)$. This means that $A$ has two eigenvalues $\lambda = 0$ (with multiplicity 2) and $\mu = -1$ (simple). It follows from (36) that*

$$e^{At} = I_3 + tA + \left(t + e^{-t} - 1\right) A^2.$$

*Simple computation we obtain*

$$e^{At} = \begin{pmatrix} 4t + \frac{2}{e^t} - 1 & 1 - \frac{1}{e^t} - 2t & 3t + \frac{1}{e^t} - 1 \\ 8t - \frac{2}{e^t} + 2 & \frac{1}{e^t} - 4t & 6t - \frac{1}{e^t} + 1 \\ 4 - \frac{4}{e^t} & \frac{2}{e^t} - 2 & 3 - \frac{2}{e^t} \end{pmatrix}.$$

# 18 On the powers of A

**Example 151** *Let*

$$A = \begin{pmatrix} a & b & c \\ 0 & a & b \\ 0 & 0 & a \end{pmatrix}.$$

*Find $A^n$ for $n \geq 0$.*

**Solution.** Setting

$$A = \begin{pmatrix} a & b & c \\ 0 & a & b \\ 0 & 0 & a \end{pmatrix} = \underbrace{\begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix}}_{D} + \underbrace{\begin{pmatrix} 0 & b & c \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix}}_{N}.$$

It is clear that $N$ is nilpotent of index $k = 3$. Moreover, $DN = ND$. By Binomial formula we have

$$A^n = (D+N)^n = C_n^0 D^n + C_n^1 D^{n-1} N + C_n^2 D^{n-2} N^2,$$

where

$$N^2 = \begin{pmatrix} 0 & 0 & b^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

That is,

$$A^n = D^n + nD^{n-1}N + \frac{n(n-1)}{2} D^{n-2} N^2.$$

**Problem 152** *Let*

$$J_n = \begin{pmatrix} 0 & \mathbf{1} & & & \\ & 0 & \mathbf{1} & & \\ & & \ddots & \ddots & \\ & & & 0 & \mathbf{1} \\ & & & & 0 \end{pmatrix}$$

*For example, we have*

$$J_2 = \begin{pmatrix} 0 & \mathbf{1} \\ 0 & 0 \end{pmatrix}, J_3 = \begin{pmatrix} 0 & \mathbf{1} & 0 \\ 0 & 0 & \mathbf{1} \\ 0 & 0 & 0 \end{pmatrix}, J_4 = \begin{pmatrix} 0 & \mathbf{1} & 0 & 0 \\ 0 & 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 & \mathbf{1} \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

*and so on. Prove that* $J_n^{n-1} \neq 0$ *and* $J_n^n = 0$. *That is,* $J_n$ *is nilpotent with index* $n$.

# 19 Nilpotent Matrices

**Definition 153** *A* ***nilpotent*** *matrix is a square matrix* $N$ *such that* $N^k = 0$ *for some positive integer* $k$.

In other words, a square matrix $N$ is said to be **nilpotent** if there exists a positive integer $k$ such that $N^k = 0$. The smallest such $k$ is called the **index** of $N$.

**Example 154** *The matrix*

$$N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

*is nilpotent with index* 2*, since* $N^2 = 0$.

**Proposition 155** *Let* $N$ *be a nilpotent matrix. Then*

- $Sp(N) = \{0\}$,
- $I - N$ *is invertible.*

**************************************************

**Proof.** Assume that $N^k = 0$ and $N^{k-1} \neq 0$ for some $k \geq 1$.

- Let $(\lambda, x)$ be an eigenpair of $N$, that is, $Nx = \lambda x$ and $x \neq 0$. It follows that $\lambda^k x = N^k x = 0$, and hence $\lambda = 0$.
- Let $x \in \mathbb{R}^n$ such that $(I - N)x = 0$. Therefore, $Nx = x$, form which it follows that $N^k x = N^{k-1} x = 0$. Since $N^{k-1} \neq 0$, then $x = 0$. Thus, $I - N$ is invertible.

The proof is finished. ■

**************************************************

**Theorem 156** *Let* $A$ *be a nonzero nilpotent matrix. Then* $A$ *is nondiagonalizable.*

**Proof.** Assume, by the way of contradiction that $A$ is diagonalizable, that is, $A = PDP^{-1}$ for some invertible matrix $P = 0$. Since $A$ is nilpotent, there exists a positive integer $k$ such that $A^k = 0$. It follows that $D = P^{-1}AP$, and so

$$D^k = P^{-1}A^kP = 0.$$

Since $D$ is diagonal, then $D = 0$. This means that $A = 0$, a contradiction. ■

**Theorem 157** *Any strictely triangular matrix is nilpotent.*

**Proof.** Setting

$$A=\begin{pmatrix} \mathbf{0} & 0 & \cdots & 0 \\ a_{21} & \mathbf{0} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & \mathbf{0} \end{pmatrix}.$$

Since $p_A(x)=x^n$. By Cayley-Hamilton theorem, $A^n =$. That is, $\exists\, k\leq n$ such that $A^k=0$, and hence $A$ is nilpotent. ∎

**Example 158** *Determine the index of the following matrix:*

$$N=\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

*It is clear that*

$$N^2=\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } N^3=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

*Since $N^3=0$ and $N^2\neq 0$, then $N$ is nilpotent of index $k=3$.*

**Remark 159** *The product of two non-zero matrices can be zero. Indeed, for a matrix $A\in\mathcal{M}_n(\mathbb{R})$, we have*

$$A^2=0 \nRightarrow A=0.$$

*For example, if $A=\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}\neq 0$ we see that*

$$A^2=\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}=\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

*But, $A\neq 0$.*

**Example 160** *Consider the matrix*

$$A=\begin{pmatrix} 3 & 9 & -9 \\ 2 & 0 & 0 \\ 3 & 3 & -3 \end{pmatrix}$$

*Show that $A$ is nilpotent.*

**Solution**. First, we determine the characteristic polynomial of $A$.

$$\begin{aligned} p_A(x) &= \begin{vmatrix} 3-x & 9 & -9 \\ 2 & -x & 0 \\ 3 & 3 & -3-x \end{vmatrix} = \begin{vmatrix} 3-x & 0 & -9 \\ 2 & -x & 0 \\ 3 & -x & -3-x \end{vmatrix} \\ &= -x\begin{vmatrix} 3-x & 0 & -9 \\ 2 & 1 & 0 \\ 3 & 1 & -3-x \end{vmatrix} \\ &= -x^3. \end{aligned}$$

By Cayley-Hamilton theorem, $A^3=0$. Since $A^2\neq 0$, then $A$ is nilpotente of index 3.

