Calculus III

Multivariate Calculus

Lecture Notes

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Contents

Preface

This booklet contains our notes for courses *Math 251 - Calculus III* at Simon Fraser University. Students are expected to use this booklet during each lecture by follow along with the instructor, filling in the details in the blanks provided, during the lecture.

Definitions of terms are stated in orange boxes and theorems appear in blue boxes.

Next to some examples you'll see [link to applet]. The link will take you to an online interactive applet to accompany the example - just like the ones used by your instructor in the lecture. Clicking the link above will take you to the following website containing all the applets:

http://www.sfu.ca/ jtmulhol/calculus-applets/html/appletsforcalculus.html

Try it now.

No project such as this can be free from errors and incompleteness. We will be grateful to everyone who points out any typos, incorrect statements, or sends any other suggestion on how to improve this manuscript.

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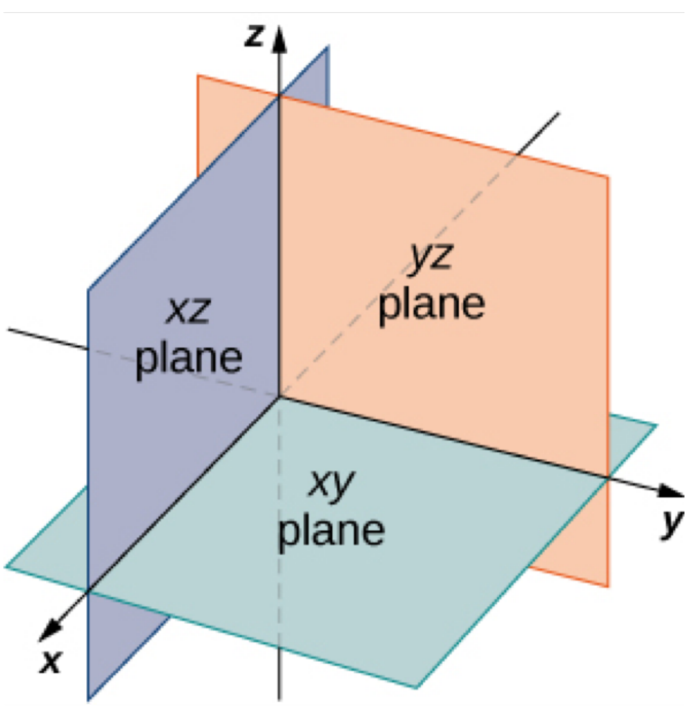
Greek Alphabet

lower case	capital	name	pronunciation	lower case	capital	name	pronunciation
α	A	alpha	(al-fah)	ν	N	nu	(new)
β	B	beta	(bay-tah)	ξ	Ξ	xi	(zie)
γ	Γ	gamma	(gam-ah)	o	O	omicron	(om-e-cron)
δ	Δ	delta	(del-ta)	π	Π	pi	(pie)
ε	E	epsilon	(ep-si-lon)	ρ	P	rho	(roe)
ζ	Z	zeta	(zay-tah)	σ	Σ	sigma	(sig-mah)
η	H	eta	(ay-tah)	τ	T	tau	(taw)
θ	Θ	theta	(thay-tah)	υ	Υ	upsilon	(up-si-lon)
ι	I	iota	(eye-o-tah)	ϕ	Φ	phi	(fie)
κ	K	kappa	(cap-pah)	χ	X	chi	(kie)
λ	Λ	lambda	(lamb-dah)	ψ	Ψ	psi	(si)
μ	M	mu	(mew)	ω	Ω	omega	(oh-may-gah)

Part 12

Vectors and the Geometry of Space



12.1 Three-Dimensional Coordinate System

1. **Two Problems.**

   (a) You wish to drill a hole in a sphere, removing points which lie within a circular cylinder whose axis goes through the center of the sphere. Suppose the sphere has radius 1. What should be the radius of the hole so that exactly half of the volume of the sphere is removed?

   (b) The depth of a lake in the xy-plane is equal to

   $$f(x, y) = 32 - x^2 - 4x - 4y^2$$

   meters.

   i. Sketch the shoreline of the lake in the xy-plane.

   ii. Dr. J is in the water at the point $(-1, 1)$. Find a unit vector which indicates in which direction he should swim in order to:

      A. stay at a constant depth;

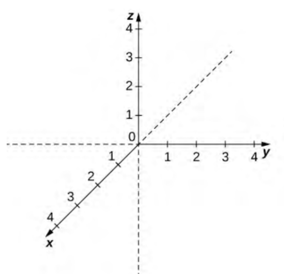
      B. increase his depth as rapidly as possible (i.e., be most likely to drown.)

2. **Rectangular Coordinates In Space.** A point in space is determined by giving its location relative to three mutually perpendicular axes that pass through the origin O.

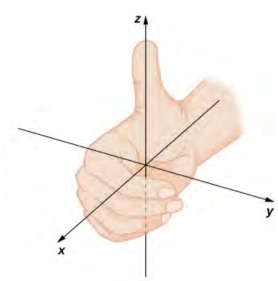
3. **Vocabulary.**

   (a) **Origin**

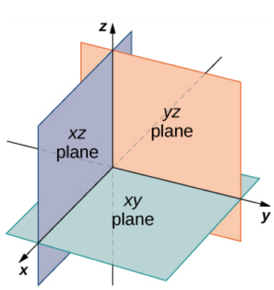
   (b) **Coordinate axes**



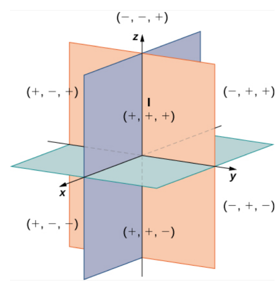
   (c) **Right-hand rule**



   (d) **Coordinate planes**



   (e) **Octants**



   (f) **Coordinates**

   (g) **Projection**

   (h) **Three-dimensional rectangular coordinate system**

4. **Examples.**

   (a) Which surface in $\mathbb{R}^3$ is represented by the equation $x = 2$?

   (b) Which surface in $\mathbb{R}^3$ is represented by the equation $y = \sqrt{x}$?

5. **Distance Formula.** The distance between two points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is given by

$$|P_1P_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}.$$

6. **Example.** Find the distance between $P_1(1, 5, 2)$ and $P_2(2, -3, -3)$

7. **Example.** Given a fixed point $C(h, k, l)$ and a number $r > 0$, find an equation of the sphere with radius r and center C.

8. **Example.** Show that the equation represents a sphere, and find its center and radius.

$$4x^2 + 4y^2 + 4z^2 - 8x + 16y = 1.$$

9. Additional Notes

12.2 Vectors

1. **Puzzle.** The arrows in this grid go in clockwise spiral starting from the top left corner. In which direction should the missing arrow point?

$\uparrow$	$\leftarrow$	$\downarrow$	$\uparrow$	$\rightarrow$
$\uparrow$	$\rightarrow$	$\uparrow$	$\uparrow$	$\uparrow$
$\downarrow$	$\uparrow$		$\leftarrow$	$\uparrow$
$\leftarrow$	$\rightarrow$	$\uparrow$	$\downarrow$	$\leftarrow$
$\uparrow$	$\uparrow$	$\rightarrow$	$\uparrow$	$\downarrow$

2. **"Help".** My house is exactly 500 m away from the bus stop.

3. **Definition.** A **vector** $\vec{\mathbf{v}}$ is a quantity with both a **magnitude** and a **direction**. In the Cartesian plane a vector $\vec{\mathbf{v}}$ can be given by an ordered pair of real numbers that has form

$$\langle a, b\rangle.$$

We write

$$\vec{\mathbf{v}} = \langle a, b\rangle$$

and call a and b the **components** of the vector $\vec{\mathbf{v}}$.

The **length** of the vector $\vec{\mathbf{v}} = \langle a, b\rangle$ is defined as follows

$$v = |\vec{\mathbf{v}}| = |\langle a, b\rangle| = \sqrt{a^2 + b^2}.$$

4. **Example:** Find a vector $\vec{\mathbf{v}} = \langle a, b\rangle$ that is represented by the directed line segment $\vec{RS}$ if $R = (5, 10)$ and $S = (-5, -10)$. Sketch both $\vec{RS}$ and the position vector of the point $P(a, b)$. Find $|\vec{\mathbf{v}}|$.

5. **Zero Vector.** The only vector with length zero is the **zero vector** with both components zero, denoted by

$$\vec{\mathbf{0}} = \langle 0, 0\rangle.$$

The zero vector is unique in that has no specific direction.

