وزارة التعليم العالي والبحث العلمي
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كلية الرياضيات والإعلام الآلي وعلوم المادة

LECTURE NOTES: ANALYSIS 3

License Mathematics

Level : Second year academic

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L I C E N S E

Contents

CHAPTER 1

Sequences and Series of Functions

In this chapter, we will use the concepts developed in Chapter 1 to define and study functions which are written as infinite series. What we will find is that several functions we know (like the exponential function, sine, cosine, arctangent, etc.) can be written as an infinite series which is relatively easy to work with. Furthermore, the representation of these and other functions by a class of infinite series called "power series" has many applications.

1.1 Sequences Functions

1.1.1 Uniform Convergence of a Sequence of Functions

Definition 1.1.1. (Pointwise Convergence)
For each $n \in \mathbb{N}$, let $f_n : A \subseteq \mathbb{R} \to \mathbb{R}$ be a real valued function on A. The sequence (f_n) of functions converges pointwise on A to a function f if, for all $x \in A$, the sequence of real numbers $(f_n(x))$ converges to the real number $f(x)$. We often write

$$\lim_{n\to\infty} f_n(x) = f(x) \qquad \text{or} \qquad \lim_{n\to\infty} f_n = f.$$

Thus we have

$$\forall x \in A,\ \forall \epsilon > 0,\ \exists n_0 \in \mathbb{N} \text{ such that, } \forall n \in \mathbb{N},\ \text{if } n > n_0,\ \text{ then } |f_n(x) - f(x)| < \epsilon.$$

Remark 1.1.1. There are several notations for the sequences of functions

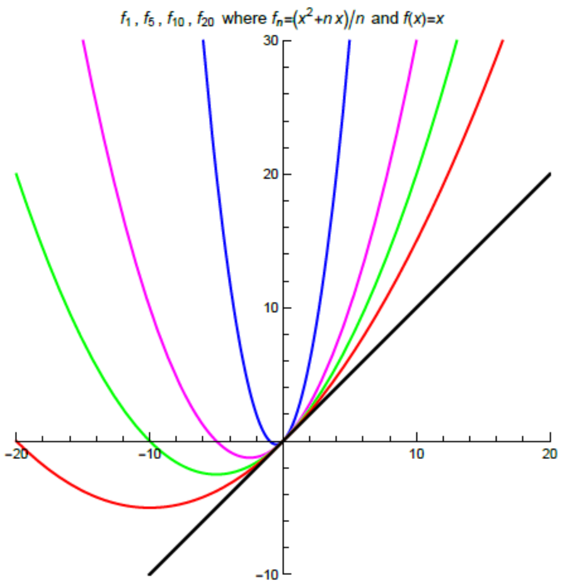
$$(f_n)_{n\in\mathbb{N}}, \qquad (f_n)_{n\geq 0}, \qquad (f_n), \quad \text{or} \quad f_0, f_1, f_2, ..., f_n, ...$$

There is a difference between (f_n) and f_n: (f_n) is the sequence and f_n is the term of rank n, or general term of this sequence.

Example 1.1.1. Let $f_n(x) = \dfrac{x^2+nx}{n} = \dfrac{x^2}{n} + x$, and

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty}\left(\frac{x^2}{n} + x\right) = 0 + x = x.$$

If $f(x) = x$, then $f_n \to f$ as $n \to \infty$. In this case, the functions f_n are everywhere continuous and differentiable, and the limit function is also everywhere continuous and differentiable.



Example 1.1.2. 1. Let $g_n(x) = x^n$ on the set $[0,1]$.

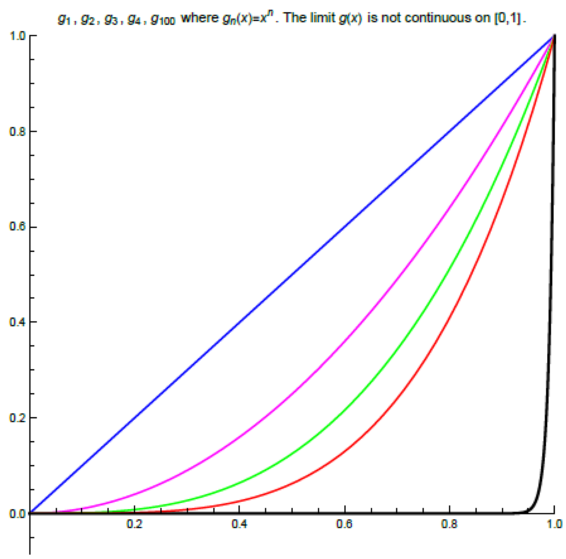
$$\lim_{n\to\infty} g_n(x) = \lim_{n\to\infty} x^n = g(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1, \\ 1 & \text{if } x = 1. \end{cases}$$

In this case, the functions $g_n(x)$ are continuous on $[0,1]$, but the limit function $g(x)$ is not continuous at $x = 1$.

2. $\forall n \in \mathbb{N},\ \forall x \in \mathbb{R}^+, \quad f_n(x) = \dfrac{x^n}{1+x^n}$

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{x^n}{1+x^n} = f(x) = \begin{cases} 0 & \text{if } 0 \le x < 1, \\ 1/2 & \text{if } x = 1, \\ 1 & \text{if } x > 1. \end{cases}$$

In this case, the functions $f_n(x)$ are continuous on $\mathbb{R}^+$, but the limit function $f(x)$ is not continuous at $x = 1$.



Example 1.1.3. 1. $\forall n \in \mathbb{N}^*,\ \forall x \in \mathbb{R}^+; \quad g_n(x) = \dfrac{\sin(nx)}{n}$: Since we have

$$\forall n \in \mathbb{N}^*,\ \forall x \in \mathbb{R}^+; \quad 0 \le |g_n(x)| \le \frac{1}{n}, \quad \text{then, } g_n(x) \to g(x) \equiv 0.$$

We have also, $\forall n \in \mathbb{N}^*,\ \forall x \in \mathbb{R}^+,\ g_n'(x) = \cos(nx)$, and this clearly shows that, in the case where $x \neq 2k\pi (k \in \mathbb{N})$ the functions $(g_n'(x))_{n\ge 1}$ has no limit when n tends to the infinity. For all $x \in \mathbb{R}^+$ and $x \neq 2k\pi (k \in \mathbb{N})$, we have

$$\frac{d}{dx}\left\{\lim_{n\to\infty} g_n\right\}(x) \neq \lim_{n\to\infty} \frac{d}{dx}\left\{(g_n)(x)\right\}.$$

2. Let $(h_n)_{n\ge 0}$ defined on $]0,1[$ such that,

$$\forall n \in \mathbb{N}, \quad \forall x \in]0,1[; \qquad h_n(x) = nx^n.$$

For all $x \in]0,1[$, we have

$$\lim_{n\to\infty} h_n(x) = \lim_{n\to\infty} nx^n = \lim_{n\to\infty} \frac{n}{e^{-n\ln x}} = \lim_{n\to\infty} \frac{-1}{e^{-n\ln x}\ln x} = 0,$$

therefore the sequence $(h_n)_n$ converges on $]0,1[$ to the function $h = 0$. But

$$\int_0^1 [\lim_{n\to\infty} h_n(x)]dx = \int_0^1 0dx = 0 \neq 1 = \lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \int_0^1 h_n(x)dx.$$

Important Question

Suppose $f(x) = \lim_{n\to\infty} f_n(x)$ for all $x \in A$. What **additional hypothesis** would ensure the following?

(i) If each f_n is continuous on A, then f is continuous on A.

(ii) if each f_n is differentiable on A, then f is differentiable on A.

