Contents

Chapter 1

Differential Calculus

1.1 Regions in the plane

Let D be a subset of the plane $\mathbb{R}^2$ and let $(a, b) \in \mathbb{R}^2$ be any point.

An **ϵ-disk** around (a, b) is the set of all points $(x, y) \in \mathbb{R}^2$ whose distance from (a, b) is less than ϵ.

(a, b) is **an interior point** of D iff *some* ϵ-disk around (a, b) is contained in D.

$(a, b) \in D$ is **an isolated point** of D iff (a, b) is the only point of D that is contained in *some* ϵ-disk around (a, b).

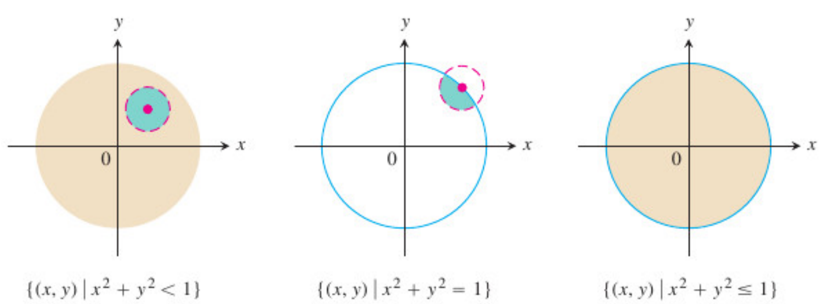
(a, b) is **a boundary point** of D iff *every* ϵ-disk around (a, b) contains points from D *and* points not from D.

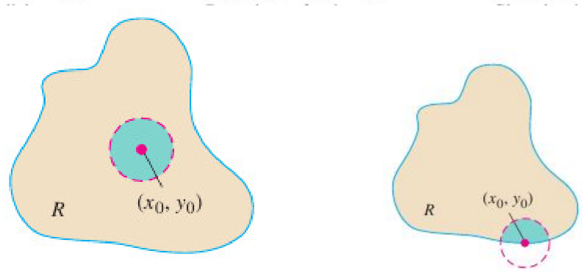
R is an **open subset** of $\mathbb{R}^2$ iff all points of D are its interior points.

D is a **closed subset** of $\mathbb{R}^2$ iff it contains all its boundary points.

$\overline{D} = D \cup$ the set of boundary points of D; It is the **closure** of D.

D is a **bounded subset** of $\mathbb{R}^2$ iff D is contained in *some* ϵ-disk. (around some point)





An interior point A boundary point

D is called a iff it contains all its interior points, possibly some of its boundary points, and satisfies the property of *connectedness* that any two points in D can be joined by a polygonal line entirely lying in D. A region is sometimes called a *domain*.

Let D be a region in the plane. Let $f : D \to \mathbb{R}$ be a function.
The **graph** of f is $\{(x, y, z) \in \mathbb{R}^3 : z = f(x, y),\ (x, y) \in D\}$.
The graph here is also called the **surface** $z = f(x, y)$.
The **domain** of f is D.
The **co-domain** of f is $\mathbb{R}$.
The **range** of f is $\{z \in \mathbb{R} : z = f(x, y)$ for some $(x, y) \in D\}$.

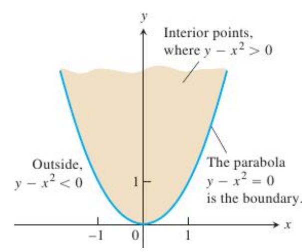
Sometimes, we do not fix the domain D of f but ask you to find it.

The function $f(x, y) = \sqrt{y - x^2}$

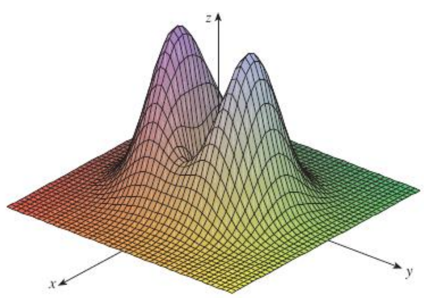
has domain $D = \{(x, y) : x^2 \le y\}$.

Its range is the set of all non-negative reals.

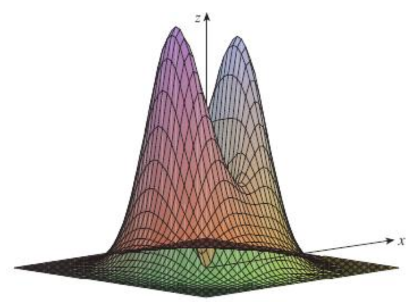
What is its graph?