6. **Algebraic Operations with Vectors.**

   (a) **Equality of Vectors:** The two vectors $\vec{\mathbf{u}} = \langle u_1, u_2 \rangle$ and $\vec{\mathbf{v}} = \langle v_1, v_2 \rangle$ are **equal** provided that

   $$u_1 = v_1 \text{ and } u_2 = v_2.$$

   (b) **Addition of Vectors:** The **sum**

   $$\vec{\mathbf{u}} + \vec{\mathbf{v}}$$

   of the two vectors $\vec{\mathbf{u}} = \langle u_1, u_2 \rangle$ and $\vec{\mathbf{v}} = \langle v_1, v_2 \rangle$ is the vector

   $$\vec{\mathbf{u}} + \vec{\mathbf{v}} = \langle u_1 + v_1, u_2 + v_2 \rangle.$$

   (c) **Multiplication of a Vector by a Scalar:** If $\vec{\mathbf{u}} = \langle u_1, u_2 \rangle$ and c is a real number, then the **scalar multiple**

   $$c\vec{\mathbf{u}}$$

   is the vector

   $$c\vec{\mathbf{u}} = \langle cu_1, cu_2 \rangle.$$

7. **Note:**

   (a)

   $$|c\vec{\mathbf{u}}| = |c| \cdot |\vec{\mathbf{u}}|$$

   (b) We say that two nonzero vectors $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ have

   - The **same direction** if

     $$\vec{\mathbf{u}} = c\vec{\mathbf{v}} \text{ for some } c > 0.$$

   - The **opposite directions** if

     $$\vec{\mathbf{u}} = c\vec{\mathbf{v}} \text{ for some } c < 0.$$

   - The **difference** of the two vectors $\vec{\mathbf{u}} = \langle u_1, u_2 \rangle$ and $\vec{\mathbf{v}} = \langle v_1, v_2 \rangle$ is the vector

     $$\vec{\mathbf{u}} - \vec{\mathbf{v}} = \vec{\mathbf{u}} + (-\vec{\mathbf{v}}) = \langle u_1 - v_1, u_2 - v_2 \rangle.$$

8. **Example** Suppose that $\vec{\mathbf{u}} = \langle -2, 1\rangle$ and $\vec{\mathbf{v}} = \langle -2, -1\rangle$.
   (a) Is it true that $\vec{\mathbf{u}} = \vec{\mathbf{v}}$?
   (b) Find $\vec{\mathbf{u}} + \vec{\mathbf{v}}$, $2\vec{\mathbf{u}}$, -3$\vec{\mathbf{v}}$, and $2\vec{\mathbf{u}} - 3\vec{\mathbf{v}}$.

9. **Unit Vectors.** A **unit vector** is a vector of length 1. If $\vec{\mathbf{a}} \neq \vec{\mathbf{0}}$ then

$$\vec{\mathbf{u}} = \frac{1}{|\vec{\mathbf{a}}|}\vec{\mathbf{a}} = \frac{\vec{\mathbf{a}}}{|\vec{\mathbf{a}}|}$$

   is the unit vector with the same direction as $\vec{\mathbf{a}}$.
   Two particular unit vectors play a special role

$$\hat{\mathbf{i}} = \langle 1, 0\rangle \text{ and } \hat{\mathbf{j}} = \langle 0, 1\rangle.$$

10. **Examples.**
   (a) Let $\vec{\mathbf{a}} = \langle 3, -2\rangle$. Express $\vec{\mathbf{a}}$ in terms of $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$.
   (b) Let $\vec{\mathbf{a}} = 3\hat{\mathbf{i}} - 2\hat{\mathbf{j}}$ and $\vec{\mathbf{b}} = -2\hat{\mathbf{i}} + 5\hat{\mathbf{j}}$. Express $\vec{\mathbf{a}} + \vec{\mathbf{b}}$ in terms of $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$.

11. **Vector in Space.** A vector $\vec{\mathbf{v}}$ in space is an ordered triple of real numbers that has form

$$\langle a, b, c\rangle.$$

   We write

$$\vec{\mathbf{v}} = \langle a, b, c\rangle.$$

   Its **length** (or **magnitude**) is given by

$$|\vec{\mathbf{v}}| = \sqrt{a^2 + b^2 + c^2}.$$

12. **Example:** Find a vector $\vec{\mathbf{v}} = \langle a, b, c\rangle$ that is represented by the directed line segment $\vec{RS}$ if $R = (5, 1, 10)$ and $S = (0, 5, -10)$. Sketch both $\vec{RS}$ and the position vector of the point $P(a, b, c)$. Find $|\vec{\mathbf{v}}|$.

13. **Algebraic Operations.** Let $\vec{\mathbf{u}} = \langle u_1, u_2, u_3\rangle$, $\vec{\mathbf{v}} = \langle v_1, v_2, v_3\rangle$, and let c be a scalar. Then

    (a) $\vec{\mathbf{u}} = \vec{\mathbf{v}}$ provided
    $$u_1 = v_1 \text{ and } u_2 = v_2 \text{ and } u_3 = v_3.$$

    (b)
    $$\vec{\mathbf{u}} + \vec{\mathbf{v}} = \langle u_1 + v_1, u_2 + v_2, u_3 + v_3\rangle.$$

    (c)
    $$c\vec{\mathbf{u}} = \langle cu_1, cu_2, cu_3\rangle.$$

14. **Example.** If $\vec{\mathbf{a}} = \langle 2, 3, 5\rangle$ and $\vec{\mathbf{b}} = \langle -5, 3, 0\rangle$, find
$$|\vec{\mathbf{a}} + \vec{\mathbf{b}}|,\ |-3\vec{\mathbf{a}}|,\ 2\vec{\mathbf{a}} - 3\vec{\mathbf{b}}.$$

15. **Properties of Vector Operations.** Let $\vec{\mathbf{u}}$, $\vec{\mathbf{v}}$, $\vec{\mathbf{w}}$, be vectors and a, b be scalars.

   (a) $\vec{\mathbf{u}} + \vec{\mathbf{v}} = \vec{\mathbf{v}} + \vec{\mathbf{u}}$
   (b) $\vec{\mathbf{u}} + \vec{\mathbf{0}} = \vec{\mathbf{u}}$
   (c) $0\vec{\mathbf{u}} = \vec{\mathbf{0}}$
   (d) $(ab)\vec{\mathbf{u}} = a(b\vec{\mathbf{u}}) = b(a\vec{\mathbf{u}})$
   (e) $(a+b)\vec{\mathbf{u}} = a\vec{\mathbf{u}} + b\vec{\mathbf{u}}$
   (f) $(\vec{\mathbf{u}} + \vec{\mathbf{v}}) + \vec{\mathbf{w}} = \vec{\mathbf{u}} + (\vec{\mathbf{v}} + \vec{\mathbf{w}})$
   (g) $\vec{\mathbf{u}} + (-\vec{\mathbf{u}}) = \vec{\mathbf{0}}$
   (h) $1\vec{\mathbf{u}} = \vec{\mathbf{u}}$
   (i) $a(\vec{\mathbf{u}} + \vec{\mathbf{v}}) = a\vec{\mathbf{u}} + a\vec{\mathbf{v}}$

16. **Three Basic Unit Vectors.**

$$
\begin{aligned}
\hat{\mathbf{i}} &= \langle 1,0,0\rangle \\
\hat{\mathbf{j}} &= \langle 0,1,0\rangle \\
\hat{\mathbf{k}} &= \langle 0,0,1\rangle
\end{aligned}
$$

For the vector $\vec{\mathbf{a}} = \langle a_1, a_2, a_3\rangle$

$$\vec{\mathbf{a}} = a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}}.$$

17. **Examples.**

   (a) Write $\vec{\mathbf{a}} = \langle -2,3,5\rangle$ in terms of $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$.
   (b) If

$$\vec{\mathbf{a}} = -2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 5\hat{\mathbf{k}}$$

   and

$$\vec{\mathbf{b}} = -3\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 5\hat{\mathbf{k}}$$

   write $\vec{\mathbf{a}} - \vec{\mathbf{b}}$ in terms of $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$.

18. Additional Notes

12.3 The Dot Product

1. **Quote.** "No more fiction, for now we calculate; but that we may calculate, we had to make fiction first."

   (Friedrich Nietzsche, German philosopher, 1844-1900)

2. **Problem.** Find the angle between $\vec{\mathbf{a}} = \langle 3, 0, 4\rangle$ and $\vec{\mathbf{b}} = \langle 0, 4, 3\rangle$.

3. **Dot product of Two Vectors.** The **dot product** of two vectors $\vec{\mathbf{a}} = \langle a_1, a_2, a_3\rangle$ and $\vec{\mathbf{b}} = \langle b_1, b_2, b_3\rangle$ is defined as
$$\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = a_1b_1 + a_2b_2 + a_3b_3.$$

4. **Example.** Let $\vec{\mathbf{a}} = \langle 5, 7, 0\rangle$ and $\vec{\mathbf{b}} = \langle -3, 4, 2\rangle$. Find $\vec{\mathbf{a}} \cdot \vec{\mathbf{b}}$.