**Theorem 161** *Let $N$ be a nilpotent matrix of index $k$ and let $x \in \mathbb{R}^n$ be a nonzero vector such that $N^{k-1}x \neq 0$. The family*

$$\left\{Ix, Nx, N^2x, ..., N^{k-1}x\right\}$$

*is free.*

**Proof.** Let $(\alpha_i)_{0\leq i\leq k-1} \in \mathbb{R}$ such that

$$\sum_{i=0}^{k-1} \alpha_i N^i x = 0,$$

from which it follows that

$$\left\{\begin{array}{c} \alpha_0 N^{k-1}x + \alpha_1 N^k x + ... + \alpha_{k-1}N^{2k-2}x = 0 \\ \alpha_0 N^{k-2}x + \alpha_1 N^{k-1}x + ... + \alpha_{k-1}N^{2k-3}x = 0 \\ \vdots \\ \alpha_0 Nx + \alpha_1 N^2 x + ... + \alpha_{k-1}N^k x = 0 \\ \alpha_0 Ix + \alpha_1 Nx + ... + \alpha_{k-1}N^{k-1}x = 0 \end{array}\right. \Rightarrow \left\{\begin{array}{c} \alpha_0 N^{k-1}x = 0 \\ \alpha_1 N^{k-1}x \\ \vdots \\ \alpha_{k-2}N^{k-1}x = 0 \\ \alpha_{k-1}N^{k-1}x = 0 \end{array}\right.$$

Since $N^{k-1}x \neq 0$, then $\alpha_0 = \alpha_1 = ... = \alpha_{k-1} = 0$. This completes the proof. ■

## 19.1 Problems

**Ex 01.** Let $A \in \mathcal{M}_n(\mathbb{R})$ be a nilpotent matrix. Prove that

$$\det(A + I_n) = 1.$$

**Ex 02.** We ask if $A^2 = 0 \Rightarrow A = 0$ ?

**Ex 03.** Verify that

$$A = \begin{pmatrix} 1 & -3 & -4 \\ -1 & 3 & 4 \\ 1 & -3 & -4 \end{pmatrix}$$

is nilpotent.

**Ex 04.** Let

$$A = \begin{pmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{pmatrix}$$

Calculate $A^3$. What do you say ?

**Ex 05.** Prove the result: If $N$ is nilpotent, then $I + N$ and $I - N$ are invertible, where $I$ is the identity matrix.

**Ex 06.** Prove that

$$A \sim 2A \Rightarrow A \text{ is nilpotent over } \mathbb{R}.$$

## 20 Trigonalization

**Definition 162** *Let $A \in \mathcal{M}_n(\mathbb{K})$. Then $A$ is called* ***trigonalizable*** *if there exists an invertible matrix $P$, that is, $P \in \mathbb{GL}_n(\mathbb{K})$, such that $A = PTP^{-1}$, where $T$ is an upper triangular matrix having the same eigenvalues of $A$. Or, equivalently, $A$ is similar to a triangular matrix $T$.*

Now, we present Schur Theorem decomposition of a square matrix $A \in \mathcal{M}_n(\mathbb{C})$.

**Theorem 163** *Any matrix with complex entries is trigonalizable over $\mathcal{M}_n(\mathbb{C})$.*

**Proof.** Let $A \in \mathcal{M}_n(\mathbb{C})$. We will show that $A$ is trigonalizable over $\mathcal{M}_n(\mathbb{C})$. We use induction on $n$. Indeed, for $n = 1$ we have

$$A = (a_{11}), \text{ where } a_{11} \in \mathbb{C}.$$

In this case, we write

$$A = I(a_{11})I^{-1} = PTP^{-1} \text{ with } P = I = (1) \text{ and } T = (a_{11}) = A.$$

Assume that every matrix $A_1 \in M_n(\mathbb{C})$ is trigonalizable. Let $(\lambda, x)$ be an eigenpair of $A$, and let $\{x, u_2, ..., u_n\}$ be a basis of $\mathbb{C}^n$. We put $U = (x, u_2, ..., u_n)$, it follows that

$$AU = \begin{pmatrix} Ax & Au_2 & \dots & Au_n \end{pmatrix} = \begin{pmatrix} \lambda x & Au_2 & \dots & Au_n \end{pmatrix}.$$

Now, calculate $U^{-1}AU$. In fact, we have

$$U^{-1} = U^{-1}Ue_1 = e_1,$$

where $e_1 = (1, 0, ..., 0)$. Therefore,

$$U^{-1}AU = U^{-1}\begin{pmatrix} \lambda x & Au_2 & \dots & Au_n \end{pmatrix} = \begin{pmatrix} \lambda e_1 & U^{-1}Au_2 & \dots & U^{-1}Au_n \end{pmatrix}.$$

Also we obtain

$$U^{-1}AU = \begin{pmatrix} \lambda & \times & ... & \times \\ 0 & * & ... & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & * & ... & * \end{pmatrix} = \begin{pmatrix} \lambda & C \\ 0 & A_1 \end{pmatrix} = T_1,$$

where $C \in M_{1,n-1}(\mathbb{C})$ and $A_1 \in M_{n-1}(\mathbb{C})$. From the hypothesis, there exists an invertible matrix $W$ such that

$$\begin{pmatrix} 1 & C \\ 0 & W^{-1} \end{pmatrix}\begin{pmatrix} \lambda & C \\ 0 & A_1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & W \end{pmatrix} = \begin{pmatrix} \lambda & CW \\ 0 & W^{-1}A_1W \end{pmatrix} = \begin{pmatrix} \lambda & CW \\ 0 & T' \end{pmatrix}.$$

Hence

$$A \sim T_1 \sim \begin{pmatrix} \lambda & CW \\ 0 & T' \end{pmatrix} = T,$$

where $T$ is upper triangular. That is, $A \sim T$. ■

**Exercise 164** *Trigonalize the following matrix:*

$$A = \begin{pmatrix} 2 & -1 \\ 1 & 4 \end{pmatrix}.$$

*Then, calculate* $A^n$*, for* $n \geq 0$.

1. *From simple computation, we haev*

$$p_A(x) = (x-3)^2.$$

*This means that* $\lambda = 3$ *is an eigenvalue of* $A$ *with multiplicity* 2*, and hence* $A$ *is not diagonalizable since* $A \neq 3I$.

*Next, we find the corresponding eigenvectors. In fact, we have*

$$\begin{aligned} E_\lambda &= \left\{(x,y) \in \mathbb{R}^2;\ \begin{matrix} 2x - y = 3x \\ x + 4y = 3y \end{matrix}\right\} \\ &= \left\{(x,y) \in \mathbb{R}^2;\ y = -x\right\} \\ &= Vect\{(1,-1)\} = Vect\{v_1\}. \end{aligned}$$

*Let* $v_2$ *be a nonzero vector for which* $\{v_1, v_2\}$ *is a basis of* $\mathbb{R}^2$*. For example, we put* $v_2 = (1,1)$*, and let*

$$P = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

*Therefore,*

$$P^{-1}AP = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & -2 \\ 0 & 3 \end{pmatrix} = T$$

*That is,* $A \sim T$.