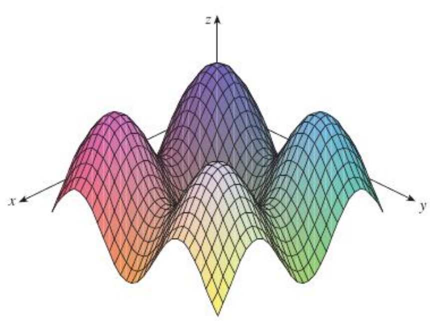
Some examples of surfaces are here:



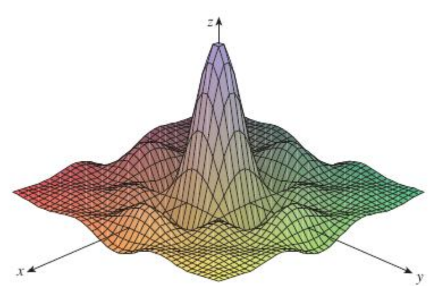
(a) $f(x, y) = (x^2 + 3y^2)e^{-x^2-y^2}$



(b) $f(x, y) = (x^2 + 3y^2)e^{-x^2-y^2}$



(c) $f(x, y) = \sin x + \sin y$



(d) $f(x, y) = \dfrac{\sin x \sin y}{xy}$

1.2 Level curves and surfaces

Let $f(x, y)$ be a function of two variables. That is, $f : D \to \mathbb{R}$, where D is a region in $\mathbb{R}^2$.

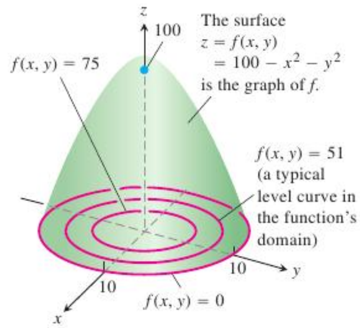
A **contour curve** of f is the curve of intersection of the surface $z = f(x, y)$ and the plane $z = c$ for some constant c in the range of f. It is the curve $f(x, y) = c$ for some constant c in the range of f.

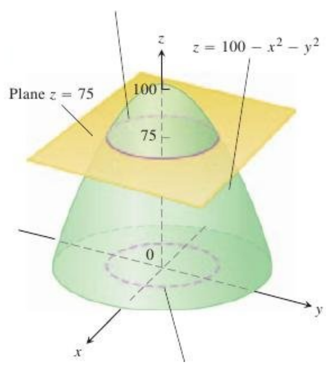
The union of all contour curves is the surface $z = f(x, y)$; it is also the graph of f.

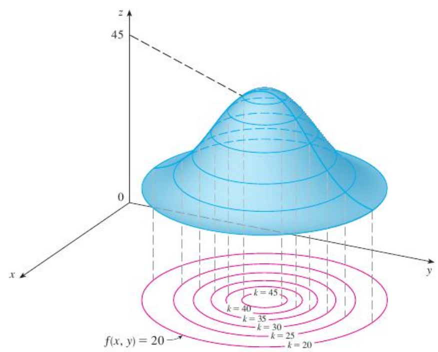
A **level curve** of f is the set of points (x, y) in the domain of f for which $f(x, y) = c$ for some constant c in the range of f.

The level curve is the projection of the contour curve on the xy-plane (with same c).

Example 1.1. Consider the function $f(x, y) = 100 - x^2 - y^2$.
Its domain is $\mathbb{R}^2$. Its range is the interval $(-\infty, 100]$.
The level curve $f(x, y) = 0$ is $\{(x, y) : x^2 + y^2 = 100\}$.
The level curve $f(x, y) = 51$ is $\{(x, y) : x^2 + y^2 = 49\}$.







Similarly, for a function $f(x, y, z)$ of three variables, the **level surfaces** are the sets of points (x, y, z) such that $f(x, y, z) = c$ for values c in the range of f.

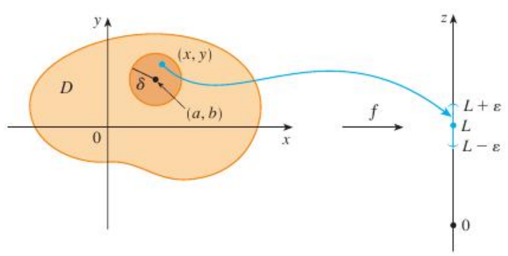
Let $f : D \to \mathbb{R}$ be a function, where D is a region in the plane. Let $(a, b) \in \overline{D}$.

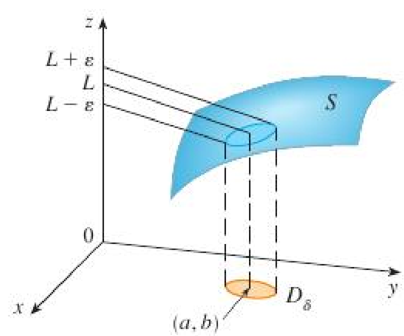
The **limit** of $f(x, y)$ as (x, y) approaches (a, b) is L iff corresponding to each $\epsilon > 0$, there exists $\delta > 0$ such that for all $(x, y) \in D$ with $0 < \sqrt{(x - a)^2 + (y - b)^2} < \delta$, we have $|f(x, y) - L| < \epsilon$.

