Math 230 Calculus II
Practice problems for Exam III

Exam III will be based on Sections 7.5, 7.7, 7.8, 11.1, 11.2, 11.3, 11.4, 11.5, 11.6, 11.8, 11.9.

1. Review all definitions from Sections 7.5, 7.7, 7.8, 11.1, 11.2, 11.3, 11.4, 11.5, 11.6, 11.8, 11.9.
2. Review all theorems from Sections 7.5, 7.7, 7.8, 11.1, 11.2, 11.3, 11.4, 11.5, 11.6, 11.8, 11.9.
3. Evaluate the integral
   (a) $\displaystyle\int_0^1 (3x+1)^{\sqrt{2}}\,dx$. *Answer:* $\frac{4^{\sqrt{2}+1}-1}{3(\sqrt{2}+1)}$.
   (b) $\displaystyle\int \frac{t}{t^4+2}\,dt$. *Answer:* $\frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{t^2}{\sqrt{2}}\right)+C$.
   (c) $\displaystyle\int t\sin t\cos t\,dt$. *Answer:* $-\frac{1}{4}t\cos 2t+\frac{1}{8}\sin 2t+C$.
   (d) $\displaystyle\int \frac{x^3}{\sqrt{1+x^2}}\,dx$. *Answer:* $\frac{1}{3}(1+x^2)^{3/2}-(1+x^2)^{1/2}+C$.
   (e) $\displaystyle\int e^{x+e^x}\,dx$. *Answer:* $e^{e^x}+C$.
   (f) $\displaystyle\int_0^1 \left(1+\sqrt{x}\right)^8\,dx$. *Answer:* $4097/45$.
   (g) $\displaystyle\int \sqrt{\frac{1+x}{1-x}}\,dx$. *Answer:* $\sin^{-1}x-\sqrt{1-x^2}+C$.
   (h) $\displaystyle\int \frac{1}{x+x\sqrt{x}}\,dx$. *Answer:* $2\ln\sqrt{x}-2\ln(1+\sqrt{x})+C$.
   (i) $\displaystyle\int \frac{4^x+10^x}{2^x}\,dx$. *Answer:* $\frac{2^x}{\ln 2}+\frac{5^x}{\ln 5}+C$.
4. Use the Midpoint Rule with $n=10$ to approximate the integral $\int_0^1 e^{x^2}\,dx$. *Answer:* 1.460393.
5. Use the Trapezoidal Rule with $n=10$ to approximate the integral $\int_1^2 e^{1/x}\,dx$. *Answer:* 2.021976.
   (a) Find the approximations T_8 and M_8 for the integral $\displaystyle\int_0^1 \cos(x^2)\,dx$. *Answer:* 0.902333, 0.905620.
   (b) Estimate the errors in the approximations of part (a). *Answer:* $\frac{1}{128}$, $\frac{1}{256}$.
   (c) How large do we have to choose n so that the approximations T_n and M_n to the integral in part (a) are accurate to within 0.0001? *Answer:* 71, 50.
6. Determine whether each integral is convergent or divergent. Evaluate those that are convergent.
   (a) $\displaystyle\int_0^\infty \frac{1}{\sqrt[4]{1+x}}\,dx$. *Answer:* Divergent.
   (b) $\displaystyle\int_{-\infty}^0 2^r\,dr$. *Answer:* $1/\ln 2$.

(c) $\int_{-\infty}^{\infty} xe^{-x^2}\,dx$. *Answer:* 0.

(d) $\int_{1}^{\infty} \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx$. *Answer:* $2e^{-1}$.

(e) $\int_{1}^{\infty} \frac{1}{x^2+x}\,dx$. *Answer:* $\ln 2$.

(f) $\int_{1}^{\infty} \frac{\ln x}{x}\,dx$. *Answer:* Divergent.

(g) $\int_{0}^{1} \frac{3}{x^5}\,dx$. *Answer:* Divergent.

(h) $\int_{0}^{1} \frac{1}{\sqrt{1-x^2}}\,dx$. *Answer:* $\pi/2$.

7. Sketch the region and find its area (if the area is finite).

   (a) $S = \{(x,y) \mid x \geq 1,\, 0 \leq y \leq e^{-x}\}$. *Answer:* $1/e$.

   (b) $S = \{(x,y) \mid x \leq 0,\, 0 \leq y \leq e^{x}\}$. *Answer:* 1.