5. **Properties of the Dot Product.**
$$\begin{aligned}
\vec{\mathbf{a}} \cdot \vec{\mathbf{a}} &= |\vec{\mathbf{a}}|^2 \\
\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} &= \vec{\mathbf{b}} \cdot \vec{\mathbf{a}} \\
\vec{\mathbf{a}} \cdot (\vec{\mathbf{b}} + \vec{\mathbf{c}}) &= \vec{\mathbf{a}} \cdot \vec{\mathbf{b}} + \vec{\mathbf{a}} \cdot \vec{\mathbf{c}} \\
(c\vec{\mathbf{a}}) \cdot \vec{\mathbf{b}} &= c(\vec{\mathbf{a}} \cdot \vec{\mathbf{b}}) = \vec{\mathbf{a}} \cdot (c\vec{\mathbf{b}}) \\
\vec{\mathbf{0}} \cdot \vec{\mathbf{a}} &= 0
\end{aligned}$$

6. **Geometric Interpretation.** If θ is the angle between the vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ then

$$\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = |\vec{\mathbf{a}}||\vec{\mathbf{b}}| \cos \theta.$$

7. **Corollary.** If θ is the angle between the nonzero vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ then

$$\cos \theta = \frac{\vec{\mathbf{a}} \cdot \vec{\mathbf{b}}}{|\vec{\mathbf{a}}||\vec{\mathbf{b}}|}.$$

8. **Example.** Find the angle between $\vec{\mathbf{a}} = \langle 3, 0, 4 \rangle$ and $\vec{\mathbf{b}} = \langle 0, 4, 3 \rangle$.

9. **Must Know!** Two nonzero vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ are **perpendicular** if and only if

$$\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = 0.$$

10. **Example.** Check if the vectors $\vec{\mathbf{a}} = \langle 3, 4, 5 \rangle$ and $\vec{\mathbf{b}} = \langle 2, -4, 2 \rangle$ are perpendicular.

11. **Direction Angles.** The **direction angles** of a nonzero vector $\vec{a}$ are the angles α, β, and γ that $\vec{a}$ makes with the positive x-, y-, and z-axes.

12. **Problem.** Let α, β, and γ be the direction angles of a nonzero vector $\vec{a}$. Find the components of the vector $\vec{a}$.

13. **Example.** Find the direction angles of the vector $\vec{b} = \langle 2, -4, 2 \rangle$.

14. **Projections.** The scalar projection of $\vec{\mathbf{b}}$ onto $\vec{\mathbf{a}}$ is defined as

$$\text{comp}_{\vec{\mathbf{a}}}\vec{\mathbf{b}} = \frac{\vec{\mathbf{a}} \cdot \vec{\mathbf{b}}}{|\vec{\mathbf{a}}|}$$

The vector projection of $\vec{\mathbf{b}}$ onto $\vec{\mathbf{a}}$ is defined as

$$\text{proj}_{\vec{\mathbf{a}}}\vec{\mathbf{b}} = \frac{\vec{\mathbf{a}} \cdot \vec{\mathbf{b}}}{|\vec{\mathbf{a}}|^2}\,\vec{\mathbf{a}}$$

15. **Example.** Find the scalar projection and the vector projection of $\vec{\mathbf{b}} = \langle 3, -2, 4 \rangle$ onto $\vec{\mathbf{a}} = \langle 2, 1, 1 \rangle$.

16. Additional Notes

12.4 The Cross Product

1. **Quote.** “When the product is right, you don’t have to be a great marketer.”

   (Lido Anthony ”Lee” Iacocca, American industrialist, 1924 -)

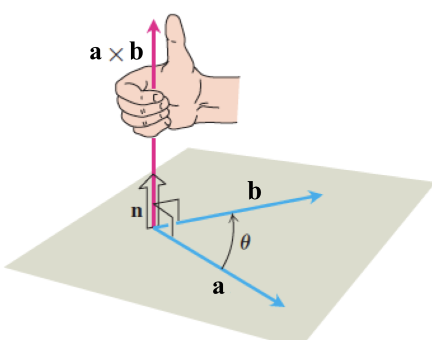
2. **Problem.** Find the area of the parallelogram determined by vectors $\vec{a} = \langle 3, 0, 4\rangle$ and $\vec{b} = \langle 0, 4, 3\rangle$.

3. **Definition.** The **cross product** (or **vector product**) of two vectors $\vec{a} = \langle a_1, a_2, a_3\rangle$ and $\vec{b} = \langle b_1, b_2, b_3\rangle$ is defined by

$$\vec{a} \times \vec{b} = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1\rangle.$$

   As we will prove shortly, the **cross product** of $\vec{a}$ and $\vec{b}$ has two defining properties: it is the vector that

   - is orthogonal to both $\vec{a}$ and $\vec{b}$ with direction given by the right-hand rule, and
   - has magnitude equal to the area of the parallelogram determined by $\vec{a}$ and $\vec{b}$.



4. **Example.** Let $\vec{a} = \langle 5, 7, 0\rangle$ and $\vec{b} = \langle -3, 4, 2\rangle$. Find

$$\vec{a} \times \vec{b}.$$

5. **Example.** Show that $\vec{a} \times \vec{a} = \vec{0}$ for any vector $\vec{a}$ in V_3.

6. **Must Know!** The cross product

$$\vec{\mathbf{a}} \times \vec{\mathbf{b}}$$

is perpendicular to both $\vec{\mathrm{a}}$ and $\vec{\mathrm{b}}$.

7. **Example.** Find a vector perpendicular to the plane that passes through the points $(1,2,3)$, $(1,0,1)$, and $(2,1,1)$.

8. **Fact 1.** If θ is the angle between $\vec{\mathrm{a}}$ and $\vec{\mathrm{b}}$ (so $0 \leq \theta \leq \pi$), then

$$|\vec{\mathbf{a}} \times \vec{\mathbf{b}}| = |\vec{\mathbf{a}}||\vec{\mathbf{b}}| \sin \theta.$$

9. **Fact 2.** Two nonzero vectors $\vec{\mathrm{a}}$ and $\vec{\mathrm{b}}$ are parallel if and only if

$$\vec{\mathbf{a}} \times \vec{\mathbf{b}} = \vec{\mathbf{0}}.$$

10. **Fact 3.** The length of the cross product $\vec{\mathrm{a}} \times \vec{\mathrm{b}}$ is equal to the area of the parallelogram determined by $\vec{\mathrm{a}}$ and $\vec{\mathrm{b}}$.

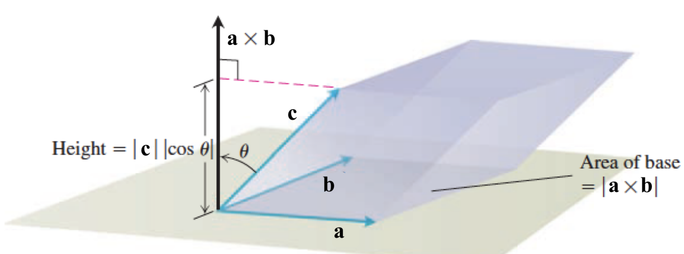
11. **Example.** Find the area of the parallelogram determined by vectors $\vec{a} = \langle 3, 0, 4 \rangle$ and $\vec{b} = \langle 0, 4, 3 \rangle$.

12. **Properties of the Cross Product.**

$$\begin{aligned}
\vec{a} \times \vec{b} &= -\vec{b} \times \vec{a} \\
(c\vec{a}) \times \vec{b} &= c(\vec{a} \times \vec{b}) = \vec{a} \times (c\vec{b}) \\
\vec{a} \times (\vec{b} + \vec{c}) &= \vec{a} \times \vec{b} + \vec{a} \times \vec{c} \\
(\vec{a} + \vec{b}) \times \vec{c} &= \vec{a} \times \vec{c} + \vec{b} \times \vec{c} \\
\vec{a} \cdot (\vec{b} \times \vec{c}) &= (\vec{a} \times \vec{b}) \cdot \vec{c} \\
\vec{a} \times (\vec{b} \times \vec{c}) &= (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}
\end{aligned}$$

13. **Triple Product.** The volume of the parallelepiped determined by the vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ is the magnitude of their triple product:

$$V = |\vec{c} \cdot (\vec{a} \times \vec{b})|.$$



14. **Example.** Use the scalar triple product to determine whether the points $P(1,0,1)$, $Q(2,4,6)$, $R(3,-1,2)$, and $S(6,2,8)$ lie in the same plane.

15. **Example.** Let P be a point not on the line L that passes through the points Q and R. Show that the distance d from the point P to the line L is

$$d = \frac{|\vec{\mathbf{a}} \times \vec{\mathbf{b}}|}{|\vec{\mathbf{a}}|}$$

where $\vec{\mathbf{a}} = \overrightarrow{QR}$ and $\vec{\mathbf{b}} = \overrightarrow{QP}$.

16. Additional Notes

12.5 Equations of Lines and Planes

1. **Quote.** "As far as the laws of mathematics refer to reality, they are not certain, and as far as they are certain, they do not refer to reality."