In this case, we write $\lim_{(x,y)\to(a,b)} f(x, y) = L$.

We also say that L is the limit of f at (a, b).

If for no real number L, the above happens, then **limit of f at (a, b) does not exist.**





The intuitive understanding of the notion of limit is as follows:

The distance between $f(x, y)$ and L can be made arbitrarily small by making the distance between (x, y) and (a, b) sufficiently small but not necessarily zero.

It is often difficult to show that limit of a function does not exist at a point. We will come back to this question soon. When limit exists, we write it in many alternative ways:

$$\text{The limit of } f(x,y) \text{ as } (x,y) \text{ approaches } (a,b) \text{ is } L.$$

$$f(x,y) \to L \text{ as } (x,y) \to (a,b).$$

$$\lim_{(x,y)\to(a,b)} f(x,y) = L.$$

$$\lim_{\substack{x\to a\\ y\to b}} f(x,y) = L.$$

Example 1.2. Determine if $\displaystyle\lim_{(x,y)\to(0,0)} \frac{4xy^2}{x^2+y^2}$ exists.

Observe that the region D of f is $\mathbb{R}^2 \setminus \{(0,0)\}$. And $f(0, y) = 0$ for $y \neq 0$; $f(x, 0) = 0$ for $x \neq 0$. We guess that if the limit exists, it would be 0. To see that it is the case, we start with any $\epsilon > 0$. We want to choose a $\delta > 0$ such that the following sentence becomes true:

$$\text{If } \; 0 < \sqrt{x^2+y^2} < \delta, \; \text{ then } \; \left|\frac{4xy^2}{x^2+y^2}\right| < \epsilon.$$

Since $|y^2| = y^2 \le x^2 + y^2$ and $|x^2| = x^2 \le x^2 + y^2$, we have

$$\left|\frac{4xy^2}{x^2+y^2}\right| \le 4|x| \le 4\sqrt{x^2+y^2}.$$

So, we choose $\delta = \epsilon/4$. Let us verify whether our choice is all right.
Assume that $0 < \sqrt{x^2+y^2} < \delta$. Then

$$\left|\frac{4xy^2}{x^2+y^2} - 0\right| \le 4\sqrt{x^2+y^2} < 4\delta = \epsilon.$$

Hence

$$\lim_{(x,y)\to(0,0)} \frac{4xy^2}{x^2+y^2} = 0.$$

Observation: Suppose we have obtained a δ corresponding to some ϵ. If we take ϵ_1 which is larger than the earlier ϵ, then the same δ will satisfy the requirement in the definition of the limit. Thus while showing that the limit of a function is such and such at a point, we are free to choose a pre-assigned upper bound for our ϵ.

Similarly, suppose for some ϵ, we have already obtained a δ such that the limit requirement is satisfied. If we choose another δ, say δ_1, which is smaller than δ, then the limit requirement is also satisfied. Thus, we are free to choose a pre-assigned upper bound for our δ provided it is convenient to us and it works.

Example 1.3. Consider $f(x, y) = \sqrt{1 - x^2 - y^2}$ where $D = \{(x, y) : x^2 + y^2 \le 1\}$.
We guess that limit $f(x, y)$ is 1 as $(x, y) \to (0, 0)$.

To show that the guess is right, let $\epsilon > 0$. Observe that $0 \le f(x, y) \le 1$ on D.
Using our observation, assume that $0 < \epsilon < 1$. Choose $\delta = \sqrt{1 - (1-\epsilon)^2}$. Let $|(x, y) - (0, 0)| < \delta$.
Then $0 < x^2 + y^2 < 1 - (1-\epsilon)^2 \Rightarrow 1 - x^2 - y^2 > (1-\epsilon)^2 \Rightarrow f(x, y) > 1 - \epsilon$.
That is, $|f(x, y) - 1| = 1 - f(x, y) < \epsilon$. Therefore, $f(x, y) \to 1$ as $(x, y) \to (0, 0)$.

Theorem 1.1. (Uniqueness of limit) *Let $f(x, y)$ be a real valued function defined on a region $D \subseteq \mathbb{R}^2$. Let $(a, b) \in \overline{D}$. If limit of $f(x, y)$ as (x, y) approaches (a, b) exists, then it is unique.*

Proof: Suppose $f(x, y) \to \ell$ and also $f(x, y) \to m$ as $(x, y) \to (a, b)$. Let $\epsilon > 0$. For $\epsilon/2$, we have $\delta_1 > 0,\ \delta_2 > 0$ such that

$$0 < (x-a)^2+(y-b)^2 < \delta_1^2 \Rightarrow |f(x, y)-\ell| < \epsilon/2, \quad 0 < (x-a)^2+(y-b)^2 < \delta_2^2 \Rightarrow |f(x, y)-m| < \epsilon/2.$$