*Next, we compute* $A^n$ : *We have*

$$A^n = PT^nP^{-1}.$$

*It suffices to compute* $T^n$ : *We write* $T$ *in the form*

$$T = \begin{pmatrix} 3 & -2 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}_D + \begin{pmatrix} 0 & -2 \\ 0 & 0 \end{pmatrix}_N, \text{ where } N^2 = 0.$$

*Hence*

$$\begin{aligned} T^n &= D^n + nD^{n-1}N \\ &= \begin{pmatrix} 3^n & 0 \\ 0 & 3^n \end{pmatrix} + n\begin{pmatrix} 3^{n-1} & 0 \\ 0 & 3^{n-1} \end{pmatrix}\begin{pmatrix} 0 & -2 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 3^n & -2n \times 3^{n-1} \\ 0 & 3^n \end{pmatrix};\ n \geq 0. \end{aligned}$$

*Finally, we deduce that*

$$\begin{aligned} A^n &= \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 3^n & -2n \cdot 3^{n-1} \\ 0 & 3^n \end{pmatrix}\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \\ &= \begin{pmatrix} 3^n - n \cdot 3^{n-1} & -n \cdot 3^{n-1} \\ n \cdot 3^{n-1} & n \cdot 3^{n-1} + 3^n \end{pmatrix};\ n \geq 0. \end{aligned}$$

**Theorem 165** *For any matrix $A \in \mathcal{M}_n(\mathbb{C})$, we have*

$$\det(A) = \prod_{\lambda \in Sp(A)} \lambda.$$

*Recall that $Sp(A)$ consists of all eigenvalues of $A$.*

**Proof.** We know that $A$ is trigonalizable, and so there exists an invertible matrix $P \in \mathbb{GL}_n(\mathbb{C})$ and an upper triangular matrix $T$ such that

$$A = PTP^{-1} \ (T = (t_{ij}) \text{ with } t_{ii} \in Sp(A)).$$

Therefore,

$$\begin{aligned} \det(A) &= \det\left(PTP^{-1}\right) \\ &= \det(P)\det(T)\det\left(P^{-1}\right) \\ &= \det(T) = t_{11}t_{22}...t_{nn} \\ &= \prod_{\lambda_i \in Sp(A)} \lambda_i. \end{aligned}$$

This completes the proof. ■

**Corollary 166** *Let $A \in \mathcal{M}_n(\mathbb{C})$. Then*

$$0 \notin Sp(A) \Rightarrow A \text{ is invertible.}$$

**Proof.** By Theorem 165, if we have $0 \notin Sp(A)$ then $\det(A) \neq 0$, and so $A$ is invertible. ■

# Addional notes

## 21 Nonsingular Matrices

**Definition 167** *Let $A$ be an $n \times n$ matrix. $A$ is nonsingular if the only solution to $Ax = 0$ is the zero solution $x = 0$.*

**Definition 168** *Let $A$ be an $n \times n$ matrix.*

- *If $A$ is nonsingular, then $A^t$ is nonsingular.*
- *$A$ is nonsingular if and only if the column vectors of $A$ are linearly independent.*
- *$Ax = b$ has a unique solution for every $n \times 1$ column vector $b$ if and only if $A$ is nonsingular.*

**Definition 169** *Nonsingular matrices are sometimes also called* ***regular*** *matrices. A square matrix is nonsingular iff its determinant is nonzero.*

**Exercise 1.** Determine whether the following matrices are nonsingular or not.

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 2 \\ 1 & 0 & -1 \end{pmatrix}, B = \begin{pmatrix} 2 & 1 & 2 \\ 1 & 0 & 1 \\ 4 & 1 & 4 \end{pmatrix}.$$

**Exercise 2.** Consider the matrix

$$M = \begin{pmatrix} 1 & 4 \\ 3 & 12 \end{pmatrix}$$

1. Show that $M$ is singular.
2. Find a non-zero vector $v$ such that $Mv = 0$, where 0 is the 2-dimensional zero vector.

**Exercise 3.** Let $A$ be the following $3 \times 3$ matrix

$$A = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 2 \\ 1 & 1 & a \end{pmatrix}.$$

Determine the values of $a$ so that the matrix $A$ is nonsingular.

# 22 Inverse Matrices

**Definition 170** *An $n \times n$ matrix $A$ is said to be **invertible** if there exists an $n \times n$ matrix $B$ such that $AB = BA = I$, where $I$ is the $n \times n$ identity matrix. Such a matrix $B$ is unique and called the inverse matrix of $A$, denoted by $A^{-1}$.*

- $A$ is invertible if and only if $A$ is nonsingular.
- Not all matrices have inverses. This is the first question we ask about a square matrix.
- If $A$ and $B$ are invertible then so is $AB$. The inverse of a product $AB$ is
$$(AB)^{-1} = B^{-1}A^{-1}.$$
- If $A$ is invertible, then $A^t$ is invertible and $(A^t)^{-1} = (A^{-1})^t$.

**Exercise 1.** Let $A$ be the matrix

$$A = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Is the matrix A invertible? If not, then explain why it isn't invertible. If so, then find the inverse.

**Exercise 2.** Find the inverse matrix of

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

if it exists. If you think there is no inverse matrix of $A$, then give a reason.

## 23 Introduction to Eigenvalues and Eigenvectors

- Let $A$ be an $n \times n$ matrix. A scalar $\lambda$ is called an eigenvalue of $A$ if the equation $Ax = \lambda x$ has a nonzero solution $x$. Such a nonzero solution $x$ is called an eigenvector corresponding to the eigenvalue $\lambda$.
- The characteristic polynomial of $A$ is the polynomial of degree $n$ given by $p(t) = det(A - tI)$.
- If $p(t) = (t - \lambda_1)^{n_1} \cdots (t - \lambda_k)^{n_k}$ is a factorization of the characteristic polynomial of $A$, where $\lambda_1, \ldots, \lambda_k$ are distinct eigenvalues of $A$, then the algebraic multiplicity of the eigenvalue $\lambda_i$ is $n_i$.

Let $A$ be an $n \times n$ matrix and let $p(t)$ be the characteristic polynomial of $A$.

- The degree of $p(t)$ is $n$.
- $\lambda$ is an eigenvalue of $A$ if and only if $p(\lambda) = det(A - \lambda I) = 0$.
- $A$ has at least one eigenvalue and has at most $n$ distinct eigenvalues.
- $A$ has at most $n$ distinct eigenvalues.
- The eigenvalues of a matrix $A$ are roots of the characteristic polynomial of $A$.
- The eigenvalues of a triangular matrix are diagonal entries.

**Exercise 1.**
(a) True or False. If each entry of an $n \times n$ matrix $A$ is a real number, then the eigenvalues of $A$ are all real numbers.
(b) Find the eigenvalues of the matrix

$$A = \begin{pmatrix} -2 & -1 \\ 5 & 2 \end{pmatrix}.$$

**Exercise 2.** Find all the eigenvalues and eigenvectors of the matrix

$$A = \begin{pmatrix} 3 & -2 \\ 6 & -4 \end{pmatrix}.$$

Show that the eigenvalues of the matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

are 0 and 2.
**Exercise 4.** Let

$$A = \begin{pmatrix} a & -1 \\ 1 & 4 \end{pmatrix}$$

be a $2 \times 2$ matrix, where $a$ is some real number. Suppose that the matrix $A$ has an eigenvalue 3.