   (Albert Einstein, German-born theoretical physicist, 1879–1955)

2. **Reminder.** A straight line is determined by any two points that lie on it.

3. **Problem.** Find an equation of a line in space that passes through two given points.

4. **Three Equations for a Line.**

   (a) The **vector equation** of the line passing through the point given by its position vector $\vec{\mathbf{r}}_0$ and parallel to the given vector $\vec{\mathbf{v}}$ is:

   $$\vec{\mathbf{r}} = \vec{\mathbf{r}}_0 + t\vec{\mathbf{v}}$$

   (b) The **parametric equations** of the line that passes through the point (x_0, y_0, z_0) and is parallel to the vector $\vec{\mathbf{v}} = \langle a, b, c\rangle$ are:

   $$x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + ct.$$

   (c) The **symmetric equations** for a line that passes through a point (x_0, y_0, z_0) and is parallel to the vector $\vec{\mathbf{v}} = \langle a, b, c\rangle$ are

   $$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}.$$

5. **Examples.**

(a) Write parametric equations of the line that passes through the points $P_1(1, 0, 2)$ and $P_2(-2, -1, 0)$.

(b) Determine whether the two lines L_1 and L_2 are parallel, skew, or intersecting.

i.

$$\begin{aligned} L_1 &: \ x = 2t, \ \ y = -1 + 2t, \ \ z = 3 + 3t \\ L_2 &: \ x = 5 + 3t, \ \ y = 1 + 2t, \ \ z = 4t \end{aligned}$$

ii.

$$\begin{aligned} L_1 &: \ x = 4 + 2t, \ \ y = 3 + 2t, \ \ z = 4 + 3t \\ L_2 &: \ x = 3t, \ \ y = 1 + 2t, \ \ z = -1 + 4t \end{aligned}$$

(c) Find parametric and symmetric equations of the line that passes through the points $(1, 2, 3)$ and $(-2, 1, -2)$. At which point does this line intersect the xz-plane?

Planes:

6. **Reminder.** A plane is determined by any three points that lie on it.

7. **Problem.** Find an equation of a plane in space that passes through three given points:

$$P_1(a_1, b_1, c_1), \quad P_2(a_2, b_2, c_2), \quad P_3(a_3, b_3, c_3).$$

8. **Two Equations for a Plane.**

   (a) The **vector equation** of the plane passing through the point given by its position vector $\vec{\mathbf{r}}_0$ and normal to the given vector $\vec{\mathbf{n}}$:

   $$\vec{\mathbf{n}} \cdot (\vec{\mathbf{r}} - \vec{\mathbf{r}}_0) = 0$$

   (b) The **scalar equation** of the plane that passes through the point (x_0, y_0, z_0) and is perpendicular to the vector $\vec{\mathbf{n}} = \langle a, b, c \rangle$:

   $$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0,$$

   or equivalently by

   $$ax + by + cz = d$$

   where $d = -(ax_0 + by_0 + cz_0)$.

9. **Examples.** Write an equation of the indicated plane.

   (a) Through $P(5, 7, -6)$ and parallel to xz-plane.

   (b) Through $P(5, 1, 4)$ and parallel to the plane $2x + 3y + 4z = 0$.

   (c) Through $P(2, 4, 6)$ and contains the line

$$x = 7 - 3t, \;\; y = 3 + 4t, \;\; z = 5 + 2t.$$

10. **Examples.**

(a) Find the angle between the following planes.

$$3x - 2y + z = 5 \text{ and } 2x + 3y - 2z = 3.$$

(b) Find parametric equations for the line of intersection.

(c) Find the distance from the point $P(1, 2, 3)$ to the plane $3x - 2y + z = 5$.

11. Additional Notes

12.6 Cylinders and Quadric Surfaces

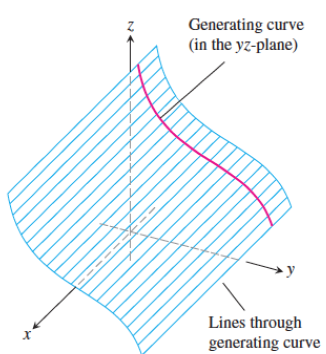
1. **Puzzle.** Which shape is the odd one out?

2. **Traces in the Coordinate Planes.** Describe and sketch the graphs of the equations given by

   (a) $x + 2y + 3z = 4$

   (b) $z = x^2 + y^2$.

3. **Cylinders.** A cylinder is a surface that consists of lines (called **rulings**) that are parallel to the given line and pass through a given plane curve.



4. **Examples.** Describe and sketch the graphs of the equations given by

   (a) $x^2 + y^2 = 4$

   (b) $y = z^2$.

5. **Quadratic Surfaces.** A **quadratic surface** is the graph of a second-degree equation in three variables x. y, and z:

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0$$

where A, B, C,..., J are constants, but by translation and rotation it can be brought to one of the the two standard forms:

$$Ax^2 + By^2 + Cz^2 + J = 0$$

or

$$Ax^2 + By^2 + Iz = 0.$$

6. **Examples.** Use traces to sketch the surfaces:

   (a) $x^2 + y^2 + z^2 = 4$

   (b) $y = x^2 + z^2$

   (c) $\dfrac{x^2}{4} + \dfrac{y^2}{9} + \dfrac{z^2}{16} = 1$

(d) $y = x^2 - z^2$

(e) $x^2 + y^2 - z^2 = 1$

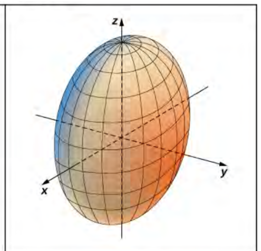
7. Graphs of common quadric surfaces

Ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

Traces
In plane $z = p$: an ellipse
In plane $y = q$: an ellipse
In plane $x = r$: an ellipse

If $a = b = c$, then this surface is a sphere.

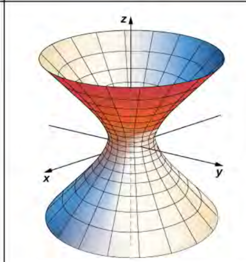


Hyperboloid of One Sheet

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

Traces
In plane $z = p$: an ellipse
In plane $y = q$: a hyperbola
In plane $x = r$: a hyperbola

In the equation for this surface, two of the variables have positive coefficients and one has a negative coefficient. The axis of the surface corresponds to the variable with the negative coefficient.

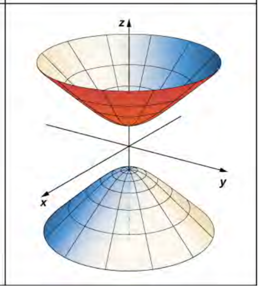


Hyperboloid of Two Sheets

$$\frac{z^2}{c^2} - \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Traces
In plane $z = p$: an ellipse or the empty set (no trace)
In plane $y = q$: a hyperbola
In plane $x = r$: a hyperbola

In the equation for this surface, two of the variables have negative coefficients and one has a positive coefficient. The axis of the surface corresponds to the variable with a positive coefficient. The surface does not intersect the coordinate plane perpendicular to the axis.

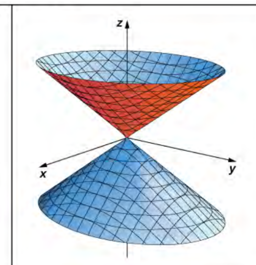


Elliptic Cone

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$$

Traces
In plane $z = p$: an ellipse
In plane $y = q$: a hyperbola
In plane $x = r$: a hyperbola
In the xz – plane: a pair of lines that intersect at the origin
In the yz – plane: a pair of lines that intersect at the origin

The axis of the surface corresponds to the variable with a negative coefficient. The traces in the coordinate planes parallel to the axis are intersecting lines.

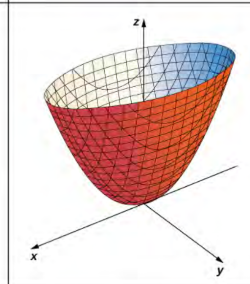


Elliptic Paraboloid

$$z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

Traces
In plane $z = p$: an ellipse
In plane $y = q$: a parabola
In plane $x = r$: a parabola

The axis of the surface corresponds to the linear variable.