The best general answer to these questions has to do with the concept of **uniform convergence.**

Definition 1.1.2. (Uniform Convergence)
Let (f_n) be a sequence of functions defined on $A \subset \mathbb{R}$. We say that (f_n) converges uniformly on A to the limit function f defined on A if for every $\epsilon > 0$ there exists an $n_0 \in \mathbb{N}$ such that

$$|f_n(x) - f(x)| < \epsilon \qquad \text{for all} \quad x \in A,$$

whenever $n \geq n_0$. Which is equivalent to

$$\forall \epsilon > 0,\ \exists n_0 \in \mathbb{N}\ (n_0 = n_0(\epsilon))/(n > n_0) \Longrightarrow (\forall x \in A, \quad |f_n(x) - f(x)| < \epsilon).$$

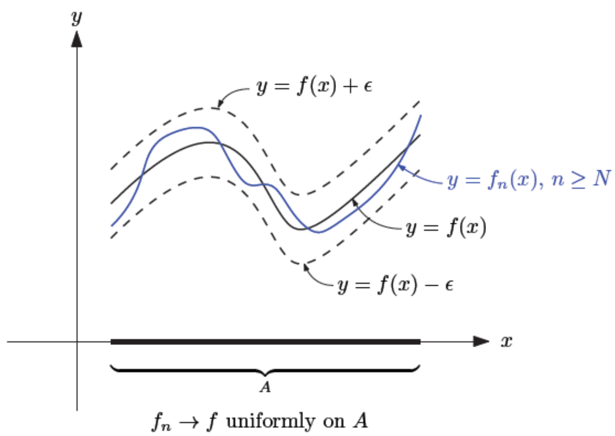
or

$$\lim_{n\to\infty} \|f_n - f\|_\infty = \lim_{n\to\infty} \sup_{x\in A} |f_n(x) - f(x)| = 0.$$

This means that for every $n \geq n_0$, the difference between $f_n(x)$ and $f(x)$ is less than ϵ for every $x \in A$.

Remark 1.1.2. In the definition, the value of n_0 is **independent of** x.

Here is a figure that graphically depicts the definition:



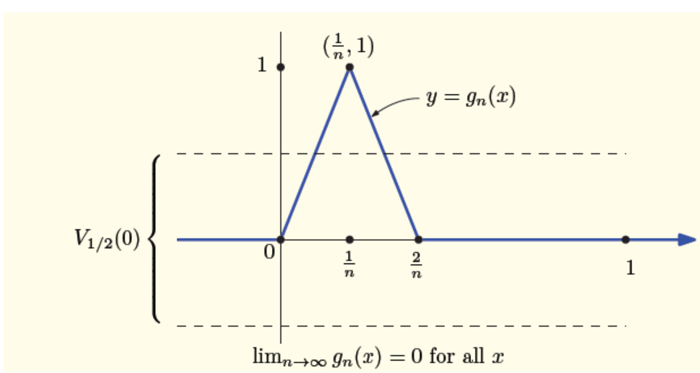
Example 1.1.4. Example of Non-Uniform Convergence

$$g_n(x) = \begin{cases} nx & \text{if } 0 \leq x \leq \dfrac{1}{n} \\ 2 - nx & \text{if } \dfrac{1}{n} \leq x \leq \dfrac{2}{n} \\ 0 & \text{otherwise.} \end{cases} \qquad \lim_{n\to\infty} g_n(x) = g(x) = 0.$$

If $g(x) := 0$, then $(g_n) \to g$ pointwise. Let $\epsilon = \dfrac{1}{2}$ and $x_n = \dfrac{1}{n}$. Then

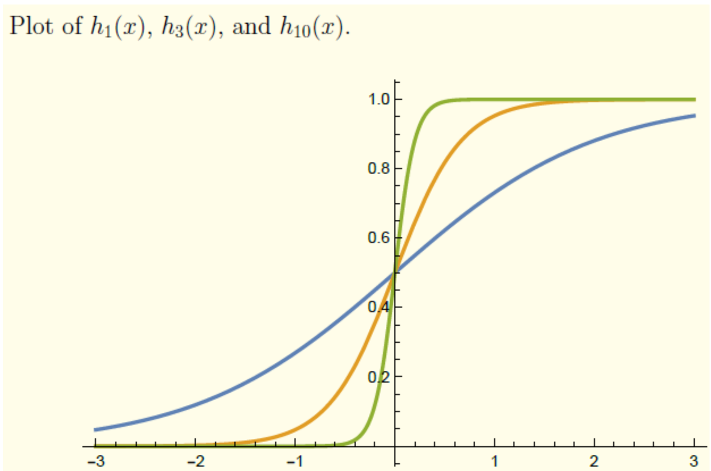
$$|g_n(x_n) - g(x_n)| = |1 - 0| = 1 > \epsilon = \frac{1}{2}.$$

So, it is not true that for all $\epsilon > 0$, there exist an $n_0 \in \mathbb{N}$ large enough such that $n \geq n_0$ implies $|g_n(x) - g(x)| < \epsilon$ for all x. So, (g_n) does not converge to g uniformly.



Example 1.1.5. Another Example of Non-Uniform Convergence For $x \in \mathbb{R}$ and $n \in \mathbb{N}$, let

$$h_n(x) = \frac{e^{nx}}{1+e^{nx}} \implies \lim_{n\to\infty} h_n(x) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{1}{2} & \text{if } x = 0 \\ 1 & \text{if } x > 0. \end{cases}$$



Example 1.1.6. Example of Uniform Convergence

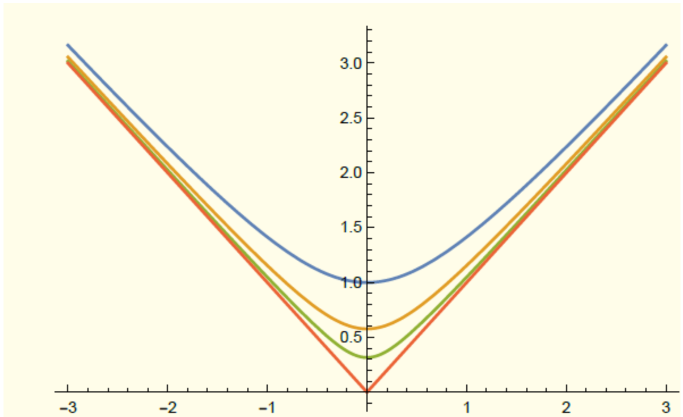
$$f_n(x) = \sqrt{x^2 + \frac{1}{n^2}} \qquad \lim_{n\to\infty} f_n(x) = \sqrt{x^2+0} = \sqrt{x^2} = |x|.$$

So, $f_n(x) \to f(x) = |x|$ pointwise. Let $\epsilon > 0$ be given. Choose $n_0 \in \mathbb{N}$ large enough such that $\frac{1}{n_0} < \epsilon$. Then for any $x \in \mathbb{R}$ and $n \geq n_0$ we have

$$\begin{aligned} |f_n(x) - f(x)| &= \left|\sqrt{x^2+\frac{1}{n^2}} - |x|\right| = \left|\sqrt{x^2+\frac{1}{n^2}} - |x|\right| \left(\frac{\sqrt{x^2+\frac{1}{n^2}}+|x|}{\sqrt{x^2+\frac{1}{n^2}}+|x|}\right) \\ &= \frac{\frac{1}{n^2}}{\sqrt{x^2+\frac{1}{n^2}}+|x|} \leq \frac{\frac{1}{n^2}}{\sqrt{0+\frac{1}{n^2}}+0} \\ &= \frac{1}{n} < \epsilon. \end{aligned}$$

This shows that $(f_n) \to f$ uniformly on $\mathbb{R}$. Note that each $f_n(x)$ is both continuous and differentiable on $\mathbb{R}$, but $f(x) = |x|$ is continuous on $\mathbb{R}$ and not differentiable at $x = 0$.

Plot of $f_1(x), f_3(x), f_{10}(x)$, and $|x|$.



Cauchy Criterion for Uniform Convergence

Theorem 1.1.1. *A sequence of functions (f_n) defined on a set $A \subseteq \mathbb{R}$ converges uniformly on A if and only if for every $\epsilon > 0$ there exists an $n_0 \in \mathbb{N}$ such that $|f_n(x) - f_m(x)| < \epsilon$ whenever $m, n \geq n_0$ and $x \in A$.*

Proof. ($\Longrightarrow$) Assume the sequence (f_n) converges uniformly on A to a limit function f. Let $\epsilon > 0$ **be given.** Then there exists an $n_0 \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \frac{\epsilon}{2}$, whenever $n \geq n_0$ and $x \in A$. Then if $n, m \geq n_0$ and $x \in A$, we have

$$\begin{aligned} |f_n(x) - f_m(x)| &= |f_n(x) - f(x) + f(x) - f_m(x)| \\ &\leq |f_n(x) - f(x)| + |f(x) - f_m(x)| = \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

($\Longleftarrow$) Conversely, assume that for every $\epsilon > 0$ there exists an $n_0 \in \mathbb{N}$ such that $|f_n(x) - f_m(x)| < \epsilon$. whenever $m, n \geq n_0$ and $x \in A$. This hypothesis implies that, for each $x \in A$, $(f_n(x))$ is a Cauchy sequence. By Cauchy's Criterion, this sequence converges to a point, which we will call $f(x)$. So, the uniformly Cauchy sequence converges pointwise to the function $f(x)$. We must show that the convergence is also uniform. For the value of ϵ given above, we use the corresponding n_0. Then for $n, m \geq n_0$ and all $x \in A$,

$$|f_n(x) - f_m(x)| < \epsilon.$$

Taking the limit as $m \to \infty$ gives

$$|f_n(x) - f(x)| \leq \epsilon \qquad \text{for all} \quad x \in A,$$

which shows that (f_n) converges uniformly to f on A. This completes the proof. □

Proposition 1.1.1. *Let $(f_n) \to f$ pointwise on $A \subseteq \mathbb{R}$. If there exists a real positive sequence (α_n) such that*

- $\lim\limits_{n\to\infty} \alpha_n = 0,$
- $|f_n(x) - f(x)| \leq \alpha_n \qquad \text{for all } x \in A.$

Then $(f_n) \to f$ uniformly on A.