8. Determine whether the integral is convergent or divergent.

   (a) $\int_{0}^{\infty} \frac{x}{x^3+1}\,dx$. *Answer:* Convergent.

   (b) $\int_{1}^{\infty} \frac{2+e^{-x}}{x}\,dx$. *Answer:* Divergent.

   (c) $\int_{0}^{\pi} \frac{\sin^2 x}{\sqrt{x}}\,dx$. *Answer:* Convergent.

9. What is a sequence? What does it mean to say that $\lim_{n\to\infty} a_n = 8$?

10. Give two example of convergent sequences. Give two examples of divergent sequences.

11. Determine whether the sequence is convergent or divergent. If it is convergent, find the limit.

    (a) $a_n = \frac{n^3}{n^3+1}$. *Answer:* 1.

    (b) $a_n = e^{1/n}$. *Answer:* 1.

    (c) $a_n = \frac{3^{n+2}}{5^n}$. *Answer:* 0.

    (d) $a_n = \sqrt{\frac{n+1}{9n+1}}$. *Answer:* $1/3$.

    (e) $a_n = e^{2n/(n+2)}$. *Answer:* e^2.

    (f) $a_n = \frac{n^3}{n+1}$. *Answer:* Divergent.

    (g) $a_n = \cos(2/n)$. *Answer:* 1.

    (h) $a_n = n^2 e^{-n}$. *Answer:* 0.

    (i) $a_n = \frac{\cos^2 n}{2^n}$. *Answer:* 0.

    (j) $a_n = 2^{-n}\cos n\pi$. *Answer:* 0.

    (k) $a_n = \frac{(\ln n)^2}{n}$. *Answer:* 0.

    (l) $a_n = \tan^{-1}(\ln n)$. *Answer:* $\pi/2$.

12. What is the difference between a sequence and a series?

13. Explain what it means to say that $\sum_{n=1}^{\infty} a_n = 5$.

14. Calculate the sum of the series $\sum_{n=1}^{\infty} a_n$ whose partial sums are given.

    (a) $s_n = 2 - 3(0.8)^n$. *Answer:* 2.

    (b) $s_n = \frac{n^2-1}{4n^2+1}$. *Answer:* $1/4$.

15. Determine whether the series is convergent or divergent. If it is convergent, find its sum.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{10^n}{(-9)^{n-1}}$. *Answer:* Divergent.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{4^n}$. *Answer:* 1/7.

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{n-1}{3n-1}$. *Answer:* Divergent.

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{e^n}{3^{n-1}}$. *Answer:* $\frac{3e}{3-e}$.

(e) $\displaystyle\sum_{n=1}^{\infty} \ln\left(\frac{n^2+1}{2n^2+1}\right)$. *Answer:* Divergent.

(f) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{1+\left(\frac{2}{3}\right)^n}$. *Answer:* Divergent.

(g) $\displaystyle\sum_{n=2}^{\infty} \frac{2}{n^2-1}$. *Answer:* 3/2.

(h) $\displaystyle\sum_{n=1}^{\infty} \tan^{-1} n$. *Answer:* Divergent.

16. Find the values of x for which the series converges. Find the sum of the series for those values of x.

(a) $\displaystyle\sum_{n=1}^{\infty} (-5)^n x^n$. *Answer:* $-\frac{1}{5} < x < \frac{1}{5}$; $\frac{-5x}{1+5x}$.

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{(x-2)^n}{3^n}$. *Answer:* $-1 < x < 5$; $\frac{3}{5-x}$.

(c) $\displaystyle\sum_{n=0}^{\infty} \frac{\sin^n x}{3^n}$. *Answer:* all values of x; $\frac{3}{3-\sin x}$.

17. Determine whether the series is convergent or divergent.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{n^3+1}$. *Answer:* Divergent.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+4}$. *Answer:* Convergent.

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+6n+13}$. *Answer:* Convergent.

(d) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\ln n}$. *Answer:* Divergent.

(e) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$. *Answer:* Convergent.

(f) $\displaystyle\sum_{n=1}^{\infty} \frac{6^n}{5^n-1}$. *Answer:* Divergent.

(g) $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n}$. *Answer:* Divergent

(h) $\displaystyle\sum_{n=1}^{\infty} \frac{n\sin^2 n}{1+n^3}$. *Answer:* Convergent.

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+1}}$. *Answer:* Divergent.

(j) $\displaystyle\sum_{n=1}^{\infty} \frac{1+4^n}{1+3^n}$. *Answer:* Divergent.