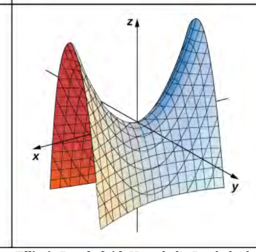


Hyperbolic Paraboloid

$$z = \frac{x^2}{a^2} - \frac{y^2}{b^2}$$

Traces
In plane $z = p$: a hyperbola
In plane $y = q$: a parabola
In plane $x = r$: a parabola

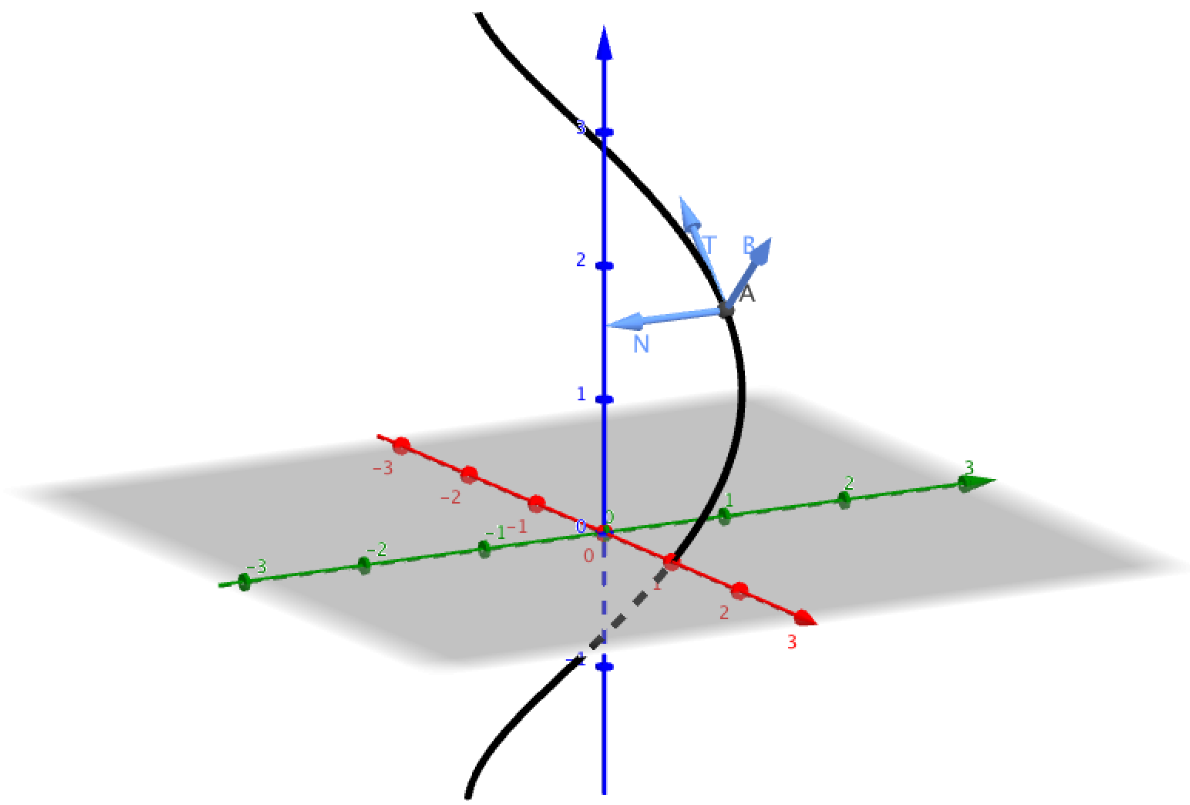
The axis of the surface corresponds to the linear variable.



8. Additional Notes

Part 13

Vector Functions



13.1 Vector Functions and Space Curves

1. **Puzzle.** Using six matches of equal length, form four equilateral triangles.

2. **Problem.** Describe the motion of the particle whose coordinates at time t are

$$x = \cos t \text{ and } y = 0 \text{ and } z = \sin t.$$

3. **Generalization.** A **parametric curve** C in space is a triple of functions

$$x = f(t),\ y = g(t),\ z = h(t)$$

that give x, y, and z as continuous functions of the real number t (the parameter) in some interval I.

4. **Step Further.** The changing location of a point moving along the parametric curve can be described by giving its **position vector**

$$\vec{\mathbf{r}}(t) = x(t)\hat{\mathbf{i}} + y(t)\hat{\mathbf{j}} + z(t)\hat{\mathbf{k}} = \langle x(t), y(t), z(t)\rangle.$$

5. **Example.** Describe the motion of the particle whose position vector is

$$\vec{\mathbf{r}}(t) = (\cos t)\hat{\mathbf{i}} + (\sin t)\hat{\mathbf{j}} + t\hat{\mathbf{k}}.$$

6. **Vector-Valued Function.** Any function that associates with the number t the vector $\vec{\mathbf{r}}(t)$ is called a **vector-valued function**.

   (a) The **limit** of a vector-valued function $\vec{\mathbf{r}} = \langle f, g, h\rangle$ is defined by

$$\begin{aligned}\lim_{t\to a} \vec{\mathbf{r}}(t) &= \langle \lim_{t\to a} f(t), \lim_{t\to a} g(t), \lim_{t\to a} h(t)\rangle \\ &= \hat{\mathbf{i}}\left(\lim_{t\to a} f(t)\right) + \hat{\mathbf{j}}\left(\lim_{t\to a} g(t)\right) + \hat{\mathbf{k}}\left(\lim_{t\to a} h(t)\right)\end{aligned}$$

   (b) We say that $\vec{\mathbf{r}} = \vec{\mathbf{r}}(t)$ is **continuous** at a provided that

$$\lim_{t\to a} \vec{\mathbf{r}}(t) = \vec{\mathbf{r}}(a).$$

7. **Example.** Find the vector equation and parametric equation for the line segment that joins points $P(1, -1, 2)$ and $Q(4, 1, 7)$.

8. **Example.** Find a vector function that represents the curve of intersection of the paraboloid $z = 4x^2 + y^2$ and the parabolic cylinder $y = x^2$.

9. Additional Notes

13.2 Derivatives and Integrals of Vector Functions

1. **Quote.** ”Why should I refuse a good dinner simply because I don’t understand the digestive processes involved?”

   (Oliver Heaviside, A self-taught English electrical engineer, mathematician, and physicist, 1850-1925)

2. **Reminder.** Any function that associates with the number t the vector $\vec{\mathbf{r}}(t)$ is called a **vector-valued function**.

3. **Reminder.** The **limit** of a vector-valued function $\vec{\mathbf{r}} = \langle f, g, h\rangle$ is defined by

$$\begin{aligned}\lim_{t\to a}\vec{\mathbf{r}}(t) &= \langle \lim_{t\to a} f(t), \lim_{t\to a} g(t), \lim_{t\to a} h(t)\rangle \\ &= \hat{\mathbf{i}}\left(\lim_{t\to a} f(t)\right) + \hat{\mathbf{j}}\left(\lim_{t\to a} g(t)\right) + \hat{\mathbf{k}}\left(\lim_{t\to a} h(t)\right)\end{aligned}$$

4. **Derivative.** The **derivative** $\vec{\mathbf{r}}'(t)$ of the vector-valued function $\vec{\mathbf{r}}(t)$ is defined by

$$\frac{d\vec{\mathbf{r}}}{dt} = \vec{\mathbf{r}}'(t) = \lim_{h\to 0}\frac{\vec{\mathbf{r}}(t+h) - \vec{\mathbf{r}}(t)}{h}.$$

5. **Componentwise Differentiation.** Suppose that

$$\vec{\mathbf{r}}(t) = \langle f(t), g(t), h(t)\rangle = f(t)\hat{\mathbf{i}} + g(t)\hat{\mathbf{j}} + h(t)\hat{\mathbf{k}},$$

   where f, g, and h are differentiable functions. Then

$$\vec{\mathbf{r}}'(t) = f'(t)\hat{\mathbf{i}} + g'(t)\hat{\mathbf{j}} + h'(t)\hat{\mathbf{k}}.$$

6. **Example.** Find $\vec{\mathbf{r}}'(t)$ if

$$\vec{\mathbf{r}}(t) = \cos 2t\hat{\mathbf{i}} + te^t\hat{\mathbf{j}} + \frac{\ln t}{t^2}\hat{\mathbf{k}}.$$

7. **Tangent Vector.**

(a) We call $\vec{\mathbf{r}}'(t)$ a **tangent vector** to the curve C at the point $\vec{\mathbf{r}}(t)$ provided that $\vec{\mathbf{r}}'(t)$ exists and is non-zero there.

(b) The **tangent line** to C at P is the line through P parallel to the tangent vector $\vec{\mathbf{r}}'(t)$.

(c) The **unit tangent vector** is the vector

$$\mathbf{T}(t) = \frac{\vec{\mathbf{r}}'(t)}{|\vec{\mathbf{r}}'(t)|}$$

8. **Example.** Find parametric equations of the line tangent to the helix

$$\vec{\mathbf{r}}(t) = \hat{\mathbf{i}}\cos t + \hat{\mathbf{j}}\sin t + t\hat{\mathbf{k}}$$

at the point $P(-1, 0, \pi)$.

9. **Smooth Curve.** A curve given by a vector function $\vec{\mathbf{r}}(t)$ on an interval I is called **smooth** if $\vec{\mathbf{r}}\,'$ is continuous and $\vec{\mathbf{r}}\,'(t) \neq \mathbf{0}$ (except possibly at any endpoint of I.)