Example 1.1.7. Consider the sequence

$$f_n(x) = \frac{e^{-nx}}{n} \qquad \text{for all } x \in \mathbb{R}^+ \text{ and } n \geq 1.$$

$(f_n) \to f(x) = 0$ pointwise on $\mathbb{R}^+$. for all $n \geq 1$, we have:

$$|f_n(x) - f(x)| = \frac{e^{-nx}}{n} \leq \frac{1}{n} = \alpha_n \qquad \text{for all } x \in \mathbb{R}^+$$

Since $\frac{1}{n} \to_{n\to\infty} 0$, we conclude that $(f_n) \to 0$ uniformly on $\mathbb{R}^+$.

1.1.2 Uniform Convergence and Continuity

Theorem 1.1.2. ***Continuous Limit Theorem***
Let (f_n) be a sequence of functions defined on $A \subseteq \mathbb{R}$ that converges uniformly on A to a function f. If each f_n is continuous at $c \in A$, then f is continuous at c.

Proof. Let $\epsilon > 0$ be given. Fix $c \in A$. Since $f_n \to f$ uniformly, there exists an $n_0 \in \mathbb{N}$ such that

$$|f_{n_0}(x) - f(x)| < \frac{\epsilon}{3} \qquad \text{for all} \quad x \in A.$$

Since f_{n_0} is continuous at c, there exists $\delta > 0$ such that $|f_{n_0}(x) - f_{n_0}(c)| < \frac{\epsilon}{3}$ whenever

$|x - c| < \delta$. If $|x - c| < \delta$, then

$$\begin{aligned} |f(x) - f(c)| &= |f(x) - f_{n_0}(x) + f_{n_0}(x) - f_{n_0}(c) + f_{n_0}(c) - f(c)| \\ &\leq |f(x) - f_{n_0}(x)| + |f_{n_0}(x) - f_{n_0}(c)| + |f_{n_0}(c) - f(c)| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon. \end{aligned}$$

The first and third $\frac{\epsilon}{3}$ are due to uniform convergence and the choice of n_0. The second $\frac{\epsilon}{3}$ is due to the choice of δ. This shows that f is continuous at c, as desired.

Remark 1.1.3. The converse of theorem 1.1.2 is generally false: A sequence of continuous functions can converge to a continuous function, without the convergence being uniform.

Example 1.1.8. Let for all $n \in \mathbb{N}$ $f_n(x) = \frac{1}{nx+1}$, $x \in I =]0,1[$ It's clear that $(f_n) \to f(x) = 0$ pointwise on I that all f_n are continuous on I and f is continuous also, but (f_n) does not converge uniformly to $f(x) = 0$ on I. Since

$$\sup_{x \in I} |f_n(x) - f(x)| = 1 \not\to 0 \text{ as } n \to \infty.$$

Theorem 1.1.3. (Dini's theorem)
Let (f_n) be a sequence of real functions defined on the bounded and closed interval $[a,b]$, and assume that

- Each (f_n) is continuous on $[a,b]$ for large n,
- $f_n \xrightarrow[n \to \infty]{P.C} f$ on $[a,b]$,[1]
- f is continuous on $[a,b]$,
- (f_n) is increasing(or decreasing) on $[a,b]$.

$$\forall n \geq N_1 \; \forall x \in [a,b], \; f_{n+1}(x) \geq f_n(x), \quad \text{or}, \; f_{n+1}(x) \leq f_n(x).$$

Then (f_n) converges uniformly on $[a,b]$.

Remark 1.1.4. The condition " $I = [a,b]$ is closed and bounded" is really important in Dini's theorem. It is thanks to her that we were able to write, in the proof of this theorem, that each function continues f_n is bounded on I and that it reaches its upper limit at a point x_n of I.

[1] P.C means Pointwise Convergence.

Proof. Fix $c \in [a,b]$ and let $\epsilon > 0$. We will show that there exists $\delta > 0$ such that

$$\left|\frac{f(x)-f(c)}{x-c} - g(c)\right| < \epsilon,$$

whenever $0 < |x-c| < \delta$ and $x \in [a,b]$.
For $x \neq c$, consider the following:

$$\begin{aligned}\left|\frac{f(x)-f(c)}{x-c} - g(c)\right| &\leq \underbrace{\left|\frac{f(x)-f(c)}{x-c} - \frac{f_n(x)-f_n(c)}{x-c}\right|}_{iii} \\ &+ \underbrace{\left|\frac{f_n(x)-f_n(c)}{x-c} - f_n'(c)\right|}_{ii} + \underbrace{\left|f_n'(c)-g(c)\right|}_{i} \qquad (1.1.1)\end{aligned}$$

Since $\lim\limits_{n\to\infty} f_n'(c) = g(c)$, there exists $n_1 \in \mathbb{N}$ such that

$$\left|f_n'(c) - g(c)\right| < \frac{\epsilon}{3} \qquad \text{for all } n \geq n_1. \qquad (1.1.2)$$

From Cauchy's Criterion for uniform convergence, since the sequence (f_n') converges uniformly to g, there exists an $n_2 \in \mathbb{N}$ such that

$$\left|f_n'(x) - f_m'(x)\right| < \frac{\epsilon}{3} \quad \text{whenever } m,n \geq n_2 \text{ and } x \in [a,b].$$

Set $N = \max\{n_1, n_2\}$, the function f_N is differentiable at c. So there exists $\delta > 0$ such that

$$\left|\frac{f_N(x)-f_N(c)}{x-c} - f_N'(c)\right| < \frac{\epsilon}{3} \qquad \text{whenever } 0 < |x-c| < \delta \text{ and } x \in [a,b]. \qquad (1.1.3)$$

We'll show this δ will suffice.
Suppose $0 < |x-c| < \delta$ and $m \geq N$. By the Mean Value Theorem applied to $f_m - f_N$ on the interval $[c,x]$ (if $x < c$ the argument is the same) there exists $\alpha \in (c,x)$ such that

$$f_m'(\alpha) - f_N'(\alpha) = \frac{[f_m(x)-f_N(x)] - [f_m(c)-f_N(c)]}{x-c}.$$

By our choice of N, $\left|f_m'(\alpha) - f_N'(\alpha)\right| < \dfrac{\epsilon}{3}$ and so

$$\left|\frac{[f_m(x) - f_N(x)] - [f_m(c) - f_N(c)]}{x - c}\right| < \frac{\epsilon}{3}.$$

Since $f_m \to f$ as $m \to \infty$, by the **Algebraic Order Limit Theorem**

$$\left|\frac{f(x) - f(c)}{x - c} - \frac{f_N(x) - f_N(c)}{x - c}\right| \leq \frac{\epsilon}{3}. \tag{1.1.4}$$

Combining inequalities (1.1.1), (1.1.2), (1.1.3), and (1.1.4), we obtain for $0 < |x - c| < \delta$ and $x \in [a, b]$

$$\left|\frac{f(x) - f(c)}{x - c} - g(c)\right| \leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3}.$$

This proves that $f = \lim_{n\to\infty} f_n$ is differentiable and that $f' = g = \lim_{n\to\infty} f_n'$. □

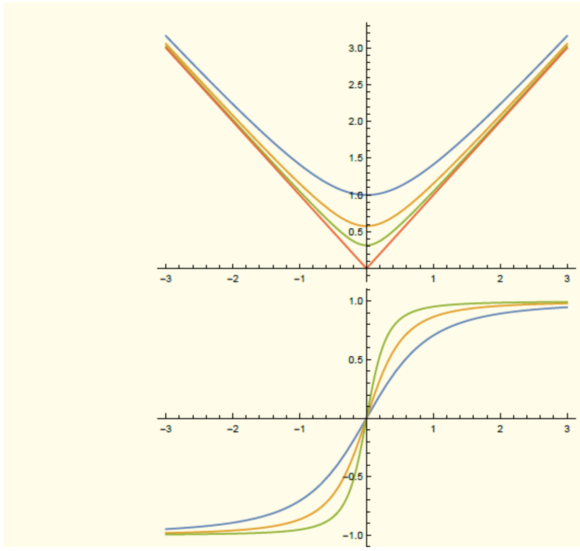
Example 1.1.10. Earlier, we studied the example

$$h_n(x) = \sqrt{x^2 + \frac{1}{n^2}}$$

We showed that $(h_n(x)) \to h(x) = |x|$ uniformly on $\mathbb{R}$. However, since the function $h(x) = |x|$ is not differentiable at $x = 0$, by the previous theorem, we know that $h_n'(x)$ does not converge uniformly to a limit function on $\mathbb{R}$. Note that

$$\lim_{n\to\infty} h_n'(x) = \lim_{n\to\infty} \frac{x}{\sqrt{x^2 + \frac{1}{n^2}}} = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0. \end{cases}$$

Plots of $h_1(x), h_3(x), h_{10}(x)$, and $|x|$. Plots of $h_1'(x), h_3'(x)$, and $h_{10}'(x)$.