1. Determine the value of $a$.

2. Does the matrix $A$ have eigenvalues other than 3?

**Exercise 5.** Determine all eigenvalues and their algebraic multiplicities of the matrix

$$A = \begin{pmatrix} 1 & a & 1 \\ a & 1 & a \\ 1 & a & 1 \end{pmatrix},$$

where $a$ is a real number.

**Exercise 6.** Suppose that $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector of a matrix $A$ corresponding to the eigenvalue 3 and that $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is an eigenvector of $A$ corresponding to the eigenvalue $-2$. Compute $A^2 \begin{bmatrix} 4 \\ 3 \end{bmatrix}$.

**Exercise 7.** Suppose that $A$ is an $n \times n$ matrix with eigenvalue $\lambda$ and corresponding eigenvector $v$.

1. If $A$ is invertible, is $v$ an eigenvector of $A^{-1}$? If so, what is the corresponding eigenvalue? If not, explain why not.

2. Is $3v$ an eigenvector of $A$? If so, what is the corresponding eigenvalue? If not, explain why not.

**Exercise 8.** Let $A$ be a $2 \times 2$ real symmetric matrix. Prove that all the eigenvalues of $A$ are real numbers by considering the characteristic polynomial of $A$.

**Exercise 9.** Let

$$A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

be a $2 \times 2$ matrix, where $a, b$ are real numbers. Suppose that $b \neq 0$. Prove that the matrix $A$ does not have real eigenvalues.

**Exercise 10.** Find all eigenvalues and corresponding eigenvectors for the matrix $A$ if

$$A = \begin{pmatrix} 2 & -3 & 0 \\ 2 & -5 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

**Exercise 11.** Let $A$ be an $n \times n$ matrix. We say that $A$ is **idempotent** if $A^2 = A$.
(a) Find a nonzero, nonidentity idempotent matrix.
(b) Show that eigenvalues of an idempotent matrix $A$ is either 0 or 1.

**Exercise 12.** Let $A$ be an $n \times n$ matrix. Suppose that all the eigenvalues $\lambda$ of $A$ are real and satisfy $\lambda < 1$. Then show that the determinant $\det(I - A) > 0$, where $I$ is the $n \times n$ identity matrix.

**Exercise 13.** Consider the $2 \times 2$ matrix

$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},$$

where $\theta$ is a real number $0 \leq \theta < 2\pi$.
(a) Find the characteristic polynomial of the matrix $A$.
(b) Find the eigenvalues of the matrix $A$.
(c) Determine the eigenvectors corresponding to each of the eigenvalues of $A$.

**Exercise 14.** Let $A$ be an $n \times n$ matrix and let $\lambda_1, \ldots, \lambda_n$ be its eigenvalues. Show that

1. $\det(A) = \prod_{i=1}^{n} \lambda_i$,
2. $tr(A) = \sum_{i=1}^{n} \lambda_i$.

**Exercise 15.**
(a) A $2 \times 2$ matrix $A$ satisfies $tr(A^2) = 5$ and $tr(A) = 3$.
Find $\det(A)$.
(b) A $2 \times 2$ matrix has two parallel columns and $tr(A) = 5$. Find $tr(A^2)$.
(c) A $2 \times 2$ matrix $A$ has $\det(A) = 5$ and positive integer eigenvalues. What is the trace of A?

**Exercise 16.** Let $n$ be an odd integer and let $A$ be an $n \times n$ real matrix. Prove that the matrix $A$ has at least one real eigenvalue.

**Exercise 17.** Let $A$ be an $n \times n$ real matrix. Prove the followings:
(a) The matrix $AA^t$ is a symmetric matrix.
(b) The set of eigenvalues of $A$ and the set of eigenvalues of $A^t$ are equal.
(c) The matrix $AA^t$ is non-negative definite.
(An $n \times n$ matrix $B$ is called non-negative definite if for any $n$ dimensional vector $x$, we have $x^t Bx \geq 0$.)
(d) All the eigenvalues of $AA^t$ is non-negative.

**Exercise 18.** Let $A$ be an $n \times n$ matrix. Suppose that $y$ is a nonzero row vector such that $yA = y$. (Here a row vector means a $1 \times n$ matrix.) Prove that there is a nonzero column vector $x$ such that $Ax = x$. (Here a column vector means an $n \times 1$ matrix.)

**Exercise 19.**

(a) Let $A$ be a real orthogonal $n \times n$ matrix. Prove that the length (magnitude) of each eigenvalue of $A$ is 1.
(b) Let $A$ be a real orthogonal $3 \times 3$ matrix and suppose that the determinant of $A$ is 1. Then prove that $A$ has 1 as an eigenvalue.

**Exercise 20.** Let $A$ and $B$ be square matrices such that they commute each other: $AB = BA$. Assume that $A - B$ is a nilpotent matrix. Then prove that the eigenvalues of $A$ and $B$ are the same.

**Exercise 21.** Let $A$ be an $n\times n$ matrix. Suppose that $A$ has real eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ with corresponding eigenvectors $u_1, u_2, \ldots, u_n$. Furthermore, suppose that $|\lambda_1| > |\lambda_2| > \ldots > |\lambda_n|$. Let

$$x_0 = c_1u_1 + c_2u_2 + \ldots + c_nu_n$$

for some real numbers $c_1, c_2, \ldots, c_n$ and $c_1 \neq 0$. Define $x_{k+1} = Ax_k$ for $k = 0, 1, 2, ...$ and let

$$\beta_k = \frac{x_k^t x_{k+1}}{x_k^t x_k}.$$

Prove that $\lim_{k\to\infty} \beta_k \to \lambda_1$.

# 24 Eigenvectors and Eigenspaces

**Definition 171** *Let $A$ be an $n \times n$ matrix. The eigenspace corresponding to an eigenvalue $\lambda$ of $A$ is defined to be*

$$E_\lambda = \{x \in \mathbb{C}^n;\ Ax = \lambda x\}.$$

Let $A$ be an $n \times n$ matrix.

- The eigenspace $E_\lambda$ consists of all eigenvectors corresponding to $\lambda$ and the zero vector.
- $A$ is singular if and only if 0 is an eigenvalue of $A$.
- The nullity of $A$ is the geometric multiplicity of $\lambda = 0$ if $\lambda = 0$ is an eigenvalue.

**Problem 172** *Let*

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

*One of the eigenvalues of the matrix $A$ is $\lambda = 0$. Find the geometric multiplicity of the eigenvalue $\lambda = 0$.*

## 24.1 Problems about Similar Matrices

Let $A, B$ be $n \times n$ matrices.

- We say that a matrix $A$ is **similar** to a matrix $B$ if there exists a nonsingular (invertible) matrix $P$ such that
$$A = PBP^{-1}.$$
- $A$ is **diagonalizable** if there exist a diagonal matrix $D$ and nonsingular matrix $P$ such that $P^{-1}AP = D$. (Namely, if $A$ is diagonalizable if it is similar to a diagonal matrix.)