Choose a point (α, β) so that both $0 < (\alpha - a)^2 + (\beta - b)^2 < \delta_1^2$, $0 < (\alpha - a)^2 + (\beta - b)^2 < \delta_2^2$ hold. Then

$$|f(\alpha, \beta) - \ell| < \epsilon/2 \quad \text{and} \quad |f(\alpha, \beta) - m| < \epsilon/2.$$

Now, $|\ell - m| \le |\ell - f(\alpha, \beta)| + |f(\alpha, \beta) - m| < \epsilon/2 + \epsilon/2 = \epsilon$. That is, for each $\epsilon > 0$, we have $|\ell - m| < \epsilon$. Hence $\ell = m$. □

For a function of one variable, there are only two directions for approaching a point; from left and from right. Whereas for a function of two variables, there are infinitely many directions, and infinite number of paths on which one can approach a point. The limit refers only to the distance between (x, y) and (a, b). It does not refer to any specific direction of approach to (a, b). If the limit exists, then $f(x, y)$ must approach the same limit no matter how (x, y) approaches (a, b). Thus, if we can find two different paths of approach along which the function $f(x, y)$ has different limits, then it follows that limit of $f(x, y)$ as (x, y) approaches (a, b) does not exist.

Theorem 1.2. *Suppose that $f(x, y) \to L_1$ as $(x, y) \to (a, b)$ along a path C_1 and $f(x, y) \to L_2$ as $(x, y) \to (a, b)$ along a path C_2. If $L_1 \ne L_2$, then the limit of $f(x, y)$ as $(x, y) \to (a, b)$ does not exist.*

Example 1.4. Consider $f(x, y) = \dfrac{x^2 - y^2}{x^2 + y^2}$ for $(x, y) \ne (0, 0)$. What is its limit at $(0, 0)$?

When $y = 0$, limit of $f(x, y)$ as $x \to 0$ is $\lim_{x\to 0} \dfrac{x^2}{x^2} = \lim_{x\to 0}(1) = 1$.
That is, $f(x, y) \to 1$ as $(x, y) \to (0, 0)$ along the x-axis.

When $x = 0$, limit of $f(x, y)$ as $y \to 0$ is $\lim_{y\to 0} \dfrac{-y^2}{y^2} = -1$.
That is, $f(x, y) \to -1$ as $(x, y) \to (0, 0)$ along the y-axis.

Hence $\lim_{(x,y)\to(0,0)} f(x, y)$ does not exist.

Example 1.5. Consider $f(x,y) = \dfrac{xy}{x^2+y^2}$ for $(x,y) \neq (0,0)$. What is its limit at $(0,0)$?

Along the x-axis, $y = 0$; then limit of $f(x,y)$ as $(x,y) \to (0,0)$ is 0.

Along the y-axis, $x = 0$; then limit of $f(x,y)$ as $(x,y) \to (0,0)$ is 0.

Does it say that limit of $f(x,y)$ as $(x,y) \to (0,0)$ is 0?

Along the line $y = x$, limit of $f(x,y)$ as $(x,y) \to 0$ is $\lim_{x\to 0} \dfrac{x^2}{x^2+x^2} = 1/2$.

Hence $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

Example 1.6. Consider $f(x,y) = \dfrac{xy^2}{x^2+y^4}$ for $(x,y) \neq (0,0)$. What is its limit at $(0,0)$?

If $y = mx$, for some $m \in \mathbb{R}$, then $f(x,y) = \dfrac{m^2 x}{1+m^4x^2}$. So, $\lim_{(x,y)\to(0,0)}$ along all straight lines is 0.

If $x = y^2,\ y \neq 0$, then $f(x,y) = \dfrac{y^4}{y^4+y^4} = 1/2$. As $(x,y) \to (0,0)$ along $x = y^2$, $f(x,y) \to 1/2$.

Hence $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

A question: are the following same?

$$\lim_{(x,y)\to(a,b)} f(x,y), \quad \lim_{x\to a}\lim_{y\to b} f(x,y), \quad \lim_{y\to b}\lim_{x\to a} f(x,y)$$

Example 1.7. Let $f(x,y) = \dfrac{(y-x)(1+x)}{(y+x)(1+y)}$ for $x + y \neq 0, -1 < x,\ y < 1$. Then

$$\lim_{y\to 0}\lim_{x\to 0} f(x,y) = \lim_{y\to 0} \frac{y}{y(1+y)} = 1.$$

$$\lim_{x\to 0}\lim_{y\to 0} f(x,y) = \lim_{x\to 0} \frac{-x(1+x)}{x} = -1.$$

$$\text{Along } y = mx, \quad \lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{x(m-1)(1+x)}{x(1+m)(1+mx)} = \frac{m-1}{m+1}.$$

For different values of m, we get the last limit value different. So, limit of $f(x,y)$ as $(x,y) \to (0,0)$ does not exist. But the two iterated limits exist and they are not equal.