10. **Example.** Determine if the curve

$$\vec{\mathbf{r}}(t) = \langle 1 + t^2, 1 - t^2, \cos t \rangle$$

is smooth.

11. **Differentiation Rules.** Suppose that $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ are differentiable vector functions, c is a scalar, and f is a real-valued function. Then

    (a) $\dfrac{d}{dt}[\vec{\mathbf{u}}(t) + \vec{\mathbf{v}}(t)] = \vec{\mathbf{u}}'(t) + \vec{\mathbf{v}}'(t)$ **(sum rule)**

    (b) $\dfrac{d}{dt}[c\vec{\mathbf{u}}(t)] = c\vec{\mathbf{u}}'(t)$ **(scalar multiple rule)**

    (c) $\dfrac{d}{dt}[f(t)\vec{\mathbf{u}}(t)] = f'(t)\vec{\mathbf{u}}(t) + f(t)\vec{\mathbf{u}}'(t)$ **(scalar multiple rule)**

    (d) $\dfrac{d}{dt}[\vec{\mathbf{u}}(t) \cdot \vec{\mathbf{v}}(t)] = \vec{\mathbf{u}}'(t) \cdot \vec{\mathbf{v}}(t) + \vec{\mathbf{u}}(t) \cdot \vec{\mathbf{v}}'(t)$ **(dot product rule)**

    (e) $\dfrac{d}{dt}[\vec{\mathbf{u}}(t) \times \vec{\mathbf{v}}(t)] = \vec{\mathbf{u}}'(t) \times \vec{\mathbf{v}}(t) + \vec{\mathbf{u}}(t) \times \vec{\mathbf{v}}'(t)$ **(cross product rule)**

    (f) $\dfrac{d}{dt}[\vec{\mathbf{u}}(f(t))] = f'(t)\vec{\mathbf{u}}'(f(t))$ **(chain rule)**

12. **Example:** Show that for a differentiable vector function $\vec{\mathbf{r}}(t)$ of constant length that the tangent vector is always orthogonal to the position vector.

13. **Integral.** The **definite integral** of a continuous vector-valued function

$$\vec{\mathbf{r}}(t) = f(t)\hat{\mathbf{i}} + g(t)\hat{\mathbf{j}} + h(t)\hat{\mathbf{k}}$$

is defined by

$$\int_a^b \vec{\mathbf{r}}(t)dt = \left(\int_a^b f(t)dt\right)\hat{\mathbf{i}} + \left(\int_a^b g(t)dt\right)\hat{\mathbf{j}} + \left(\int_a^b h(t)dt\right)\hat{\mathbf{k}}.$$

14. Find $\vec{\mathbf{r}}(t)$ if $\vec{\mathbf{r}}\,'(t) = t^2\,\hat{\mathbf{i}} + \cos t\,\hat{\mathbf{j}} + e^t\,\hat{\mathbf{k}}$ and $\vec{\mathbf{r}}(0) = \hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$.

15. Additional Notes

13.3 Arc Length and Curvature

1. **Quote.** ”To the pure geometer the radius of curvature is an incidental characteristic - like the grin of the Cheshire cat.”

   (Arthur Eddington, British astrophysicist, 1882-1944)

2. **Reminder.** The length of a plane curve C with parametric equations $x = f(t)$, $y(t) = g(t)$, $a \le t \le b$ is given by

$$s = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2} dt.$$

3. **Definition.** Suppose that the curve has the vector equation $\vec{\mathbf{r}}(t) = \langle f(t), g(t), h(t) \rangle$, $a \le t \le b$, where f', g', and h' are continuous. If the curve is traversed exactly once as t increases from a to b then its length is

$$\begin{aligned} s &= \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2} dt \\ &= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt \\ &= \int_a^b |\vec{\mathbf{r}}\,'(t)| dt \end{aligned}$$

4. **Example.** Find the length of the arc of the circular helix with vector equation

$$\vec{\mathbf{r}}(t) = \cos t\hat{\mathbf{i}} + \sin t\hat{\mathbf{j}} + t\hat{\mathbf{k}}$$

   from the point $(1, 0, 0)$ to the point $(1, 0, 2\pi)$.

5. **Arc Length Function.** Suppose that C is a piecewise-smooth curve given by a vector function function

$$\vec{\mathbf{r}}(t) = f(t)\hat{\mathbf{i}} + g(t)\hat{\mathbf{j}} + h(t)\hat{\mathbf{k}},\ a \le t \le b$$

and C is traversed exactly once at t increases from a to b. We define its **arc length function** s by

$$\begin{aligned} s(t) &= \int_a^t |\vec{\mathbf{r}}\,'(u)| du \\ &= \int_a^t \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2 + \left(\frac{dz}{du}\right)^2}\, dt \end{aligned}$$

6. **Fact.**

$$\frac{ds}{dt} = |\vec{\mathbf{r}}\,'(t)|$$

7. **Problem.** Reparametrize the helix $\vec{\mathbf{r}}(t) = \cos t\hat{\mathbf{i}} + \sin t\hat{\mathbf{j}} + t\hat{\mathbf{k}}$ with respect to the arc length measured from $(1, 0, 0)$ in the direction of increasing t.

8. **Curvature.** The curvature of the curve is the magnitude of the change of the unit tangent vector with respect to arc length.

$$\kappa = \left| \frac{d\vec{\mathbf{T}}}{ds} \right|$$

9. **Example.**

(a) Show that

$$\kappa(t) = \frac{\left|\vec{\mathbf{T}}\,'(t)\right|}{|\vec{\mathbf{r}}\,'(t)|}$$

(b) Find the curvature of the helix

$$\vec{\mathbf{r}}(t) = \hat{\mathbf{i}}\cos t + \hat{\mathbf{j}}\sin t + t\hat{\mathbf{k}}$$

at the point $P(-1, 0, \pi)$.

10. **Theorem.** The curvature of the curve given by the vector function $\vec{\mathbf{r}}$ is

$$\kappa(t) = \frac{|\vec{\mathbf{r}}\,'(t) \times \vec{\mathbf{r}}\,''(t)|}{|\vec{\mathbf{r}}\,'(t)|^3}$$

11. **Example.** Find the curvature of the curve

$$\vec{\mathbf{r}}(t) = \langle 3t, 4\sin t, 4\cos t\rangle.$$

12. **Fact.** If C is a plane curve with equation $y = f(x)$ then

$$\kappa(x) = \frac{|f''(x)|}{[1 + (f'(x))^2]^{3/2}}.$$

13. **Example.** At what point does the curve $y = e^x$ have the maximum curvature? What happens to the curvature as $x \to \infty$?

14. Additional Notes

13.4 Motion in Space: Velocity and Acceleration

1. **Quote.** ”We believe that mere movement is life, and that the more velocity it has, the more it expresses vitality.”

   (Rabindranath Tagore, Indian Poet, Playwright and Essayist, Won the Nobel Prize for Literature in 1913, 1861-1941)

2. **Velocity Vector I.** The **velocity vector** of a moving object is a vector $\vec{\mathbf{v}}$ such that:

   - The magnitude of $\vec{\mathbf{v}}$ is the speed of the object.
   - The direction of $\vec{\mathbf{v}}$ is the direction of motion.

3. **Velocity Vector II.** We define the **velocity vector** $\vec{\mathbf{v}}(t)$ at time t of a point moving along a curve with position vector $\vec{\mathbf{r}}(t)$ as the derivative

$$\begin{aligned}\vec{\mathbf{v}}(t) &= \vec{\mathbf{r}}'(t)\\ &= \lim_{h\to 0}\frac{\vec{\mathbf{r}}(t+h)-\vec{\mathbf{r}}(t)}{h}\\ &= f'(t)\hat{\mathbf{i}}+g'(t)\hat{\mathbf{j}}+h'(t)\hat{\mathbf{k}}\end{aligned}$$

   In differential notation

$$\vec{\mathbf{v}} = \frac{d\vec{\mathbf{r}}}{dt} = \frac{dx}{dt}\hat{\mathbf{i}} + \frac{dy}{dt}\hat{\mathbf{j}} + \frac{dz}{dt}\hat{\mathbf{k}}.$$

4. **Speed.** The **speed** of the particle at time t is the magnitude of the velocity vector, that is, $|\vec{\mathbf{v}}(t)|$.

5. **Acceleration.** The **acceleration** of the particle is defined as the derivative of velocity:

$$\vec{\mathbf{a}}(t) = \vec{\mathbf{v}}\,'(t) = \vec{\mathbf{r}}\,''(t).$$

6. **Example.** The position vector of a particle is given by

$$\vec{\mathbf{r}}(t) = e^{2t}\,\hat{\mathbf{i}} + \sqrt{t}\,\hat{\mathbf{j}} + \frac{1}{t^2+1}\,\hat{\mathbf{k}}.$$

Find its velocity, speed, and acceleration at time $t = 2$.