(k) $\displaystyle\sum_{n=1}^{\infty} \frac{n+4^n}{n+6^n}$. *Answer:* Convergent.

(l) $\displaystyle\sum_{n=1}^{\infty} \frac{5+2n}{(1+n^2)^2}$. *Answer:* Convergent.

(m) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\sqrt{n^2-1}}$. *Answer:* Convergent.

(n) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n}{\sqrt{n^3+2}}$. *Answer:* Convergent.

(o) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} n e^{-n}$. *Answer:* Convergent.

18. Determine whether the series is absolutely convergent, conditionally convergent, or divergent.

(a) $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n}{n^2+4}$.
Answer: Conditionally convergent.

(b) $\sum_{n=1}^{\infty}\frac{n!}{100^n}$. *Answer:* Divergent.

(c) $\sum_{n=1}^{\infty}(-1)^n\frac{n}{\sqrt{n^3+2}}$.
Answer: Conditionally convergent.

(d) $\sum_{n=1}^{\infty}\frac{n^{10}}{(-10)^{n+1}}$.
Answer: Absolutely convergent.

(e) $\sum_{n=1}^{\infty}\frac{(-2)^n}{n^n}$. *Answer:* Absolutely convergent.

(f) $\sum_{n=1}^{\infty}\left(\frac{n^2+1}{2n^2+1}\right)^n$.
Answer: Absolutely convergent.

(g) $\sum_{n=2}^{\infty}\left(\frac{-2n}{n+1}\right)^{5n}$. *Answer:* Divergent.

(h) $\sum_{n=1}^{\infty}\frac{(2n)!}{(n!)^2}$. *Answer:* Divergent.

19. Determine the radius of convergence and the interval of convergence of the series.

(a) $\sum_{n=1}^{\infty}\frac{(-1)^n x^n}{n^2}$. *Answer:* $R=1$; $[-1,\ 1]$.

(b) $\sum_{n=0}^{\infty}\frac{x^n}{n!}$. *Answer:* $R=\infty$; $(-\infty,\ \infty)$.

(c) $\sum_{n=1}^{\infty}\frac{n^n}{x^n}$. *Answer:* $R=0$; $\{0\}$.

(d) $\sum_{n=1}^{\infty}\frac{10^n x^n}{n^3}$. *Answer:* $R=\frac{1}{10}$; $(-\frac{1}{10},\ \frac{1}{10})$.

(e) $\sum_{n=0}^{\infty}\frac{(-1)^n(x-3)^n}{2n+1}$. *Answer:* $R=1$; $(2,\ 4]$.

(f) $\sum_{n=1}^{\infty}\frac{3^n(x+4)^n}{\sqrt{n}}$. *Answer:* $R=\frac{1}{3}$; $[-\frac{13}{3},-\frac{11}{3})$.

20. Find a power series representation for the function and determine the interval of convergence.

(a) $\frac{1}{1+x}$. *Answer:* $\sum_{n=0}^{\infty}(-1)^n x^n$; $(-1,\ 1)$.

(b) $\frac{5}{1-4x^2}$. *Answer:* $5\sum_{n=0}^{\infty}4^n x^{2n}$; $(-\frac{1}{2},\ \frac{1}{2})$.

(c) $\frac{1}{x+10}$. *Answer:* $\sum_{n=0}^{\infty}\frac{(-1)^n x^n}{10^{n+1}}$; $(-10,\ 10)$.

(d) $\frac{x}{9+x^2}$. *Answer:* $\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{9^{n+1}}$; $(-3,\ 3)$.

21. (a) Use differentiation to find a power series representation for $f(x)=\frac{1}{(1+x)^2}$. What is the radius of convergence?

(b) Use part (a) to find a power series representation for $g(x)=\frac{1}{(1+x)^3}$.

(c) Use part (b) to find a power series representation for $h(x)=\frac{x^2}{(1+x)^3}$.

Answer: (a) $\sum_{n=1}^{\infty}(-1)^{n+1}nx^{n-1}$, $R=1$; (b) $-\frac{1}{2}\sum_{n=1}^{\infty}(-1)^n(n+1)nx^{n-1}$; (c) $-\frac{1}{2}\sum_{n=1}^{\infty}(-1)^n(n+1)nx^{n+1}$.

22. Find a power series representation for the function $f(x)=\ln(5-x)$ and determine the radius of convergence. *Answer:* $\ln 5-\sum_{n=1}^{\infty}\frac{x^n}{n5^n}$; $R=5$.