Example 1.8. Let $f(x,y) = x \sin\dfrac{1}{y} + y \sin\dfrac{1}{x}$ for $x \neq 0,\ y \neq 0$. Then

$$\lim_{x\to 0} y \sin\frac{1}{x} \text{ and } \lim_{y\to 0} x \sin\frac{1}{y} \text{ do not exist.}$$

So, neither $\lim_{y\to 0}\lim_{x\to 0} f(x,y)$ exists nor $\lim_{x\to 0}\lim_{y\to 0} f(x,y)$ exists.

However, $|f(x,y) - 0| \le |x| + |y| = \sqrt{x^2} + \sqrt{y^2} \le 2\sqrt{x^2+y^2} = 2|(x,y)|$. Take $\delta = \epsilon/2$. Now, If $|(x,y) - (0,0)| < \delta = \epsilon/2$, then $|f(x,y) - 0| < \epsilon$. Therefore,

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0.$$

That is, the two iterated limits do not exist, but the limit exists.

Hence existence of the limit of $f(x, y)$ as $(x, y) \to (a, b)$ and the two iterated limits have no connection.

The usual operations of addition, multiplication etc have the expected effects as the following theorem shows. Its proof is analogous to the single variable limits.

Theorem 1.3. *Let* $L, M, c \in \mathbb{R}$; $\lim_{(x,y)\to(a,b)} f(x, y) = L$; $\lim_{(x,y)\to(a,b)} g(x, y) = M$. *Then*

1. Constant Multiple *:* $\lim_{(x,y)\to(a,b)} cf(x, y) = cL$.

2. Sum *:* $\lim_{(x,y)\to(a,b)} (f(x, y) + g(x, y)) = L + M$.

3. Product *:* $\lim_{(x,y)\to(a,b)} (f(x, y)\, g(x, y)) = LM$.

4. Quotient *: If* $M \neq 0$ *and* $g(x, y) \neq 0$ *in an open disk around the point* (a, b), *then* $\lim_{(x,y)\to(a,b)} (f(x, y)/g(x, y)) = L/M$

5. Power *: If* $r \in \mathbb{R}$, $L^r \in \mathbb{R}$ *and* $\lim_{(x,y)\to(a,b)} f(x, y) = L$, *then* $\lim_{(x,y)\to(a,b)} (f(x, y))^r = L^r$.

1.3 Continuity

Let $f(x, y)$ be a real valued function defined on a subsets D of $\mathbb{R}^2$. We say that $f(x, y)$ is **continuous** at a point $(a, b) \in D$ iff for each $\epsilon > 0$, there exists $\delta > 0$ such that for all points $(x, y) \in D$ with $\sqrt{(x - a)^2 + (y - b)^2} < \delta$ we have $|f(x, y) - f(a, b)| < \epsilon$.

Observe that if (a, b) is an isolated point of D, then f is continuous at (a, b). When D is a region, (a, b) is not an isolated point of D; and then f is continuous at $(a, b) \in D$ iff the following are satisfied:

1. $f(a, b)$ is well defined, that is, $(a, b) \in D$;

2. $\lim_{(x,y)\to(a,b)} f(x, y)$ exists; and

3. $\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b)$.

The function $f(x, y)$ is said to be **continuous on** a subset of D iff $f(x, y)$ is continuous at all points in the subset.

Therefore, constant multiples, sum, difference, product, quotient, and rational powers of continuous functions are continuous whenever they are well defined.

Polynomials in two variables are continuous functions.

Rational functions, i.e., ratios of polynomials, are continuous functions provided they are well defined.

Example 1.9. $f(x, y) = \begin{cases} \frac{3x^2y}{x^2+y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$ is continuous on $\mathbb{R}^2$.

At any point other than the origin, $f(x,y)$ is a rational function; therefore, it is continuous. To see that $f(x,y)$ is continuous at the origin, let $\epsilon > 0$ be given. Take $\delta = \epsilon/3$. Assume that $\sqrt{x^2+y^2} < \delta$. Then

$$\Big|\frac{3x^2y}{x^2+y^2} - f(0,0)\Big| \le \Big|\frac{3(x^2+y^2)y}{x^2+y^2}\Big| \le 3|y| \le 3\sqrt{x^2+y^2} < 3\delta = \epsilon.$$

Example 1.10. $f(x,y) = \begin{cases} \frac{xy(x^2-y^2)}{x^2+y^2} & \text{if } (x,y) \ne (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$ is continuous on $\mathbb{R}^2$. Why?