Example 1.1.11. Let $g_n(x) = \frac{x}{2} + \frac{x^2}{2n}$, then

$$g(x) = \lim_{n\to\infty} g_n(x) = \lim_{n\to\infty}\left(\frac{x}{2} + \frac{x^2}{2n}\right) = \frac{x}{2} \qquad \text{(pointwise)},$$

and $g'(x) = \frac{1}{2}$. On the other hand,

$$g_n'(x) = \frac{1}{2} + \frac{x}{n} \to h(x) = \frac{1}{2} = g'(x) \quad (\text{pointwise for all} x \in \mathbb{R}).$$

We'll now examine how the previous theorem applies to this example.
Consider the interval $[-M, M]$ where $M > 0$. Let $h(x) = \frac{1}{2}$ for all $x \in \mathbb{R}$. Let $\epsilon > 0$ be given. Let $N \in \mathbb{N}$ be large enough such that $\frac{M}{N} < \epsilon$. Then if $x \in [-M, M]$ and $n \geq N$, we have

$$\left|g_n'(x) - h(x)\right| = \left|\left(\frac{1}{2} + \frac{x}{n}\right) - \frac{1}{2}\right| = \left|\frac{x}{n}\right| \leq \frac{M}{N} < \epsilon.$$

This shows that g_n' converges uniformly to h on $[-M, M]$. Because we verified that (g_n') converges uniformly on $[-M, M]$, the theorem tells us that

$$\lim_{n\to\infty} g_n'(x) = h(x) = g'(x) \quad \text{for } x \in [-M, M].$$

Since M is arbitrary we can conclude that

$$\lim_{n\to\infty} g_n'(x) = h(x) = g'(x) \quad \text{for } x \in \mathbb{R}.$$

Theorem 1.1.6. ***Theorem Related to Differentiable Limit Theorem***
Let (f_n) be a sequence of differentiable functions defined on the closed interval $[a,b]$, and assume (f_n') converges uniformly on $[a,b]$. If there exists a point $x_0 \in [a,b]$ where the sequence $(f_n(x_0))$ converges, then (f_n) converges uniformly on $[a,b]$.

Proof. Let $x \in [a,b]$ where $x \neq x_0$. Both x and x_0 will be fixed real numbers throughout the proof. Without loss of generality, we may assume $x > x_0$. (If $x < x_0$, the argument is the same.) By the Mean Value Theorem applied the function $f_n - f_m$ on the interval $[x_0, x]$, there exists some $\alpha \in (x_0, x)$ (α depends on m and n) such that

$$f_n'(\alpha) - f_m'(\alpha) = \frac{[f_n(x) - f_m(x)] - [f_n(x_0) - f_m(x_0)]}{x - x_0},$$

which implies

$$[f_n(x) - f_m(x)] - [f_n(x_0) - f_m(x_0)] = \left(f_n'(\alpha) - f_m'(\alpha)\right)(x - x_0). \tag{1.1.5}$$

Let $\epsilon > 0$ be given. Since (f_n') converges uniformly, by Cauchy's Criterion for uniformly convergent sequences of functions, there exists some $n_1 \in \mathbb{N}$ such that

$$|f_n(c) - f_m(c)| < \frac{\epsilon}{2(b-a)} \qquad \text{for all } n, m \geq n_1 \text{ and } c \in [a,b].$$

By hypothesis, the sequence $(f_n(x_0)$ converges. So there exists an $n_2 \in \mathbb{N}$ such that

$$|f_n(x_0) - f_m(x_0)| < \frac{\epsilon}{2} \qquad \text{for all } m, n \geq n_2.$$

Let $N = \max\{n_1, n_2\}$. Then if $m, n \geq N$, we have

$$\begin{aligned} |f_n(x) - f_m(x)| &\leq |(f_n(x) - f_m(x)) - (f_n(x_0) - f_m(x_0))| + |f_n(x_0) - f_m(x_0)| \\ &=^{(1.1.5)} \left(f_n'(\alpha) - f_m'(\alpha)\right)(x - x_0) + |f_n(x_0) - f_m(x_0)| \\ &< \frac{\epsilon}{2}(x - x_0) + \frac{\epsilon}{2} \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Since the choice of N was independent of x, this proves that (f_n) converges uniformly on $[a,b]$. □

Remark 1.1.5. Combining the previous two theorems gives a stronger version of the Differentiable Limit Theorem.

Theorem 1.1.7. ***Better Version of Differentiable Limit Theorem***
Let (f_n) be a sequence of differentiable functions defined on the closed interval $[a,b]$, and assume that the sequence (f'_n) converges uniformly to a function g on $[a,b]$. If there exists a point $x_0 \in [a,b]$ for which $(f_n(x_0))$ converges, then (f_n) converges uniformly. Moreover, the limit function $f = \lim f_n$ is differentiable and satisfies $f' = g$.

1.2 Series of Functions

Definition 1.2.1. 1. Let f and f_n for $n \in \mathbb{N}$ be functions defined on a set $A \subseteq \mathbb{R}$.

(a) The infinite series $\sum_{n\geq 1} f_n(x) = f_1(x) + f_2(x) + f_3(x) + \ldots\ldots$ **converges pointwise** on A to $f(x)$ if the sequence of partial sums $s_k(x) = f_1(x) + f_2(x) + \ldots.. + f_k(x)$ converges pointwise to $f(x)$ on A.

(b) The infinite series converges uniformly on A to $f(x)$ if the sequence of partial sums converges uniformly on A to $f(x)$.

2. Let

$$D = \left\{ x \in A : \quad \text{such that } \sum_{n\geq 1} f_n(x) \text{ converges} \right\}.$$

D is the domain of pointwise convergence of the series $\sum_{n\geq 1} f_n(x)$.

3. We say that the series $\sum_{n\geq 1} f_n(x)$ is absolutely convergeent on A if the series $\sum_{n\geq 1} |f_n(x)|$ is pointwise convergent for all $x \in A$.

Remark 1.2.1. Since an infinite series of functions is defined in terms of the limit of a sequence of partial sums, everything we already know about sequences applies to series. For the sum $\sum_{n\geq 1} f_n(x)$, we merely restate all of the previous theorems for the sequence of k^{th} partial sums

$$s_k(x) = f_1(x) + \ldots.. + f_k(x).$$

Example 1.2.1. • We consider the series of functions $\sum_{n\geq 1} \frac{e^{nx}}{n}$. The functions $f_n(x) =$

$\frac{e^{nx}}{n}$ are positive. We have

$$\lim_{n\to\infty}\frac{f_{n+1}(x)}{f_n(x)} = \lim_{n\to\infty}\frac{n}{n+1}e^x = e^x.$$

Therefore, according to d'Alembert's test(LCT), the series is pointwise convergent if $e^x < 1$, in other words the domain of pointwise convergence is $\mathbb{R}_-^*$.

- Let $\sum_{n\geq 1}\frac{\sin(nx)}{n^2+|x|}$ be a function series defined on $\mathbb{R}$. We have for all $n \in \mathbb{N}^*$ and $x \in \mathbb{R}$

$$\left|\frac{\sin(nx)}{n^2+|x|}\right| \leq \frac{1}{n^2}.$$

The series $\sum_{n\geq 1}\frac{1}{n^2}$ is convergent (Riemann series), therefore according to the comparison test the given series is absolutely convergent on $\mathbb{R}$.