- $A$ is said to be **defective** if there is an eigenvalue $\lambda$ of $A$ such that the geometric multiplicity of $\lambda$ is less than the algebraic multiplicity of $\lambda$.
- If $A$ and $B$ are similar, then the characteristic polynomials of $A$ and $B$ are the same. Hence the eigenvalues of $A, B$ and their algebraic multiplicities are the same.
- $A$ is diagonalizable if and only if $A$ is not defective.
- $A$ is diagonalizable if and only if $\mathbb{R}^n$ has an eigenbasis of $A$ (a basis consisting of eigenvectors).
- $A$ is diagonalizable if and only if there are $n$ linearly independent eigenvectors of $A$.
- If $A$ has $n$ distinct eigenvalues, then $A$ is diagonalizable.
- If $v_1, \ldots, v_n$ are linearly independent eigenvectors of $A$ corresponding to the eigenvalues $\lambda_1, \ldots, \lambda_n$ (not necessarily distinct), then $S^{-1}AS = D$, where $S = [v_1, \ldots, v_n]$ and $D = diag(\lambda_1, \ldots, \lambda_n)$.

**Definition 173** *An $n \times n$ matrix $A$ is said to be diagonalizable if it can be written on the form*

$$A = PDP^{-1},$$

*where $D$ is a diagonal $n \times n$ matrix with the eigenvalues of $A$ as its entries and $P$ is a nonsingular $n \times n$ matrix consisting of the eigenvectors corresponding to the eigenvalues in $D$.*

The diagonalization theorem states that an $n \times n$ matrix $A$ is diagonalizable if and only if $A$ has $n$ linearly independent eigenvectors, i.e., if the matrix rank of the matrix formed by the eigenvectors is $n$. Matrix diagonalization (and most other forms of matrix decomposition) are particularly useful when studying linear transformations, discrete dynamical systems, continuous systems, and so on.

**How to Diagonalize a Matrix. Step by Step Explanation.**

Diagonalization Procedure of a square matrix $A$
Step 1: Find the characteristic polynomial
Step 2: Find the eigenvalues
Step 3: Find the eigenspaces
Step 4: Determine linearly independent eigenvectors
Step 5: Define the invertible matrix $P$ and find $P^{-1}$
Step 6: Define the diagonal matrix $D$
Step 7: Finish the diagonalization: We verify that $A = PDP^{-1}$

**Definition 174** *A square matrix $D$ is **diagonal** if the only nonzero entries in $D$ are on the diagonal of $D$.*

**Example.**

$$D = \begin{pmatrix} \mathbf{1} & 0 & 0 & 0 \\ 0 & \mathbf{-1} & 0 & 0 \\ 0 & 0 & \mathbf{4} & 0 \\ 0 & 0 & 0 & \mathbf{3} \end{pmatrix}$$

**Digonalisability (an idea)**

For a given $n \times n$ matrix $A$, we would like to write $A = PDP^{-1}$ for some invertible matrix $P$ and some diagonal matrix $D$. Why? Finding powers of diagonal matrices is easy.

**Powers of a diagonal matrix**

**Example.** Consider

$$A = \begin{pmatrix} 7 & 2 \\ -4 & 1 \end{pmatrix},$$

where $A = PDP^{-1}$ with $P = \begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix}$ and $D = \begin{pmatrix} 5 & 0 \\ 0 & 3 \end{pmatrix}$. Find an expression for $A^k$ for any positive integer $k$.

**Theorem 175** *We have the following notions:*

1. *If $A$ is similar to $B$, then $B$ is similar to $A$.*
2. *$A$ is similar to itself.*
3. *If $A$ is similar to $B$ and $B$ is similar to $C$, then $A$ is similar to $C$.*
4. *If $A$ is similar to the identity matrix $I$, then $A = I$.*
5. *If $A$ or $B$ is nonsingular, then $AB$ is similar to $BA$.*
6. *If $A$ is similar to $B$, then $A^k$ is similar to $B^k$ for any positive integer $k$.*

**Problem 176** *Let $A$,$B$, and $C$ be $n \times n$ matrices and $I$ be the $n \times n$ identity matrix. Prove the following statements.*

**Problem 177** *Show that if $A$ and $B$ are similar matrices, then they have the same eigenvalues and their algebraic multiplicities are the same.*

1. If $A$ is similar to $B$, then $B$ is similar to $A$.

**Proof.** If $A$ is similar to $B$, then there exists a nonsingular matrix $P$ such that $B = P^{-1}AP$. Let $Q = P^{-1}$. Since $P$ is nonsingular, so is $Q$. Then we have

$$Q^{-1}BQ = (P^{-1})^{-1}BP^{-1} = PBP^{-1} = P(P^{-1}AP)P^{-1} = IAI = A.$$

Hence $B$ is similar to $A$. ■

2. We show that $A$ is similar to itself.

**Proof.** Since the identity matrix $I$ is nonsingular and we have

$$A = I^{-1}AI,$$

the matrix $A$ is similar to $A$ itself. ■

3. If $A$ is similar to $B$ and $B$ is similar to $C$, then $A$ is similar to $C$.

**Proof.** If $A$ is similar to $B$, we have

$$B = P^{-1}AP,$$

for some nonsingular matrix $P$. Also, if $B$ is similar to $C$, we have

$$C = Q^{-1}BQ,$$

for some nonsingular matrix $Q$. Then we have

$$C = Q^{-1}BQ = Q^{-1}(P^{-1}AP)Q = (PQ)^{-1}A(PQ).$$

Let $R = PQ$. Since both $P$ and $Q$ are nonsingular, $R = PQ$ is also nonsingular. The above computation yields that we have

$$C = R^{-1}AR,$$

hence $A$ is similar to $C$. ■

**Theorem 178** *Part* $(1), (2), (3)$ *show that similarity is an equivalence relation.*

**Proposition 179** *If $A$ is similar to the identity matrix $I$, then $A = I$.*

**Proof.** Since $A$ is similar to $I$, there exists a nonsingular matrix $P$ such that

$$A = P^{-1}IP.$$

Since $P^{-1}IP$, we have $A = I$. ■

**Proposition 180** *If $A$ or $B$ is nonsingular, then $AB$ is similar to $BA$.*

**Proof.** Suppose first that $A$ is nonsingular. Then $A$ is invertible, hence the inverse matrix $A^{-1}$ exists. Then we have

$$A^{-1}(AB)A = A^{-1}ABA = IBA = BA,$$

hence $AB$ and $BA$ are similar. Analogously, if $B$ is nonsingular, then the inverse matrix $B^{-1}$ exists. We have

$$B^{-1}(BA)B = B^{-1}BAB = IAB = AB,$$

hence $AB$ and $BA$ are similar. ■

**Proposition 181** *If $A$ is similar to $B$, then $A^k$ is similar to $B^k$ for any positive integer $k$.*

**Proof.** If $A$ is similar to $B$, then we have

$$B = P^{-1}AP$$

for some nonsingular matrix $P$. Then we have for a positive integer $k$

$$\begin{aligned} B^k &= (P^{-1}AP)^k = \underbrace{(P^{-1}AP)(P^{-1}AP)\cdots(P^{-1}AP)}_{k-times} \\ &= P^{-1}A^kP, \end{aligned}$$

since we can cancel $P$ and $P^{-1}$ in between. Hence $A^k$ and $B^k$ are similar. ■

## 24.2 Problems

**Exercise 1.**

Is the matrix $A = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$ similar to the matrix $B = \begin{pmatrix} 3 & 0 \\ 1 & 2 \end{pmatrix}$?