Being a rational function, it is continuous at all nonzero points. For the point $(0,0)$, let $\epsilon > 0$ be given. Choose $\delta = \sqrt{\epsilon}$. Notice that $xy \le x^2+y^2$ and $x^2-y^2 \le x^2+y^2$.
For all (x,y) with $\sqrt{x^2+y^2} < \delta$, we have

$$|f(x,y) - 0| \le \frac{(x^2+y^2)(x^2+y^2)}{x^2+y^2} < \delta^2 = \epsilon.$$

Hence $\lim\limits_{(x,y)\to(0,0)} f(x,y) = 0 = f(0,0)$.

Example 1.11. $f(x,y) = \dfrac{x^2-y^2}{x^2+y^2}$ is continuous on $D = \mathbb{R}^2 \setminus \{(0,0)\}$.

$f(x,y)$ is not continuous at $(0,0)$ since $(0,0) \notin D$.

What about the function $g(x,y)$, where

$$g(x,y) = \begin{cases} \frac{x^2-y^2}{x^2+y^2} & \text{if } (x,y) \ne (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

By Example 1.4, $\lim\limits_{(x,y)\to(0,0)} g(x,y)$ does not exist. Hence $g(x,y)$ is not continuous at $(0,0)$.

As in the single variable case, *composition of continuous functions is continuous*:

> Let $f : D \to \mathbb{R}$ be continuous at (a,b) with $f(a,b) = c$. Let $g : I \to \mathbb{R}$ be continuous at $c \in I$ for some interval I in $\mathbb{R}$. Then $g(f(x,y))$ from D to $\mathbb{R}$ is continuous at (a,b).

Proof of this fact is left to you as an exercise.

For example,

e^{x-y} is continuous at all points in the plane.

$\cos \dfrac{xy}{1+x^2}$ and $\ln(1+x^2+y^2)$ are continuous on $\mathbb{R}^2$.

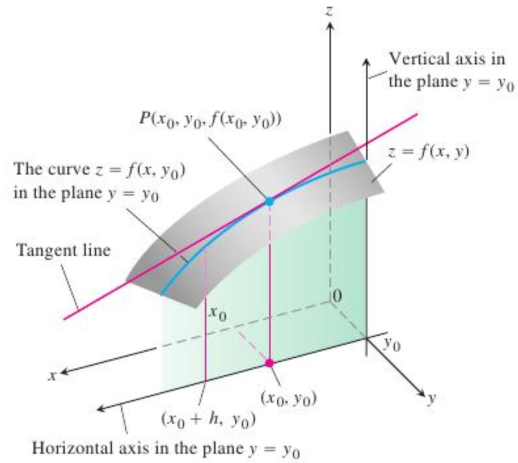
At which points is $\tan^{-1}(y/x)$ continuous?

Well, the function y/x is continuous everywhere except when $x = 0$.
The function $\tan^{-1}$ is continuous everywhere on $\mathbb{R}$.
So, $\tan^{-1}(y/x)$ is continuous everywhere except when $x = 0$.

The function $(x^2+y^2+z^2-1)^{-1}$ is continuous everywhere except on the sphere $x^2+y^2+z^2 = 1$, where it is not defined.

1.4 Partial Derivatives

Let $f(x, y)$ be a real valued function defined on a region $D \subseteq \mathbb{R}^2$. Let $(a, b) \in D$.



If C is the curve of intersection of the surface $z = f(x, y)$ with the plane $y = b$, then the slope of the tangent line to C at $(a, b, f(a, b))$ is the partial derivative of $f(x, y)$ with respect to x at (a, b). In the figure take $x_0 = a$, $y_0 = b$. A formal definition of the partial derivative follows.

The **partial derivative of** $f(x, y)$ **with respect to** x **at the point** (a, b) is

$$f_x(a, b) = \frac{\partial f}{\partial x}\Big|_{(a,b)} = \frac{df(x, b)}{dx}\Big|_{x=a} = \lim_{h\to 0} \frac{f(a + h, b) - f(a, b)}{h},$$

provided this limit exists. Notice that $f(x, b)$ must be continuous at $x = a$.
The **partial derivative of** $f(x, y)$ **with respect to** y **at the point** (a, b) is

$$f_y(a, b) = \frac{\partial f}{\partial y}\Big|_{(a,b)} = \frac{df(a, y)}{dy}\Big|_{y=b} = \lim_{k\to 0} \frac{f(a, b + k) - f(a, b)}{k},$$

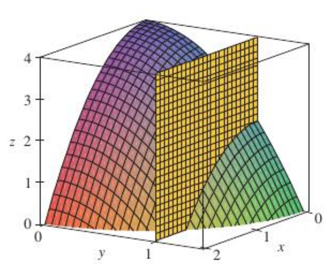
provided this limit exists. Again, $f(a, y)$ must be continuous at $y = b$.