Definition 1.2.2. We say that the series $\sum_{n\geq 1} f_n(x)$ is normally convergent on A if and only if the series $\sum_{n\geq 1}\|f_n\|_\infty$ converges such that $\|f_n\|_\infty = \sup_{x\in A}|f_n(x)|$.

Example 1.2.2.

- The normal convergence of the series of the general term $f_n(x) = \frac{\cos(nx)}{n^2\ln(n)}$ on $\mathbb{R}$ for $n \geq 2$. the function cos is bounded for all $x \in \mathbb{R}$, thus $\|f_n\|_\infty = \frac{1}{n^2\ln(n)}$. As the series $\sum_{n\geq 2}\frac{1}{n^2\ln(n)}$ is convergent (Bertrand series), we conclude that the series of functions $\sum_{n\geq 2}\frac{\cos(nx)}{n^2\ln(n)}$ converges normally on $\mathbb{R}$.

- We consider the function series defined by $\sum_n nx^2e^{-x\sqrt{n}}$ for all $x \in \mathbb{R}^+$.
To study the normal convergence we must calculate $\|f_n\|_\infty = \sup_{x\in\mathbb{R}^+}|f_n(x)|$. The function f_n is differentiable on $\mathbb{R}^+$ and for all $x \in \mathbb{R}^+$

$$f_n'(x) = nx(2-x\sqrt{n})e^{-x\sqrt{n}} = 0 \Rightarrow x = \frac{2}{\sqrt{n}},$$

then $f_n \nearrow$ for $x \in [0, \frac{2}{\sqrt{n}}]$ and $f_n \searrow$ for $x \in [\frac{2}{\sqrt{n}}, \infty]$, which implies that

$$\|f_n\|_\infty = \sup_{x\in\mathbb{R}^+}|f_n(x)| = f_n\left(\frac{2}{\sqrt{n}}\right) = 4e^{-2}.$$

The series $\sum_n 4e^{-2}$ is divergent, then the function series $\sum_n nx^2e^{-x\sqrt{n}}$ does not converge normally on $\mathbb{R}^+$.

1.2.1 Weierstrass M-test

Theorem 1.2.1. *Let $A \subset \mathbb{R}$. Suppose that there exists positive constants M_n, $n = 1, 2, ...,$ such that $|f_n(x)| \leq M_n$ for $x \in A$. If $\sum_{n\geq 1} M_n < \infty$, then $\sum_{n\geq 1} f_n(x)$ converges uniformly on A.*

Proof. Since $|f_n(x)| \leq M_n$ and $\sum_{n\geq 1} M_n$ is convergent, it is clear that[2] $f(x) = \sum_{n\geq 1} f_n(x)$ exists for every $x \in A$. Now

$$\begin{aligned}
\|f(x) - s_n\|_\infty &= \left\| \sum_{k=1}^{\infty} f_k(x) - \sum_{k=1}^{n} f_k(x) \right\|_\infty = \left\| \sum_{k=n+1}^{\infty} f_k(x) \right\|_\infty \\
&\leq \sum_{k=n+1}^{\infty} \|f_k(x)\|_\infty \\
&\leq \sum_{k=n+1}^{\infty} M_k \longrightarrow 0,
\end{aligned}$$

as $k \to \infty$. By definition, this implies that the series is uniformly convergent. □

Example 1.2.3. Solution. Let $0 < a < 1$ and $ab > 1$. Show that $f(x) = \sum_{k=1}^{\infty} a^k \sin(b^k \pi x)$ is uniformly convergent.

We see that $\left| a^k \sin(b^k \pi x) \right| \leq a^k, \qquad k = 1, 2, 3, ...,$ since $|\sin(b^k \pi x)| \leq 1$. As $\sum_{k=1}^{\infty} a^k$ is a geometric series with quotient a and $|a| < 1$, we know that this series is convergent. Thus, by Weierstrass' $M-$test, it follows that the original series is uniformly convergent

1.2.2 Abel Uniform Criterion

Theorem 1.2.2. *Let (f_n) and (g_n) two sequences of functions defined from A to $\mathbb{R}$ such that:*

1. *There exists $M > 0$ such that for all $n \in \mathbb{N}$,* $\quad \sup_{x\in A} \left| \sum_{k=0}^{n} f_k(x) \right| \leq M,$

[2] The absolute convergence implies the pointwise convergence

2. *for all* $x \in A$, (g_n) *is decreasing.*

3. $(g_n) \to 0$ *uniformly on* A.

Then, the series $\sum_n f_n(x)g_n(x)$ *converges uniformly on* A.

1.2.3 Term-by-term Continuity Theorem

Theorem 1.2.3. *Let* f_n *be continuous functions defined on a set* $A \subseteq \mathbb{R}$, *and assume* $\sum_{n\geq 1} f_n$ *converges uniformly on* A *to a function* f. *Then* f *is continuous on* A.

Proof. Apply the Continuous Limit Theorem (1.1.2) to the partial sums $s_k = f_1 + f_2 + ... + f_k$. □

Example 1.2.4. Show that $f(x) = \sum_{k=1}^{\infty} \frac{2x}{x^2+k^4}$ is continuous.

Solution. Let $f_k(x) = \frac{2x}{x^2+k^4}$. Then

$$|f_k(x)| \leq \frac{2|x|}{x^2+k^4}, \quad k = 1, 2, 3, ...$$

We need to find constants M_k such that $|f_k(x)| \leq M_k$, so we maximize $f(x) = \frac{2x}{x^2+k^4}$ on $[\,0, \infty\,[$. Note that $f(0) = 0$ and $f(x) \to 0$ as $x \to \infty$. Moreover,

$$f'(x) = \frac{2k^4 - 2x^2}{(x^2+k^4)^2} \Rightarrow f'(x) = 0 \left[\Leftrightarrow x = \pm k^2\right],$$

so

$$|f(x)| \leq \sup_x f(x) = f(k^2) = \frac{2k^2}{k^4+k^4} = \frac{1}{k^2} =: M_k.$$

Since $\sum_{k=1}^{\infty} \frac{1}{k^2} < \infty$, the $M-$test proves that $\sum_{k=1}^{\infty} f_k(x)$ is uniformly convergent. Since f_k are continuous, the uniform convergence proves that also f is continuous.

1.2.4 Term-by-term Integration Theorem

Theorem 1.2.4. *Suppose that* $f(x) = \sum_n u_n(x)$ *is uniformly convergent for* $x \in [a, b]$. *If* $f_0, f_1, f_2, ...$ *are continuous functions on* $[a, b]$, *then we can exchange the order of sum-*

mation and integration:

$$\int_a^b f(x)dx = \int_a^b \sum_n f_n(x)dx = \sum_n \int_a^b f_n(x)dx = \sum_n \int_a^b f_n(x)dx.$$

Proof. Apply the theorem 1.1.4 of the uniform convergence and integration to the partial sums $s_n(x) = \sum_{k=0}^{n} f_k(x)$. □

Example 1.2.5. Let $f(x) = \sum_{k=1}^{\infty} \frac{2x}{x^2+k^4}$ prove that $\int_0^1 f(x)dx = \sum_{k=1}^{\infty} \ln\left(1 + 1/k^4\right)$.

Solution. From the above example (1.2.4) f is uniformly convergent. Moreover, the uniform convergence implies that we can integrate the series termwise, so

$$\int_0^1 f(x)dx = \sum_{k=1}^{\infty} \int_0^1 \frac{2x}{x^2+k^4}dx = \sum_{k=1}^{\infty} \left[\ln(x^2+k^4)\right]_{x=0}^{x=1} = \sum_{k=1}^{\infty} \ln\left(1 + 1/k^4\right).$$

1.2.5 Term-by-term Differentiability Theorem

Theorem 1.2.5. *Suppose the following three statements:*

1. *Let f_n be differentiable functions defined on an interval $A = [a, b]$.*
2. *Assume $\sum_n f_n'(x)$ converges uniformly to a limit $g(x)$ on A.*
3. *$\sum_n f_n(x)$ converges pointwise on A.*

Then, the series $\sum_n f_n(x)$ converges uniformly to a differentiable function $f(x)$ satisfying $f'(x) = g(x)$ on A. In other words,

$$f(x) = \sum_n f_n(x) \qquad \text{and} \qquad f'(x) = \sum_n f_n'(x).$$

Proof. Apply the stronger version of the Differentiable Limit Theorem 1.1.7 to the partial sums $s_k = f_1 + f_2 + ... + f_k$. □

Example 1.2.6. Show that $f(x) = \sum_{k=1}^{\infty} \frac{1}{x^2+k^2}$ is continuously differentiable (that is, show that $f \in C^1$).