Is the matrix $A = \begin{pmatrix} 0 & 1 \\ 5 & 3 \end{pmatrix}$ similar to the matrix $B = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$?

Is the matrix $A = \begin{pmatrix} -1 & 6 \\ -2 & 6 \end{pmatrix}$ similar to the matrix $B = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$?

Is the matrix $A = \begin{pmatrix} -1 & 2 \\ -2 & 6 \end{pmatrix}$ similar to the matrix $B = \begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix}$?

**Exercise 2.** If two matrices are similar, then their determinants are the same.

**Exercise 3.** Determine whether the matrix

$$A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$$

is diagonalizable. If so, find a nonsingular matrix $S$ and a diagonal matrix $D$ such that $S^{-1}AS = D$.

**Exercise 4.** Diagonalize the $2 \times 2$ matrix $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$ by finding a nonsingular matrix $S$ and a diagonal matrix $D$ such that $S^{-1}AS = D$.

**Exercise 5.** Diagonalize the matrix

$$A = \begin{pmatrix} 4 & -3 & -3 \\ 3 & -2 & -3 \\ -1 & 1 & 2 \end{pmatrix}$$

by finding a nonsingular matrix $S$ and a diagonal matrix $D$ such that $S^{-1}AS = D$.

**Exercise 6.** Suppose that $A$ and $P$ are $3 \times 3$ matrices and $P$ is invertible matrix. If

$$P^{-1}AP = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{pmatrix}$$

then find all the eigenvalues of the matrix $A^2$.

**Exercise 7.** Let $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$. Compute $A^n$ for any $n \in \mathbb{N}$.

**Exercise 8.** Let $A, B$ be matrices. Show that if $A$ is diagonalizable and if $B$ is similar to $A$, then $B$ is diagonalizable.

1. Is every diagonalizable matrix invertible?
2. Is every invertible matrix diagonalizable?

**Exercise 9.** Determine whether the matrix

$$A = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

is diagonalizable. If it is diagonalizable, then find the invertible matrix $S$ and a diagonal matrix $D$ such that $S^{-1}AS = D$.

**Exercise 10.** For which values of constants $a, b$ and $c$ is the matrix

$$A = \begin{pmatrix} 7 & a & b \\ 0 & 2 & c \\ 0 & 0 & 3 \end{pmatrix}$$

diagonalizable?

**Exercise 11.** Let

$$A = \begin{pmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 2 & 4 & 3 \\ -4 & -6 & -3 \\ 3 & 3 & 1 \end{pmatrix}.$$

For this problem, you may use the fact that both matrices have the same characteristic polynomial:

$$P_A(\lambda) = P_B(\lambda) = -(\lambda - 1)(\lambda + 2)^2.$$

1. Find all eigenvectors of $A$.
2. Find all eigenvectors of $B$.
3. Which matrix $A$ or $B$ is diagonalizable?
4. Diagonalize the matrix stated in (3), i.e., find an invertible matrix $P$ and a diagonal matrix $D$ such that $A = PDP^{-1}$ or $B = PDP^{-1}$.

**Exercise 12.** Consider the matrix $A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$, where $a$ and $b$ are real numbers and $b \neq 0$.

1. Find all eigenvalues of $A$.

2. For each eigenvalue of $A$, determine the eigenspace $E_\lambda$.

3. Diagonalize the matrix $A$ by finding a nonsingular matrix $S$ and a diagonal matrix $D$ such that $S^{-1}AS = D$.

**Exercise 13.** Determine all $2 \times 2$ matrices $A$ such that $A$ has eigenvalues 2 and $-1$ with corresponding eigenvectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$, respectively.

**Exercise 14.** Let $A$ and $B$ be $n \times n$ matrices. Suppose that $A$ and $B$ have the same eigenvalues $\lambda_1, \ldots, \lambda_n$ with the same corresponding eigenvectors $x_1, \ldots, x_n$. Prove that if the eigenvectors $x_1, \ldots, x_n$ are linearly independent, then $A = B$.

**Exercise 15.** Suppose that $A$ is a diagonalizable $n \times n$ matrix and has only 1 and $-1$ as eigenvalues. Show that $A^2 = I_n$, where $I_n$ is the $n \times n$ identity matrix.

**Exercise 16.** Let

$$A = \begin{pmatrix} \frac{1}{7} & \frac{3}{7} & \frac{3}{7} \\ \frac{3}{7} & \frac{1}{7} & \frac{3}{7} \\ \frac{3}{7} & \frac{3}{7} & \frac{1}{7} \end{pmatrix}$$

be $3 \times 3$ matrix. Find $\lim\limits_{n \to +\infty} A^n$.

**Exercise 17.** Let

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

1. Find the characteristic polynomial and all the eigenvalues (real and complex) of $A$. Is $A$ diagonalizable over the complex numbers?

2. Calculate $A^{2009}$.

**Exercise 18.** Let $A$ be an $n \times n$ matrix with real number entries. Show that if $A$ is diagonalizable by an orthogonal matrix, then $A$ is a symmetric matrix.

**Exercise 19.** Let

$$A = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}.$$

Determine whether the matrix $A$ is diagonalizable. If it is diagonalizable, then diagonalize $A$.

**Exercise 20.** Let $A$ be an $n \times n$ matrix with the characteristic polynomial

$$p(t) = t^3 (t-1)^2 (t-2)^5 (t+2)^4.$$

Assume that the matrix $A$ is diagonalizable.

1. Find the size of the matrix $A$.

2. Find the dimension of the eigenspace $E_2$ corresponding to the eigenvalue $\lambda = 2$.

3. Find the nullity of $A$.

**Exercise 21.** Let $A$ be an $n \times n$ real symmetric matrix whose eigenvalues are all non-negative real numbers. Show that there is an $n \times n$ real matrix $B$ such that $B^2 = A$.

**Exercise 22.** Find a square root of the matrix

$$A = \begin{pmatrix} 1 & 3 & -3 \\ 0 & 4 & 5 \\ 0 & 0 & 9 \end{pmatrix}.$$

How many square roots does this matrix have?

**Exercise 23.** Suppose the following information is known about a $3 \times 3$ matrix $A$.

$$A\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = 6\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, A\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = 3\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, A\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} = 3\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$$

(a) Find the eigenvalues of $A$.
(b) Find the corresponding eigenspaces.
(c) Is $A$ a diagonalizable matrix? Is $A$ an invertible matrix? Is $A$ an idempotent matrix?

**Exercise 24.** Diagonalize the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

Namely, find a nonsingular matrix $S$ and a diagonal matrix $D$ such that $S^{-1}AS = D$.