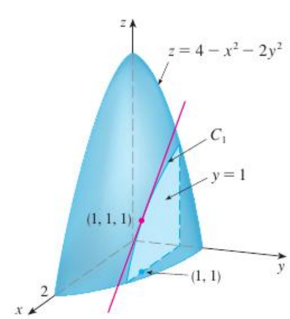
Example 1.12. Find $f_x(1, 1)$ where $f(x, y) = 4 - x^2 - 2y^2$.

$$f_x(1, 1) = \lim_{h\to 0} \frac{(4 - (1 + h)^2 - 2) - (4 - 1 - 2)}{h} = \lim_{h\to 0} \frac{-2h - h^2}{h} = -2.$$

That is, treat y as a constant and differentiate with respect to x.

$$f_x(1, 1) = f_x(x, y)|_{(1,1)} = -2x|_{(1,1)} = -2.$$





The vertical plane $y = 1$ crosses the paraboloid in the curve $C_1 : z = 2 - x^2, y = 1$. The slope of the tangent line to this parabola at the point $(1, 1, 1)$ (which corresponds to $(x, y) = (1, 1)$) is $f_x(1, 1) = -2$.

Example 1.13. Find f_x and f_y, where $f(x, y) = y \sin(xy)$.

Treating y as a constant and differentiating with respect to x, we get f_x. Similarly, f_y.

$$f_x(x, y) = y \cos(xy)\, y, \quad f_y(x, y) = yx \cos(xy) + \sin(xy).$$

Example 1.14. Find $\partial z/\partial x$ and $\partial z/\partial y$ where $z = f(x, y)$ is defined by $x^3 + y^3 + z^3 - 6xyz = 1$.

Differentiate $x^3 + y^3 + z^3 - 6xyz - 1 = 0$ with respect to x treating y as a constant:

$$3x^2 + 0 + 3z^2 \frac{\partial z}{\partial x} - 6y \left(z + x \frac{\partial z}{\partial x} \right) - 0 = 0.$$

Solving this for $\partial z/\partial x$, we have

$$\frac{\partial z}{\partial x}(3z^2 - 6xy) + (3x^2 - 6yz) = 0, \quad \text{that is,}$$

$$\frac{\partial z}{\partial x} = -\frac{x^2 - 2yz}{z^2 - 2xy}.$$

Similarly,

$$\frac{\partial z}{\partial y} = -\frac{y^2 - 2xz}{z^2 - 2xy}.$$

Example 1.15. The plane $x = 1$ intersects the surface $z = x^2 + y^2$ in a parabola. Find the slope of the tangent to the parabola at the point $(1, 2, 5)$.

The asked slope is $\partial z/\partial y$ at $(1, 2)$. It is

$$\frac{\partial(x^2 + y^2)}{\partial y}(1, 2) = (2y)(1, 2) = 4.$$

Alternatively, the parabola is $z = x^2 + y^2, x = 1$ OR, $z = 1 + y^2$. So, the slope at $(1, 2, 5)$ is

$$\frac{dz}{dy}\Big|_{y=2} = \frac{d(1 + y^2)}{dy}\Big|_{y=2} = (2y)|_{y=2} = 4.$$

For a function $f(x, y)$, partial derivatives of second order are:

$$\begin{aligned}
f_{xx} &= (f_x)_x = \frac{\partial}{\partial x}\frac{\partial f}{\partial x} = \frac{\partial^2 f}{\partial x^2}. \\
f_{xy} &= (f_x)_y = \frac{\partial f_x}{\partial y} = \frac{\partial}{\partial y}\frac{\partial f}{\partial x} = \frac{\partial^2 f}{\partial y \partial x}. \\
f_{yx} &= (f_y)_x = \frac{\partial f_y}{\partial x} = \frac{\partial}{\partial x}\frac{\partial f}{\partial y} = \frac{\partial^2 f}{\partial x \partial y}. \\
f_{yy} &= (f_y)_y = \frac{\partial}{\partial y}\frac{\partial f}{\partial y} = \frac{\partial^2 f}{\partial y^2}.
\end{aligned}$$

Similarly, higher order partial derivatives are defined. For example,

$$f_{xxy} = \frac{\partial}{\partial y}\frac{\partial}{\partial x}\frac{\partial f}{\partial x} = \frac{\partial^3 f}{\partial y \partial x \partial x}.$$

Observe that $f_x(a, b)$ is not the same as $\lim_{(x,y)\to(a,b)} f_x(x, y)$. To see this, let

$$f(x, y) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x \le 0. \end{cases}$$

Then $f_x(x, y) = 0$ for all $x > 0$. Also, $f_x(x, y) = 0$ for all $x < 0$. Now, $\lim_{(x,y)\to(0,0)} f_x(x, y) = 0$. But $f_x(0, 0)$ does not exist. Reason?

$$f_x(0, 0) = \lim_{h\to 0} \frac{f(h, 0) - f(0, 0)}{h} = \lim_{h\to 0} \frac{1 \text{ or } 0}{h} \text{ does not exist}$$

On the other hand, $f_x(a, b)$ can exist though $\lim_{(x,y)\to(a,b)} f_x$ does not.
However, if $f_x(x, y)$ is continuous at (a, b), then

$$f_x(a, b) = \lim_{(x,y)\to(a,b)} f_x(x, y).$$

Similarly, f_{xy} need not be equal to f_{yx}. See the following example.