Solution. Let $f_k(x) = \dfrac{1}{x^2+k^2}$. Clearly $|f_k(x)| \le \dfrac{1}{k^2}, \quad k = 1, 2, 3, \ldots$, so the series defining $f(x)$ is convergent for all $x \in \mathbb{R}$ (actually uniformly convergent by the $M-$test). To show that $f(x)$ is differentiable, we prove the uniform convergence of the series

$$\sum_{k=1}^{\infty} f_k'(x) = \sum_{k=1}^{\infty} \frac{-2x}{(x^2+k^2)^2}.$$

Clearly $f_k'(x) \to 0$ as $x \to \pm\infty$ and

$$f_k''(x) = \frac{6x^2 - 2k^2}{(x^2+k^2)^3} \Rightarrow \left[f_k''(x) = 0 \Leftrightarrow x^2 = k^2/3 \right],$$

so

$$|f'k(x)| \le |\sup f_k'(x)|$$

where the $a_0, a_1, ...$ are real numbers called the coefficients of the power serie.

Example 1.3.1.

$$\sum_{n\geq 0}\frac{x^n}{n!} \text{ centered at } 0, \qquad \sum_{n\geq 1}\frac{(x-1)^n}{n} \text{ centered at } 1$$
$$\sum_{n\geq 0} nx^n \text{ centered at } 0, \quad \sum_{n\geq 1}\frac{(x+2)^n}{n^2} \text{ centered at } -2.$$

Important Question: For which $x \in \mathbb{R}$ does the series $\sum_n a_n(x-a)^n$ converge?

Theorem 1.3.1. *If a power series $\sum_n a_n x^n$ converges at some nonzero point $x_0 \in \mathbb{R}$, then it converges absolutely for any x satisfying $|x| < |x_0|$.*

Proof. If $\sum_n a_n x_0^n$ converges, then $\lim_{n\to\infty} a_n x_0^n = 0$. So there exists some $M > 0$ such that $|a_n x_0^n| < M$ for all $n \geq 0$. If x satisfies $|x| < |x_0|$, then

$$|a_n x^n| = |a_n x_0^n| \left|\frac{x}{x_0}\right|^n \leq M \left|\frac{x}{x_0}\right|^n.$$

By the Comparison Test the series $\sum_n a_n x^n$ converges absolutely as follows:

$$\left|\sum_n a_n x^n\right| \leq \sum_n |a_n x^n| \leq \sum_n M\left|\frac{x}{x_0}\right|^n = \frac{M}{1-\left|\frac{x}{x_0}\right|} < \infty.$$

Thus, $\sum_n a_n x^n$ converges absolutely for x satisfying $|x| < |x_0|$. □

Example 1.3.2. For each power series, state where the power series is centered, identify its second coefficient, its first term, its sixth term, and its ninth coefficient:

$$(a)\ f(x) = \sum_{n=0}^{\infty}\frac{(x-2)^n}{n^2+1}, \ (b)\ f(x) = \sum_{n=0}^{\infty}\frac{x^{2n}}{3n+1}.$$

Definition 1.3.2. (Radius of Convergence) Let

$$R = \sup\left\{|x_0| : \quad \sum_n a_n x_0^n \text{ converges}\right\}.$$

Then R is called the radius of convergence of the series $\sum_n a_n x^n$.

Remark 1.3.1. From the previous theorem and the definition of the radius of convergence, it is clear that if $0 < R < \infty$, then the series converges for $|x| < R$ and diverges for $|x| > R$.

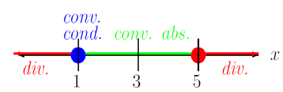
Example 1.3.3. For what x does the given series converge?

$$(a)\ \sum_{n=1}^{\infty} \frac{(x-3)^n}{n2^n},\quad (b)\ \sum_{n=0}^{\infty} n!(x+4)^n.$$

Solution: (a) We can apply the **Ratio Test** to the terms of this series. First, we have

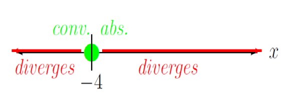
$$\rho = \lim_{n\to\infty} \frac{|f_{n+1}(x)|}{|f_n(x)|} = \lim_{n\to\infty} \frac{\left|\frac{(x-3)^{n+1}}{(n+1)2^{n+1}}\right|}{\left|\frac{(x-3)^n}{n2^n}\right|} = \lim_{n\to\infty} (1/2)|x-3|\frac{n}{n+1} = (1/2)|x-3|.$$

By the Ratio Test, this series converges absolutely when $\rho < 1$. Since $\rho = (1/2)|x-3|$, this corresponds to the inequality $(1/2)|x-3| < 1 \Leftrightarrow 1 < x < 5$. So the power series converges absolutely when $1 < x < 5$. Similarly, the series diverges when $x < 1$ or $x > 5$.
We analyze the cases $x = 1$ and $x = 5$ individually: If $x = 1 \Rightarrow f(1) = \sum_n \frac{(-1)^n}{n}$. This series converges conditionally (it is alternating harmonic). If $x = 5 \Rightarrow f(5) = \sum_n \frac{1}{n}$. This series diverges since it is harmonic. **In conclusion**, we have determined that the series converges absolutely when $x \in (1,5)$, the series converges conditionally when $x = 1$, and the series diverges for all other x. We summarize this with the following picture,



(b) we have

$$\rho = \lim_{n\to\infty} \frac{\left|\frac{(x-3)^{n+1}}{(n+1)2^{n+1}}\right|}{\left|\frac{(x-3)^n}{n2^n}\right|} = \lim_{n\to\infty} (n+1)|x+4| = \begin{cases} 0 & \text{if } x = -4 \\ 1 & \text{else} \end{cases}$$



Corollary 1.3.1. ***(Cauchy-Hadamard (C-H) Theorem).***
If the series $\sum_n a_n x^n$ *has radius of convergence* R*, then the set of all* x *for which the series converges is one of the following intervals:*

- *If* $R = 0$*, the series converges only for* $x = 0$*.*
- *If* $0 < R < \infty$*, the series converges for all* x *in one of the following four intervals:*

$$(-R, R), \ [-R, R), \ (-R, R], \ [-R, R].$$

- *If* $R = \infty$*, then the series converges for all* $x \in \mathbb{R}$*.*

Proof. This is an immediate consequence of the previous theorem and the definition of the radius of convergence. If the corollary is not obvious, go back and review the previous theorem and definition. □

Theorem 1.3.2. ***(Abel's Formula).***
Let $\sum_{n\geq 0} a_n(x-a)^n$ *be a power series centered at* a*. Then, the radius of convergence of this power series is given by*

$$R = \lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|},$$

(assuming this limit exists).

Proof. **(Proof of the Cauchy-Hadamard Theorem and Abel's Formula):** Given power series $\sum_{n=0}^{\infty} a_n(x-a)^n$, let $R = \lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|}$. We'll begin by proving statement (1) of the C-H Theorem, so we assume for now that $0 < R < 1$. Now try to determine the convergence of the power series using the Ratio Test; first compute:

Theorem 1.3.3. *If a power series* $\sum_{n\geq 0} a_n x^n$ *converges absolutely at a point* x_0*, then it converges uniformly on the closed interval* $[-|x_0|, |x_0|]$*.*

Proof. For each $n \geq 0$, let $M_n = |a_n x_0^n|$. By hypothesis the series $\sum_{n\geq 0} a_n x_0^n$ converges absolutely and so $\sum_{n\geq 0} |a_n x_0^n| = \sum_{n\geq 0} M_n$ converges. Then for any $x \in [-|x_0|, |x_0|]$, we have

$$\left|\sum_{n\geq 0} a_n x^n\right| \leq \sum_{n\geq 0} |a_n x^n| \leq \sum_{n\geq 0} |a_n x_0^n| = \sum_{n\geq 0} M_n < \infty.$$

By the Weierstrass M-test, the series converges uniformly on the closed interval $[-|x_0|, |x_0|]$. □

Theorem 1.3.4. ***Abel's Theorem about Uniform Convergence***
Let $g(x) = \sum_{n\geq 0} a_n x^n$ be a power series that converges at the point $x = R > 0$. Then the series converges uniformly on the interval $[0, R]$. A similar result holds if the series converges at $x = -R$.

We have seen that the sum $f(x) = \sum_{n\geq 0} a_n(x-a)^n$ of a power series is continuous in the interior $(a-R, a+R)$ of its interval of convergence. But what happens if the series converges at an endpoint $a \pm R$?

Before we turn to the proof, we need a lemma that can be thought of as a discrete version of integration by parts.