**Exercise 25.** Prove that the matrix

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

is diagonalizable.
Prove, however, that $A$ cannot be diagonalized by a real nonsingular matrix.
That is, there is no real nonsingular matrix $S$ such that $S^{-1}AS$ is a diagonal matrix.

**Exercise 26.** Let

$$A = \begin{pmatrix} 1-a & a \\ -a & 1+a \end{pmatrix}$$

be a $2 \times 2$ matrix, where $a$ is a complex number. Determine the values of $a$ such that the matrix $A$ is diagonalizable.

**Exercise 27.** Consider the $2 \times 2$ complex matrix

$$A = \begin{pmatrix} a & b-a \\ 0 & b \end{pmatrix}$$

(a) Find the eigenvalues of $A$.
(b) For each eigenvalue of $A$, determine the eigenvectors.
(c) Diagonalize the matrix $A$.
(d) Using the result of the diagonalization, compute and simplify $A^k$ for each positive integer $k$.

**Exercise 28.** Consider the complex matrix

$$A = \begin{pmatrix} \sqrt{2}\cos x & i\sin x & 0 \\ i\sin x & 0 & -i\sin x \\ 0 & -i\sin x & -\sqrt{2}\cos x \end{pmatrix},$$

where $x$ is a real number between 0 and $2\pi$. Determine for which values of $x$ the matrix $A$ is diagonalizable. When $A$ is diagonalizable, find a diagonal matrix $D$ so that $P^{-1}AP = D$ for some nonsingular matrix $P$.

**Exercise 29.** Consider the Hermitian matrix

$$A = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}.$$

(a) Find the eigenvalues of $A$.
(b) For each eigenvalue of $A$, find the eigenvectors.
(c) Diagonalize the Hermitian matrix $A$ by a unitary matrix. Namely, find a diagonal matrix $D$ and a unitary matrix $U$ such that $U^{-1}AU = D$.

**Exercise 30.** Let $A$ be an $n \times n$ complex matrix. Let $S$ be an invertible matrix.
(a) If $SAS^{-1} = \lambda A$ for some complex number $\lambda$, then prove that either $\lambda^n = 1$ or $A$ is a singular matrix.
(b) If $n$ is odd and $SAS^{-1} = -A$, then prove that 0 is an eigenvalue of $A$.
(c) Suppose that all the eigenvalues of $A$ are integers and $det(A) > 0$. If $n$ is odd and $SAS^{-1} = A^{-1}$, then prove that 1 is an eigenvalue of $A$.

**Exercise 31.** Let $A$ be a real skew-symmetric matrix, that is, $A^t = -A$. Then prove the following statements.
(a) Each eigenvalue of the real skew-symmetric matrix $A$ is either 0 or a purely imaginary number.
(b) The rank of $A$ is even.

**Exercise 32.** Let $A$ be an $n \times n$ real symmetric matrix. Prove that there exists an eigenvalue $\lambda$ of $A$ such that for any vector $v \in \mathbb{R}^n$, we have the inequality $v \cdot Av \leq \lambda \|v\|^2$.

**Exercise 33.** A real symmetric $n \times n$ matrix $A$ is called positive definite if $x^t Ax > 0$ for all nonzero vectors $x$ in $\mathbb{R}^n$.
(a) Prove that the eigenvalues of a real symmetric positive-definite matrix $A$ are all positive.
(b) Prove that if eigenvalues of a real symmetric matrix $A$ are all positive, then $A$ is positive-definite

**Exercise 34.** Suppose $A$ is a positive definite symmetric $n \times n$ matrix.
(a) Prove that $A$ is invertible.
(b) Prove that $A^{-1}$ is symmetric.
(c) Prove that $A^{-1}$ is positive-definite.

**Exercise 35.** Let

$$A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$$

(a) Find eigenvalues of the matrix $A$.
(b) Find eigenvectors for each eigenvalue of $A$.
(c) Diagonalize the matrix $A$. That is, find an invertible matrix $S$ and a diagonal matrix $D$ such that $S^{-1}AS = D$.
(d) Diagonalize the matrix $A^3 - 5A^2 + 3A + I$, where $I$ is the $2 \times 2$ identity matrix.
(e) Calculate $A^{100}$. (You do not have to compute $5^{100}$.)
(f) Calculate $(A^3 - 5A^2 + 3A + I)^{100}$. Let $w = 2^{100}$. Express the solution in terms of $w$.

**Exercise 36.** Prove that if $A$ is a diagonalizable nilpotent matrix, then $A$ is the zero matrix $O$.

**Exercise 37.** Let $A$ be a square matrix. A matrix $B$ satisfying $B^2 = A$ is call a **square root** of $A$. Find all the square roots of the matrix

$$A = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}.$$

**Exercise 38.**
Let $A$ be an $n \times n$ idempotent complex matrix. Then prove that $A$ is diagonalizable.

**Exercise 39.** Let $A$ be an $n \times n$ real skew-symmetric matrix.
(a) Prove that the matrices $I - A$ and $I + A$ are nonsingular.
(b) Prove that $B = (I - A)(I + A)^{-1}$ is an orthogonal matrix.

**Exercise 40.** Let $A$ be a real symmetric $n \times n$ matrix with 0 as a simple eigenvalue (that is, the algebraic multiplicity of the eigenvalue 0 is 1), and let us fix a vector $v \in \mathbb{R}^n$.
(a) Prove that for sufficiently small positive real $\varepsilon$, the equation $Ax + \varepsilon x = v$ has a unique solution $x = x(\varepsilon) \in \mathbb{R}^n$.
(b) Evaluate $\lim_{\varepsilon \to 0} \varepsilon x(\varepsilon)$ in terms of $v$, the eigenvectors of $A$, and the inner product $\langle ., . \rangle$ on $\mathbb{R}^n$.

**Exercise 41.** Prove that a positive definite matrix has a unique positive definite square root.

# 25 Cayley-Hamilton Theorem

**Theorem 182 (The Cayley-Hamilton Theorem)** *If $p(t)$ is the characteristic polynomial for an $n \times n$ matrix $A$, then the matrix $p(A)$ is the $n \times n$ zero matrix.*

**Example 183** *Let* $A = \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}$. *The characteristic polynomial* $p(t)$ *of* $A$ *is*

$$p(t) = \det(A - tI) = \begin{vmatrix} 1-t & 1 \\ 1 & 3-t \end{vmatrix} = t^2 - 4t + 2.$$

*Then the Cayley-Hamilton theorem says that the matrix* $p(A) = A^2 - 4A + 2I$ *is the* $2 \times 2$ *zero matrix. In fact, we can directly check this*

$$\begin{aligned} p(A) &= A^2 - 4A + 2I = \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix} - 4\begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix} + 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 4 \\ 4 & 10 \end{bmatrix} + \begin{bmatrix} -4 & -4 \\ -4 & -12 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \end{aligned}$$

**Exercise 1.** Let

$$T = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}.$$

Calculate and simplify the expression $-T^3 + 4T^2 + 5T - 2I$, where $I$ is the $3 \times 3$ identity matrix.