7. **Example.** Find the position vector of a particle that has the acceleration

$$\vec{\mathbf{a}}(t) = -10\,\hat{\mathbf{k}}$$

with the initial conditions

$$\vec{\mathbf{v}}(0) = \hat{\mathbf{i}} + \hat{\mathbf{j}} - \hat{\mathbf{k}} \text{ and } \vec{\mathbf{r}}(0) = 2\hat{\mathbf{i}} + 3\hat{\mathbf{j}}.$$

8. **Example.** A projectile is fired with an initial speed of 500 m/s and angle of elevation 30^0. Find
   (a) the range of the projectile,
   (b) the maximum height reached,
   (c) the speed at impact.

9. **Tangent and Normal Vectors.**

   (a) Tangent vector:

   $$\vec{\mathbf{T}}(t) = \frac{\vec{\mathbf{r}}'(t)}{|\vec{\mathbf{r}}'(t)|} = \frac{\vec{\mathbf{v}}(t)}{|\vec{\mathbf{v}}(t)|} = \frac{\vec{\mathbf{v}}}{v}$$

   (b) Normal vector:

   $$\vec{\mathbf{N}}(t) = \frac{\vec{\mathbf{T}}'(t)}{|\vec{\mathbf{T}}'(t)|}$$

10. **Components of Acceleration.**

$$\vec{\mathbf{a}} = a_T\vec{\mathbf{T}} + a_N\vec{\mathbf{N}}$$

where

$$a_T = v' \text{ and } a_N = \kappa v^2.$$

11. **Fact.**

$$a_T = \frac{\vec{\mathbf{r}}\,'(t) \cdot \vec{\mathbf{r}}\,''(t)}{|\vec{\mathbf{r}}\,'(t)|} \qquad \text{and} \qquad a_N = \frac{|\vec{\mathbf{r}}\,'(t) \times \vec{\mathbf{r}}\,''(t)|}{|\vec{\mathbf{r}}\,'(t)|}$$

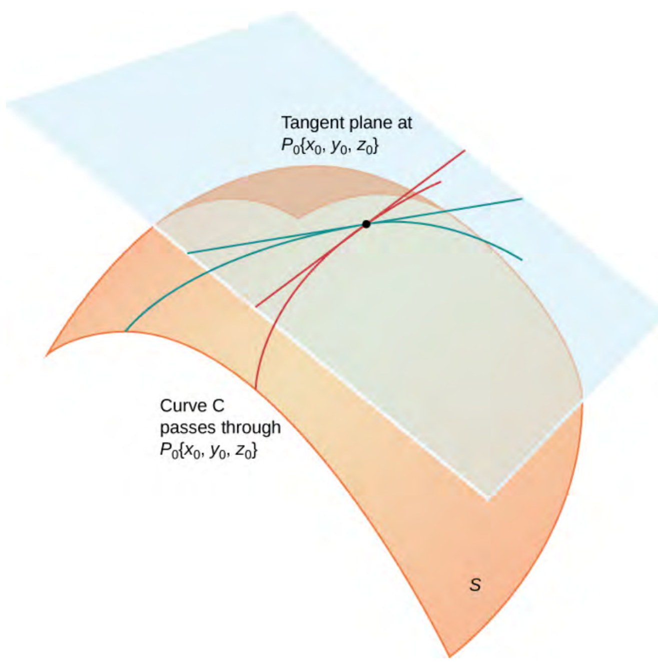
12. **Example.** Find the tangential and normal components of the acceleration vector

$$\vec{\mathbf{r}}(t) = t\hat{\mathbf{i}} + t^2\hat{\mathbf{j}} + 3t\hat{\mathbf{k}}.$$

13. Additional notes.

Part 14

Partial Derivatives



14.1 Functions of Several Variables

1. **Problem.** Find the maximum possible product of three positive numbers whose sum is 120.

2. **Definition.** A **function of two variables**, defined on the **domain** D in the plane, is a rule f that associates with each point (x, y) in D a unique real number, denoted by $f(x, y)$. A **function of three variables**, defined on the **domain** D in space, is a rule f that associates with each point (x, y, z) in D a unique real number $f(x, y, z)$.

3. **Examples.**

   (a) Let
   $$f(x, y) = \sqrt{x^2 + y^2 - 16}.$$
   Is it possible to evaluate $f(2, 1)$? $f(5, 0)$? $f(10, 10)$? Find the domain of f.

   (b) Let
   $$g(x, y, z) = \frac{\sqrt{x}}{\sqrt{y + z}}.$$
   Find the domain of g. Evaluate $g(16, 50, 50)$.

4. **Graphs.** Sketch the graph of f if

   (a)
$$f(x, y) = x + y + 2$$

   (b)
$$f(x, y) = \sqrt{36 - 4x^2 - 9y^2}$$

5. **Contour and Level Curves.**

   (a) The **contour curve** of **height** k on the surface $z = f(x, y)$ is the intersection of the surface and the plane $z = k$. The vertical projection of this contour curve into xy-plane is the **level curve** $f(x, y) = k$.

Example. Sketch some typical level curves of the functions

$$f(x,y) = x^2 + y^2, \; g(x,z) = \sqrt{x^2 + z^2}$$

(b) The **level surfaces** of the function $f(x,y,z)$ are surfaces of the form $f(x,y,z) = k$.

Example. Describe the level surfaces of the function

$$f(x,u,z) = x^2 + y^2 - z.$$

6. **Example.** Sketch the surface

$$z = \sin \sqrt{x^2 + y^2}.$$

7. Additional Notes

14.2 Limits and Continuity

1. **Problem.** Let

$$f(x,y) = \frac{x^2}{x+y} \text{ and } g(x) = \frac{x}{x+y}.$$

Complete the following table

m	1	2	3
$\lim_{x\to 0} f(x,mx)$			
$\lim_{x\to 0} g(x,mx)$			

2. **Definition.** Let f be a function of two variables whose domain D includes points arbitrarily close to (a,b). We say that the **limit of** $\mathbf{f(x,y)}$ **as** $\mathbf{(x,y)}$ **approaches** $\mathbf{(a,b)}$ **is L** provided that for every $\varepsilon > 0$, there exists a number $\delta > 0$ with the following property: If (x,y) is a point of the domain of f such that

$$0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta,$$

then it follows

$$|f(x,y) - L| < \varepsilon.$$

3. **Examples.** Evaluate the limit or show that it does not exist.

   (a)
$$\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{\sqrt{x^2 + y^2}}.$$

   (b)
$$\lim_{(x,y)\to(2,-2)} \frac{4 - xy}{4 + xy}.$$

4. **The Limit Laws.** If

$$\lim_{(x,y)\to(a,b)} f(x,y) = L \text{ and } \lim_{(x,y)\to(a,b)} g(x,y) = M$$

then

- $\lim_{(x,y)\to(a,b)} (f(x,y) + g(x,y)) = L + M$
- $\lim_{(x,y)\to(a,b)} (f(x,y) \cdot g(x,y)) = L \cdot M$
- $\lim_{(x,y)\to(a,b)} \dfrac{f(x,y)}{g(x,y)} = \dfrac{L}{M},\ M \neq 0$

Also, the Squeeze Theorem holds for multi-variable functions.

5. **Example.** Find

$$\lim_{(x,y)\to(0,0)} \frac{2xy^2}{x^2+y^2}$$

if it exists.

6. **Definition.** We say that f is **continuous at the point** (a,b) if

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b).$$

The function is said to be **continuous on the set** D if it is continuous at each point of D.

7. **Example.** Find k so that the function

$$f(x,y) = \begin{cases} \exp(-\frac{1}{x^2+y^2}) & (x,y) \neq (0,0) \\ k & x = y = 0 \end{cases}$$

is continuous on $\mathbb{R}^2$.

8. **Example.** Test for continuity at the origin.

(a)

$$\begin{cases} \dfrac{xy}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & x = y = 0 \end{cases}$$

(b)

$$\begin{cases} \dfrac{xy^2}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & x = y = 0 \end{cases}$$

(c)

$$f(x,y,z) = \begin{cases} \dfrac{xy}{x^2+y^2+z^2} & (x,y,z) \neq (0,0,0) \\ 0 & x = y = z = 0 \end{cases}$$

(d)

$$f(x,y,z) = \begin{cases} \dfrac{xyz}{x^2+y^2+z^2} & (x,y,z) \neq (0,0,0) \\ 0 & x = y = z = 0 \end{cases}$$

9. Additional Notes

14.3 Partial Derivatives

1. **Quote.** ”Young man, in mathematics you don’t understand things, you just get used to them.”
   John von Neumann, Hungarian-born American mathematician, 1903 - 1957)

2. **Definition.** The **partial derivatives (with respect to** x **and with respect to** y**)** of the function $f(x, y)$ are the two function defined by

$$\frac{\partial f}{\partial x} = f_x(x, y) = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

$$\frac{\partial f}{\partial y} = f_y(x, y) = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h}$$

3. **Calculating Partial Derivatives.**
   - To calculate $\dfrac{\partial f}{\partial x}$, regard y as a constant and differentiate with respect to x.
   - To calculate $\dfrac{\partial f}{\partial y}$, regard x as a constant and differentiate with respect to y.