Lemma 1.3.1. ***(Abel's Summation Formula)*** *Let $(a_n)_n$ and $(b_n)_n$ be two sequences of real numbers, and let $s_n = \sum_{k=0}^{n} a_k$. Then*

$$\sum_{n=0}^{N} a_n b_n = s_N b_N + \sum_{n=0}^{N-1} s_n(b_n - b_{n+1}).$$

If the series $\sum_n a_n$ converges, and $b_n \to 0$ as $n \to \infty$, then

$$\sum_{n=0}^{\infty} a_n b_n = \sum_{n=0}^{\infty} s_n(b_n - b_{n+1}).$$

Proof. Note that $a_n = s_n - s_{n-1}$ for $n \geq 1$, and that this formula even holds for $n = 0$ if we define $s_{-1} = 0$. Hence

$$\sum_{n=0}^{N} a_n b_n = \sum_{n=0}^{N} (s_n - s_{n-1}) b_n = \sum_{n=0}^{N} s_n b_n - \sum_{n=0}^{N} s_{n-1} b_n,$$

Changing the index of summation and using that $s_{-1} = 0$, we see that $\sum_{n=0}^{N} s_{n-1} b_n =$

$\sum_{n=0}^{N-1} s_n b_{n+1}$ Putting this into the formula above, we get

$$\sum_{n=0}^{N} a_n b_n = \sum_{n=0}^{N} s_n b_n - \sum_{n=0}^{N-1} s_n b_{n+1} = s_N b_N + \sum_{n=0}^{N-1} s_n (b - n - b_{n+1})$$

and the first part of the lemma is proved. The second follows by letting $N \to \infty$. □We are now ready to prove:

Theorem 1.3.5. *The sum of a power series $f(x) = \sum_n a_n (x-a)^n$ is continuous in its entire interval of convergence. This means in particular that if R is the radius of convergence, and the power series converges at the right endpoint $a+R$, then $\lim_{x \uparrow a+R} f(x) = f(a+R)$, and if the power series converges at the left endpoint $a-R$, then $\lim_{x \downarrow a-R} f(x) = f(a-R)$.*

Example 1.3.4. Summing a geometric series, we clearly have

$$\frac{1}{1+x^2} = \frac{1}{1-\underbrace{(-x^2)}_{=u}} = \frac{1}{1-u} = \sum_{n=0}^{\infty} u^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}, \quad \text{for } |u| = |-x^2| < 1 \Leftrightarrow |x| < 1.$$

Integrating, we get

$$\int_0^x \frac{1}{1+t^2} dt = \arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \quad \text{for } |x| < 1$$

we see that the series converges even for $x = 1$. By Abel's Theorem

$$\pi/4 = \arctan 1 = \lim_{x \to 1} \arctan x = \lim_{x \to 1} \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$$

Hence we have proved

$$\pi/4 = 1 - 1/3 + 1/5 - 1/7 + ...$$

This is often called Leibniz' or Gregory's formula for π.

Uniqueness of power series

Recall that we asked in the last section if a function could be represented by two different power series centered at a. To address this question, suppose that function f can be represented by some power series on an open interval containing a, i.e. that

$$f(x)=\sum_{n=0}^{\infty}a_n(x-a)^n=a_0+a_1(x-a)+a_2(x-2)^2+a_3(x-a)^3+a_4(x-a)^4+....$$

on $(a-R,a+R)$ where $R>0$ is the radius of convergence of the series. Then, $f'(x)$ can be expressed as a power series centered at a with the same radius of convergence, so f is infinitely differentiable on $(a-R,a+R)$ (i.e. it is a function which can be repeatedly differentiated without anything becoming undefined). Furthermore

$$\begin{aligned}
f'(x) &= a_1+2a_2(x-a)+3a_3(x-a)^2+4a_4(x-a)^3+.....\\
f''(x) &= 2a_2+3.2a_3(x-a)+4.3a_4(x-a)^2+....\\
f^{(3)}(x) &= f'''(x)=3.2a_3+4.3.2a_4(x-a)+....\\
\vdots &= \vdots\\
f^{(n)}(x) &= n!a_n+(n+1)!a_{n+1}(x-a)+....
\end{aligned}$$

Now, plug in a for x in each of the following formulas above. We obtain

$$\begin{aligned}
f'(a) &= a_1+2a_2(a-a)+3a_3(a-a)^2+4a_4(a-a)^3+.....=a_1\\
f''(a) &= 2a_2+3.2a_3(a-a)+4.3a_4(a-a)^2+....=2a_2\\
f^{(3)}(a) &= f'''(a)=3.2a_3+4.3.2a_4(a-a)+....=3.2a_3\\
\vdots &= \vdots\\
f^{(n)}(a) &= n!a_n+(n+1)!a_{n+1}(a-a)+....=n!a_n
\end{aligned}$$

The key formula that has been derived is in the last line above:

$$f^{(n)}(a)=n!a_n\Leftrightarrow a_n=\frac{f^{(n)}(a)}{n!}.$$

We have proven the following theorem:

Theorem 1.3.6. ***(Formula for coefficients of a power series).*** *Suppose* $f(x)=\sum_{n=0}^{\infty}a_n(x-a)^n$ *where this series converges on an open interval containing* a *(equivalently, the series has positive radius of convergence). Then, for every* n *the coefficients* a_n *of the power* *series must satisfy*

$$a_n = \frac{f^{(n)}(a)}{n!}.$$

Theorem 1.3.7. ***(Uniqueness of coefficients).*** *Suppose* $\sum_{n=0}^{\infty} a_n(x-a)^n = \sum_{n=0}^{\infty} b_n(x-a)^n$ *on an open interval containing* $x = a$*. Then* $a_n = b_n$ *for all* n*.*

Proof. By Theorem 1.3.6, both a_n and b_n must be equal to $\frac{f^{(n)}(a)}{n!}$ for all n, thus they are equal to one another. □

Example 1.3.5. Suppose $f(x) = \sum_{n=0}^{\infty} \frac{3}{(n+1)^2} x^n$. Find $f^{(9)}(0)$.

By Theorem 1.3.6 with $n = 9$, we know that $a_9 = \frac{f^{(9)}(0)}{9!}$. Now a_9 can be found by the formula for f that is given; it is the coefficient on the x^9 term which is $\frac{3}{(9+1)^2} = 3/100$. Thus we have $f^{(9)}(0) = \frac{3.9!}{100}$.

The next question we ask is the converse: if you start with a function f which is infinitely differentiable on $(a-R, a+R)$, is it the case that f is representable by a power series? **This leads to the discussion in the next section.**

1.3.2 Taylor Series

Definition 1.3.3. Given a function f which is infinitely differentiable on some open interval containing a, the Taylor series of f centered at a is the power series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + ...$$

If $a = 0$, then the series in this definition, namely

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + ...$$

is called the Taylor series of f or the Maclaurin series of f.

Remark 1.3.2. It is easy to confuse the terms "power series" and "Taylor series". A power series is any expression of the form $\sum_{n=0}^{\infty} a_n(x-a)^n$. A Taylor series is a particular power series associated to some function f which is specified in advance.

Main questions related to Taylor series:

1. For what x does the Taylor series of a function f centered at a converge?
2. What function does the Taylor series of f converge to?

At this point, we know enough to answer the first question. The Taylor series of a function f centered at a is an example of a power series centered at a. Therefore, by the Cauchy-Hadamard Theorem, the Taylor series converges (absolutely) to $f(a)$ when $x = a$. This is because

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \bigg|_{x=a} = f(a) + 0 + 0 + 0 + ... = f(a).$$

We also know that there is some interval $(a-R, a+R)$ centered at a on which the Taylor series of f converges to something. Ideally, the Taylor series of f should converge to f itself (since it is the only possible power series representation of f). But we don't know at this point whether or not this happens, or under what circumstances this happens.

Example 1.3.6. Prototype Example 1: $f(x) = e^x$, and $a = 0$.

Here, we see that $f^{(n)}(x) = e^x$ for all n. Therefore $f^{(n)}(a) = f^{(n)}(0) = 1$ for all n and therefore the Taylor series of f is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = \sum_{n=0}^{\infty} \frac{1}{n!}x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

To find where this series converges, we use Abel's Formula:

$$R = \lim_{n\to\infty} |a_n|/|a_{n+1}| = \lim_{n\to\infty}(n+1) = \infty$$

Since $R = \infty$, this series converges for all x by the Cauchy-Hadamard Theorem.

Example 1.3.7. Prototype Example 2: $g(x) = \sin x$, and $a = 0$.