**Exercise 2.** Find the inverse matrix of the matrix

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 9 & 2 & 0 \\ 5 & 0 & 3 \end{pmatrix}$$

using the Cayley–Hamilton theorem.

**Exercise 3.** Find the inverse matrix of the $3 \times 3$ matrix

$$A = \begin{pmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{pmatrix}$$

using the Cayley-Hamilton theorem.

**Exercise 4.** Let

$$A = \begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}.$$

Find the eigenvalues and the eigenvectors of the matrix

$$B = A^4 - 3A^3 + 3A^2 - 2A + 8I.$$

**Exercise 5.** Let $A, B$ be complex $2 \times 2$ matrices satisfying the relation $A = AB - BA$.

Prove that $A^2 = O$, where $O$ is the $2 \times 2$ zero matrix.

**Exercise 6.** In each of the following cases, can we conclude that $A$ is invertible? If so, find an expression for $A^{-1}$ as a linear combination of positive powers of $A$. If $A$ is not invertible, explain why not.

(a) The matrix $A$ is a $3\times 3$ matrix with eigenvalues $\lambda = i, \lambda = -i$, and $\lambda = 0$.
(b) The matrix $A$ is a $3\times 3$ matrix with eigenvalues $\lambda = i, \lambda = -i$, and $\lambda = -1$.

**Exercise 7.** Suppose that $A$ is $2\times 2$ matrix that has eigenvalues $-1$ and $3$. Then for each positive integer $n$ find $a_n$ and $b_n$ such that $A^{n+1} = a_n A + b_n I$, where $I$ is the $2\times 2$ identity matrix.

**Exercise 8.** Suppose that the $2\times 2$ matrix $A$ has eigenvalues 4 and $-2$. For each integer $n \geq 1$, there are real numbers $b_n$,$c_n$ which satisfy the relation $A^n = b_n A + c_n I$, where $I$ is the identity matrix. Find $b_n$ and $c_n$ for $2 \leq n \leq 5$, and then find a recursive relationship to find $b_n$,$c_n$ for every $n \geq 1$.

**Exercise 9.** Let $n > 1$ be a positive integer. Let $V = M_{n\times n}(\mathbb{C})$ be the vector space over the complex numbers $\mathbb{C}$ consisting of all complex $n \times n$ matrices. The dimension of $V$ is $n^2$. Let $A \in V$ and consider the set

$$S_A = \{I = A^0, A, A^2, \ldots, A^{n^2-1}\}$$

of $n^2$ elements. Prove that the set $S_A$ cannot be a basis of the vector space $V$ for any $A \in V$.

**Exercise 10**. Let $A$ be a $3\times 3$ real orthogonal matrix with $\det(A) = 1$.

1. If $\frac{-1+\sqrt{3}i}{2}$ is one of the eigenvalues of $A$, then find all the eigenvalues of $A$.

2. Let $A^{100} = aA^2 + bA + cI$, where $I$ is the $3\times 3$ identity matrix.

Using the Cayley-Hamilton theorem, determine $a, b, c$.

**Exercise 11**. Let $A$ and $B$ be $2\times 2$ matrices such that $(AB)^2 = O$, where $O$ is the $2\times 2$ zero matrix. Determine whether $(BA)^2$ must be $O$ as well. If so, prove it. If not, give a counter example.

# 26 Nilpotent Matrices and Non-Singularity of Such Matrices

**Definition 184** *In linear algebra, a **nilpotent matrix** is a square matrix $N$ such that*

$$N^k = 0,$$

*for some positive integer $k$. The smallest such $k$ is sometimes called the index of $N$.*

**Example 185** *The matrix*

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

*is nilpotent with index 2, since $A^2 = 0$.*

More generally, any triangular matrix with zeros along the main diagonal is nilpotent, with index $\leq n$. For example, the matrix

$$B = \begin{pmatrix} 0 & 2 & 1 & 6 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

is nilpotent, with $B^4 = 0$. The index of $B$ is therefore 4.

Although the examples above have a large number of zero entries, a typical nilpotent matrix does not. For example,

$$C = \begin{pmatrix} 5 & -3 & 2 \\ 15 & -9 & 6 \\ 10 & -6 & 4 \end{pmatrix}, \; C^2 = 0,$$

although the matrix has no zero entries.

**Theorem 186** *For an $n \times n$ square matrix $N$ with real (or complex) entries, the following are equivalent:*

1. *$N$ is nilpotent.*
2. *The minimal polynomial for $N$ is $x^k$ for some positive integer $k \leq n$.*
3. *he characteristic polynomial for $N$ is $x^n$.*
4. *The only complex eigenvalue for $N$ is $\lambda = 0$.*
5. *tr $\left(N^k\right) = 0$ for all $k > 0$.*

This theorem has several consequences, including:

- The determinant and trace of a nilpotent matrix are always zero. Consequently, a nilpotent matrix cannot be invertible.

- The only nilpotent diagonalizable matrix is the zero matrix.

## 26.1 Problems

**Exercise 1**. Let $A$ be an $n \times n$ nilpotent matrix, that is, $A^m = O$ for some positive integer $m$, where $O$ is the $n \times n$ zero matrix. Prove that $A$ is a singular matrix and also prove that $I - A, I + A$ are both nonsingular matrices, where $I$ is the $n \times n$ identity matrix.

**Exercise 2**. Suppose that $A$ is an $n \times n$ nilpotent matrix and $B$ is an $n \times n$ invertible matrix. Is the matrix $B - A$ invertible? If so, give a proof. Otherwise, give a counterexample.

**Exercise 3**. Is the sum of a nilpotent matrix and an invertible matrix invertible?

**Exercise 4**. A square matrix $A$ is called nilpotent if there exists a positive integer $k$ such that $A^k = O$, where $O$ is the zero matrix.

1. If $A$ is a nilpotent $n \times n$ matrix and $B$ is an $n \times n$ matrix such that $AB = BA$. Show that the product $AB$ is nilpotent.

2. Let $P$ be an invertible $n \times n$ matrix and let $N$ be a nilpotent $n \times n$ matrix. Is the product $PN$ nilpotent? If so, prove it. If not, give a counterexample.

**Theorem 187** *Every singular matrix can be written as a product of nilpotent matrices.*

**Theorem 188** *If $N$ is nilpotent, then $\det(I+N) = 1$, where $I$ denotes the $n \times n$ identity matrix. Conversely, if $A$ is a matrix and $\det(I+N) = 1$ for all values of $t$, then $A$ is nilpotent. In fact, since $p(t) = \det(I+tA) - 1$ is a polynomial of degree $n$, it suffices to have this hold for $n+1$ distinct values of $t$.*

**Theorem 189** *If $N$ is nilpotent, then {\displaystyle $I+N$ is invertible, where $I$ is the $n \times n$ identity matrix. The inverse is given by*

$$(I+N)^{-1} = \sum_{k=0}^{\infty} (-N)^k = I - N + N^2 - N^3 + N^4 - N^5 + ...,$$

*where only finitely many terms of this sum are nonzero.*

**End.**