Fourier Series

EXERCICE 1. Let f_1 and f_2 be $2\pi-$periodic function defined on $]-\pi,\pi]$ as follows.

$$f_1(x)=\begin{cases}1, \text{ if } |x|\leq \pi/2,\\ 0, \text{ if } -\pi<x<-\pi/2 \text{ or } \pi/2<x\leq\pi,\end{cases}\qquad f_2(x)=\begin{cases}1, \text{ if } 0\leq x\leq \pi/2,\\ -1, \text{ if } -\pi/2\leq x<0,\\ 0, \text{ if } -\pi<x<-\pi/2 \text{ or } \pi/2<x\leq\pi.\end{cases}$$

(a): Sketch f_1 on the interval $]-\pi,3\pi[$.

(b): Determine the Fourier series of f_1

(c): Determine the Fourier series of f_2, giving the values of $b_1, b_2, ..., b_5$.

(d): State for what values of $x\in\,]-\pi,\pi[$ the Fourier series for $f_1(x)$ is the same as $f_1(x)$ and for what values, if any, they differ.

EXERCICE 2.

$$f_1(x)=\begin{cases}1, \text{ if } -\pi<x\leq -\pi/2,\\ 0, \text{ if } -\pi/2<x<\pi/2,\\ 1, \text{ if } \pi/2\leq x\leq\pi.\end{cases}\qquad f_2(x)=-\frac{x}{2}+\int_0^x f_1(t)dt.$$

(a): Obtain the Fourier coefficients of f_1 in their simplest form.

(b): Show that $f-2$ can be written in the form

$$f_2(x)=\begin{cases}\frac{x+\pi}{2}, \text{ if } -\pi<x\leq -\pi/2,\\ -\frac{x}{2}, \text{ if } -\pi/2<x<\pi/2,\\ \frac{x-\pi}{2}, \text{ if } \pi/2\leq x\leq\pi.\end{cases}$$

and sketch f_2 on the interval $-\pi\leq x\leq\pi$.

(c): Obtain the Fourier coefficients of f_2 in their simplest form.

(d): For each f_1 and f_2 state the set of points, if any, where the values of the Fourier series is not the same with the value of the represented function.

EXERCICE 3.Consider the function $g:[0,2\pi]\to\mathbb{R}$ defined by

$$g(x)=\begin{cases}2, \text{ if } 0\leq x<\pi/2, \text{ or } 3\pi/2<x\leq 2\pi,\\ 1, \text{ if } \pi/2\leq x\leq 3\pi/2.\end{cases}$$

(a): Denote by $f:\mathbb{R}\to\mathbb{R}$ the $2\pi-$periodic extension of g over $\mathbb{R}$. Sketch f over the interval $x\in[-2\pi,2\pi]$

(b): Show that the Fourier series S of f is

$$S(x)=3/2+2/\pi\left(\sum_{n=0}^{\infty}(-1)^n\frac{\cos((2n+1)x)}{2n+1}\right).$$

(c): For what values of x do we have $S(x)\neq f(x)$ on $[-\pi,\pi]$?

(d): Explain why the Fourier series suggests that

$$\pi/4=1-1/3+1/5-1/7+....=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n-1}$$

and use the Leibniz Alternating Series Test (AST) to test this series for convergence

EXERCICE 4.

— Find the sine Fourier series for (periodic extension of) $f(t)=t-1$ for $t\in[\,0,2\,[$ and $f(t)=3-t$ for $t\in[\,2,4\,[$. Determine the sum of this series.

— Find the cosine Fourier series for (periodic extension of) $f(t)=1$ for $t\in[\,0,1\,[$ and $f(t)=0$ for $t\in[\,1,4\,[$. Determine the sum of this series.