So the Taylor series of $\sin x$ is

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!}x^n &= g(0) + g'(0)x + \frac{g''(0)}{2}x^2 + \frac{g'''(0)}{3!}x^3 + ... \\
&= 0 + x + 0x^2 - \frac{1}{3!}x^3 + 0x^4 + \frac{1}{5!}x^5 + 0x^6 - \frac{1}{7!}x^7 + ... \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1}
\end{aligned}$$

By an argument similar to the previous example (Abel's Formula gives $R = \infty$), this series converges absolutely for all x.

Remark 1.3.3. To study the convergence of Taylor series for arbitrary functions, we return to the basics of infinite series. Recall from Chapter 1 that a series converges if the limit of its partial sums exists and is finite. Therefore, to understand the convergence of Taylor series, it makes sense to talk about the partial sums of a Taylor series. These partial sums are called **Taylor polynomials**

Definition 1.3.4. Let $n \geq 0$. Given a function f which can be differentiated n times on an open interval containing a, we can define the Taylor polynomial of order n centered at a, also called the nth Taylor polynomial centered at a to be

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + ... + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Properties of Taylor polynomials

Given any function f, where P_n denotes the nth Taylor polynomial centered at a, the following hold:

1. $P_n(x)$ is a polynomial of degree $\leq n$,
2. If $f^{(n)}(a) \neq 0$, then $P_n(x)$ is a polynomial whose degree is exactly n,
3. $P_0(x)$ is the constant function $f(a)$,
4. $P_1(x) = f(a) + f'(a)(x-a)$ is the tangent line to f when $x = a$,
5. $P_n(x)$ is the nth partial sum of the Taylor series of f centered at a, therefore

$$\lim_{n\to\infty} P_n(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n,$$

   if the limit exists.

Example 1.3.8. $f(x) = e^x,\ a = 0$, recall that the Taylor series of f is

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + x^2/2 + x^3/(3!) + x^4/(4!) + ...$$

Given this, we see that

$$P_0(x) = 1, \quad P_1(x) = 1 + x, \quad P_2(x) = 1 + x + x^2/2, \quad P_3(x) = 1 + x + x^2/2 + x^3/(3!),$$

$$....P_n(x) = 1 + x + x^2/2 + x^3/(3!) + ... + x^n/(n!).$$

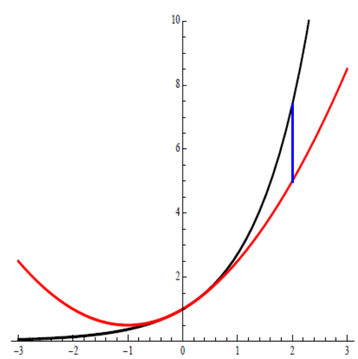
Next, we turn to the problem of determining whether the Taylor series of a function f converges to f, or to something else. To do this, we introduce a new function, called the nth remainder, which measures the difference between the original function f and its nth Taylor polynomial

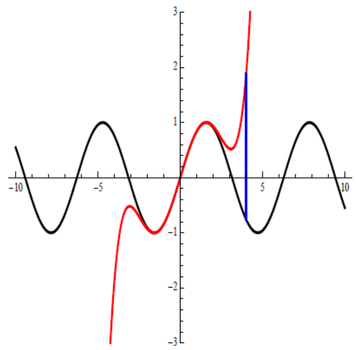
Definition 1.3.5. Let f be infinitely differentiable on $(a - R, a + R)$ and let P_n be the nth Taylor polynomial of f, centered at $x = a$. Define the nth remainder (of f centered at a) to be the function

$$R_n(x) = f(x) - P_n(x).$$

Example 1.3.9.

1. $f(x) = e^x$, $a = 0$, recall that $P_2(x) = 1 + x + x^2/2$. In the picture below, f is graphed in black, P_2 is graphed in red, and $R_2(2)$ is the length of the blue line segment
2. $g(x) = \sin x$, $a = 0$, recall that $P_5(x) = x - x^3/(3!) + x^5/(5!)$. In the picture below, f is graphed in black, P_5 is graphed in red, and $R_5(4)$ is the length of the blue line segment





Theorem 1.3.8. ***(Remainder Theorem).*** *Let f be infinitely differentiable on $(a-R, a+R)$ and let P_n and R_n be the nth Taylor polynomial and nth remainder of f, centered at $x=a$. Then if $\lim_{n\to\infty} R_n(x) = 0$, we have*

$$f(x) = \sum_{n=0}^{n} \frac{f^{(n)}(a)}{n!}(x-a)^n,$$

i.e. f is equal to its Taylor series on $(a-R, a+R)$.

Proof. Recall that $P_n(x)$ is the nth partial sum of the Taylor series of f. Therefore, since any infinite series is defined to be the limit of its partial sums, we have

$$\begin{aligned} \sum_{n=0}^{n} \frac{f^{(n)}(a)}{n!}(x-a)^n &= \lim_{n\to\infty} P_n(x) = \lim_{n\to\infty} (f(x) - R_n(x)) \quad \text{(by definition of } R_n) \\ &= f(x) - \lim_{n\to\infty} R_n(x) \\ &= f(x). \qquad \text{(by hypothesis)} \end{aligned}$$

□The Remainder Theorem sufficiently (for our purposes) answers (at least theoretically) the second main question related to Taylor series, because it gives a condition under which the Taylor series of f converges to f itself. In particular, the Remainder Theorem tells us that to show an infinitely differentiable function is equal to its Taylor series, we need only to show that $\lim_{n\to\infty} R_n(x) = 0$. However, the definition of $R_n(x)$ alone is insufficient to evaluate this limit. We need an alternate representation of the nth remainders which will allow us to show that $\lim_{n\to\infty} R_n(x) = 0$. To get this alternate representation, we first recall a theorem from Calculus I

Theorem 1.3.9. ***(Mean Value Theorem (MVT)).*** *Let f be differentiable on the interval*

$[a, x]$. *Then, there exists a* $z \in (a, x)$ *such that*

$$\frac{f(x) - f(a)}{x - a} = f'(z)$$

Theorem 1.3.10. ***(Taylor's Theorem).*** *Suppose* f *can be differentiated* $n + 1$ *times in an open interval* $(a - R, a + R)$ *(where* $R > 0$*). Then, for all* $x \in (a - R, a + R)$ *and all* $n \geq 0$*, there exists a* z *between* a *and* x *such that*

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x - a)^{n+1}.$$

Proof. **First, a remark:** this proof will use the Mean Value Theorem. **The proof of the Mean Value Theorem is deep; take an advanced calculus course if you want to see that.** Now, let's prove the theorem. Fix $x \in (a - R, a + R)$ and recall that $R_n(x) = f(x) - P_n(x)$. Define a new function g, whose input variable will be called t, by setting $g(t)$ equal to

$$f(x) - \left[f(t) + f'(t)(x - t) + \frac{f''(t)}{2!}(x - t)^2 + ... + \frac{f^{(n)}(t)}{n!}(x - t)^n \right] - R_n(x)\frac{(x - t)^{n+1}}{(x - a)^{n+1}}.$$

Observe that $g(a) = 0$ and $g(x) = 0$. Now apply the Mean Value Theorem to g to find a point z between a and x such that

$$g'(z) = \frac{g(x) - g(a)}{x - a} = \frac{0 - 0}{x - a} = 0,$$

Last, evaluate the derivative of g. We have

$$\begin{aligned} g'(z) &= \frac{d}{dt}g(t)\,|_{t=z} \\ &= \Big(0 - \Big[f'(t) + (f''(t)(x - t) - f'(t)) + \Big(f'''(t)/(2!)(x - t)^2 - f''(t)(x - t)\Big) + \\ &\vdots \\ &+ \left(\frac{f^{(n+1)}(t)}{n!}(x - t)^n - \frac{f^{(n)}(t)}{(n-1)!}(x - t)^{n+1}\right)\Big] + \frac{R_n(x)}{(x - a)^{n+1}}(n + 1)(x - t)^n\Big)\Big|_{t=z} \end{aligned}$$

notice that the terms inside the brackets cancel out to leave

$$g'(z) = -\frac{f^{(n+1)}(t)}{n!}(x - t)^n + \frac{R_n(x)}{(x - a)^{n+1}}(n + 1)(x - t)^n.$$

Since $g'(z) = 0$, we have

$$0 = -\frac{f^{(n+1)}(t)}{n!}(x-t)^n + \frac{R_n(x)}{(x-a)^{n+1}}(n+1)(x-t)^n \Leftrightarrow R_n(x) = \frac{f^{(n+1)}(t)}{(n+1)!}(x-a)^{n+1}.$$

$\square$