Example 1.16. Consider $f(x, y) = \dfrac{xy(x^2 - y^2)}{x^2 + y^2}$ for $(x, y) \neq (0, 0)$, and $f(0, 0) = 0$.

$$\begin{aligned}
f(x, 0) &= f(0, y) = f(0, 0) = 0. \\
f_x(x, 0) &= f_y(0, y) = f_{xx}(0, 0) = f_{yy}(0, 0) = 0. \\
f_x(0, y) &= \lim_{h\to 0} \frac{f(h, y) - f(0, y)}{h} = -y, \ f_y(x, 0) = \lim_{k\to 0} \frac{f(x, k) - f(x, 0)}{k} = x. \\
f_{xy}(0, 0) &= \lim_{k\to 0} \frac{f_x(0, k) - f_x(0, 0)}{k} = \lim_{k\to 0} \frac{-k - 0}{k} = -1. \\
f_{yx}(0, 0) &= \lim_{h\to 0} \frac{f_y(h, 0) - f_y(0, 0)}{h} = \lim_{h\to 0} \frac{h - 0}{h} = 1.
\end{aligned}$$

That is, $f_{xy} \neq f_{yx}$.
But continuity of both of f_{xy} and f_{yx} implies their equality.

Theorem 1.4. (Clairaut) *Let D be a region in $\mathbb{R}^2$. Let the function $f : D \to \mathbb{R}$ have continuous first and second order partial derivatives on D. Then $f_{xy} = f_{yx}$.*

Proof: Let $(a, b) \in D$. Let $h \neq 0$. Write $g(x) = f(x, b + h) - f(x, b)$. Then

$$\phi(f) := g(a + h) - g(a) = [f(a + h, b + h) - f(a + h, b)] - [f(a, b + h) - f(a, b)].$$

By MVT, we have c between a and $a + h$ such that

$$\phi(f) = g'(c)h = h[f_x(c, b + h) - f_x(c, b)].$$

Again, by MVT (on f_x with the second variable), we have d between b and $b + h$ such that

$$\phi(f) = h \cdot h \cdot f_{xy}(c, d) = h^2 f_{xy}(c, d).$$

Due to continuity of f_{xy}, we have

$$\lim_{h\to 0} \frac{\phi(f)}{h^2} = \lim_{(c,d)\to(a,b)} f_{xy}(c, d) = f_{xy}(a, b).$$

Write

$$\phi(f) = [f(a + h, b + h) - f(a, b + h)] - [f(a + h, b) - f(a, b)]$$

and apply MVT twice as above to get $f_{yx}(a, b) = \lim_{h\to 0} \frac{\phi(f)}{h^2}$. But the two limits with $\phi(f)/h^2$ are equal. So, $f_{xy}(a, b) = f_{yx}(a, b)$. □

In one variable, $f'(t)$ exists at $t = a$ implies that $f(t)$ is continuous at $t = a$. We have seen similarly that existence of $f_x(a, b)$ and $f_y(a, b)$ guarantees continuity of $f(x, b)$ and of $f(a, y)$ at (a, b). But for $f(x, y)$, even both $f_x(x, y)$ and $f_y(x, y)$ exist at (a, b), the function $f(x, y)$ need not be continuous at (a, b). See the following example.

Example 1.17. Let $f(x, y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$

Here, $f(x, 0) = 0 = f(0, y)$. So, $f_x(0, 0) = 0 = f_y(0, 0)$. And limit of $f(x, y)$ as $(x, y) \to (0, 0)$ does not exist. Hence $f(x, y)$ is not continuous at $(0, 0)$.

Further, we find that $f_{xx}(x, 0) = 0 = f_{yy}(0, y)$. What about $f_{xy}(0, 0)$?

$$f_x(0, y) = \lim_{h\to 0} \frac{f(h, y) - f(0, y)}{h} = \lim_{h\to 0} \frac{y}{h^2 + y^2} = \frac{1}{y}.$$

$f_x(0, y)$ is not continuous at $y = 0$.

Notice that the second partial derivatives $f_{xy}(0, 0)$ and $f_{yx}(0, 0)$ do not exist.

1.5 Increment Theorem

In order to see the connection between continuity of a function and the partial derivatives, the associated geometry may help.

Let S be the surface $z = f(x, y)$, where f_x, f_y are continuous on the region D, the domain of f. Let $(a, b) \in D$. Let C_1 and C_2 be the curves of intersection of the planes $x = a$ and of $y = b$ with S.