4. **Example.** Compute the first-order partial derivatives if

$$z = \frac{e^x - y \sin x}{1 - \cos y}$$

5. **Example.** The volume (in cubic centimeters) of 1 mole (mol) of an ideal gas is given by

$$V = \frac{(82.06)T}{p},$$

where p is the pressure (in atmospheres) and T is the absolute temperature in kelvins (K), where K=^{0}C $+273$). Find the rates of change of the volume of 1 mol of an ideal gas with respect to pressure and with respect to temperature when $T = 300K$ and $p = 5$atm.

6. **Higher-Order Partial Derivatives** The partial derivatives of $f_x(x, y)$ and $f_y(x, y)$ are called the **second-order partial derivatives** of f. There are four second order partial derivatives:

$$\begin{aligned}
(f_x)_x &= f_{xx} = \frac{\partial f_x}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} \\
(f_x)_y &= f_{xy} = \frac{\partial f_x}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} \\
(f_y)_x &= f_{yx} = \frac{\partial f_y}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} \\
(f_y)_y &= f_{yy} = \frac{\partial f_y}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}
\end{aligned}$$

7. **Example.** Find the second and third-order partial derivatives of the function

$$f(x, y) = x^3 - 3x^2 y + y^2.$$

8. **Equality of Mixed Partials:** If f_{xy} and f_{yx} are continuous on a circular disk centered at (a, b), then

$$f_{xy}(a, b) = f_{yx}(a, b).$$

9. **More than Two Variables.**

$$f_x(x, y, z) = \lim_{h \to 0} \frac{f(x+h, y, z) - f(x, y, z)}{h}$$

Example. If $w = \dfrac{x}{y+2z}$ find $\dfrac{\partial^3 w}{\partial z \partial y \partial x}$.

10. **Partial Differential Equations.** Verify that the function

$$u = \frac{1}{\sqrt{x^2 + y^2 + z^2}}$$

is a solution of three-dimensional Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0.$$

11. Additional Notes

14.4 Tangent Planes and Linear Approximation

1. **Problem.** Estimate
$$2\sin\left(\frac{29\pi}{60}\right) + 3\cos\left(\frac{\pi}{50}\right).$$

2. **Reminder.** Any plane passing through the point $P(x_0, y_0, z_0)$ has an equation in the form
$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0.$$

3. **Tangent Plane.** Suppose f has continuous partial derivatives. An equation of the tangent plane to the surface $z = f(x, y)$ is
$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

4. **Example.** Find the equation of the tangent plane to the surface

$$z = \sqrt{4 - x^2 - 2y^2}$$

at the point $(1, -1, 1)$.

5. **Linearization.** Let $z = f(x, y)$ has continuous partial derivatives at the point (a, b). The linear function whose graph is the tangent plane to the graph of the the function f at the point $(a, b, f(a, b))$

$$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

is called the **linearization** of f at (a, b).

6. **Linear Approximation.**

$$f(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

is called the **linear approximation** of f at (a, b).

7. **Problem.** Find the linear approximation to z at the point $(1, 2)$ if

$$z = 2x^3 + 2y^3 - 9xy.$$

Approximate the y coordinate of the point $P(1.1, y)$ that belongs to the curve $2x^3 + 2y^3 - 9xy = 0$.

8. **Increment.** The increment of $z = f(x, y)$ when x changes from a to $a + \Delta x$ and y changes from b to $b + \Delta y$ is

$$\Delta z = \Delta f(x, y) = f(x + \Delta x, y + \Delta y) - f(x, y).$$

9. **Definition.** If $z = f(x, y)$, then f is **differentiable** at (a, b) if Δz can be expressed in the form

$$\Delta z = f_x(a, b)\Delta x + f_y(a, b)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

where ε_1 and $\varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$.

10. **Theorem.** If the partial derivatives f_x and f_y exist near (a, b) and are continuous at (a, b), then f is differentiable at (a, b).

11. **Warning!** A function of two variables is **differentiable** at the point (a, b) if it can be approximated sufficiently closely near (a, b) by a **linear function**.

- Theorem implies that if the partial derivatives of a function f are continuous then f is differentiable.
- If a function is differentiable at a point, then both partial derivatives exist there.
- Having both partial derivatives at a point does not guarantee that a function is differentiable (or even continuous).

12. **Example.** Explain why the function $f(x, y) = \sqrt{x + e^{4y}}$ is differentiable at the point $(3, 0)$. Find the linearization $L(x, y)$ of the function at the point and use it to approximate $f(3.1, -0.1)$.

13. **Differential.** For a differentiable function in two variables, $z = f(x, y)$, we define the **differential**

$$\begin{aligned} dz &= f_x(x,y)\cdot\Delta x + f_y(x,y)\cdot\Delta y \\ &= f_x(x,y)\ dx + f_y(x,y)\ dy \\ &= \frac{\partial z}{\partial x}\ dx + \frac{\partial z}{\partial y}\ dy \end{aligned}$$

14. **Examples.**

(a) Find the differential of the function

$$f(x, y) = 2\sin x + 3\cos y.$$

(b) For

$$f(x, y) = 2\sin x + 3\cos y$$

calculate $f(\pi/2, 0)$ and the differential df to approximate the value of

$$f\left(\frac{29\pi}{60}, \frac{\pi}{50}\right).$$

15. **More than Three Variables.** Let $w = f(x, y, z)$

    - **Linear approximation:**
      $f(x, y, z) \approx f(a, b, c) + f_x(a, b, c)(x - a) + f_y(a, b, c)(y - b) + f_z(a, b, c)(z - c)$
    - **Increment:**
      $\Delta w = f(x + \Delta x, y + \Delta y, z + \Delta z) - f(x, y, z)$
    - **Differential:**
      $dw = \frac{\partial w}{\partial x} dx + \frac{\partial w}{\partial y} dy + \frac{\partial w}{\partial z} dz$

16. **Example.** The dimensions of a closed rectangular are measured as 80 cm, 60 cm, and 50 cm, respectively, with a possible error of 0.2 cm in each dimension. Use differentials to estimate the maximum error in calculating the surface area of the box.

17. Additional Notes

14.5 The Chain Rule

1. **Problem.** Falling sand forms a conical sandpile. When the sandpile has a height of 5 m and its base radius is 2 m, its height is increasing at 0.4 m/min and its base radius is increasing at 0.7 m/min. At what rate is the volume of the sandpile increasing at that moment?

2. **The Chain Rule.** Suppose that $z = f(x, y)$ has continuous first-order partial derivatives and that $x = g(t)$ and $y = h(t)$ are differentiable functions. Then z is a differentiable function of t and

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}.$$

3. **Examples.** Find dz/dt if

   (a) $z = \arctan(x^2 + y^2),$ with $x = t^2,\ y = t^3$

   (b) $w = \arctan(x^2 + y^2 + z^2),$ with $x = t^2,\ y = t^3,\ z = t^4$

4. **Chain Rule: Several Independent Variables..** If

$$w = f(x, y, z) = f(g(u, v), h(u, v), z(u, v))$$

then

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial u} + \frac{\partial w}{\partial z} \cdot \frac{\partial z}{\partial u}$$

and

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial v} + \frac{\partial w}{\partial z} \cdot \frac{\partial z}{\partial v}.$$

5. **Example.** Find $\partial w/\partial u$ and $\partial w/\partial v$ if

$$w = \arctan(x/y) \text{ and } x = u^2 + v^2,\ y = u^2 - v^2.$$

6. **Example.** Suppose that $w = f(u, v, x, y)$, where u and v are functions of x and y. Find $\partial w/\partial x$ and $\partial w/\partial y$.

7. **Implicit Partial Differentiation.** Suppose that $F(x, y, z)$ has continuous first-order partial derivatives and that the equation $F(x, y, z) = 0$ implicitly defines a function $z = f(x, y)$ that has continuous first-order partial derivatives. Then

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} \quad \text{and} \quad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}.$$

8. **Three Problems.**

(a) Let z be a function of x and y such that

$$z^3 - z + 2xy - y^2 = 0, \quad z(2,4) = 1.$$

i. Find the linear approximation to z at the point $(2,4)$.

ii. Use your answer in (a) to estimate the value of z at $(2.02, 3.96)$.

(b) The equations

$$\begin{aligned} x &= uv + v^2 \\ y &= u^2 - uv \end{aligned}$$

define u and v implicitly as functions of x and y near the points $(x,y) = (2,0)$ and $(u,v) = (1,1)$.

i. Compute $\dfrac{\partial u}{\partial y}$ at the point $(x,y) = (2,0)$.

ii. If $z = u^2v^3$, compute $\dfrac{\partial z}{\partial y}$ at the point $(x,y) = (2,0)$.

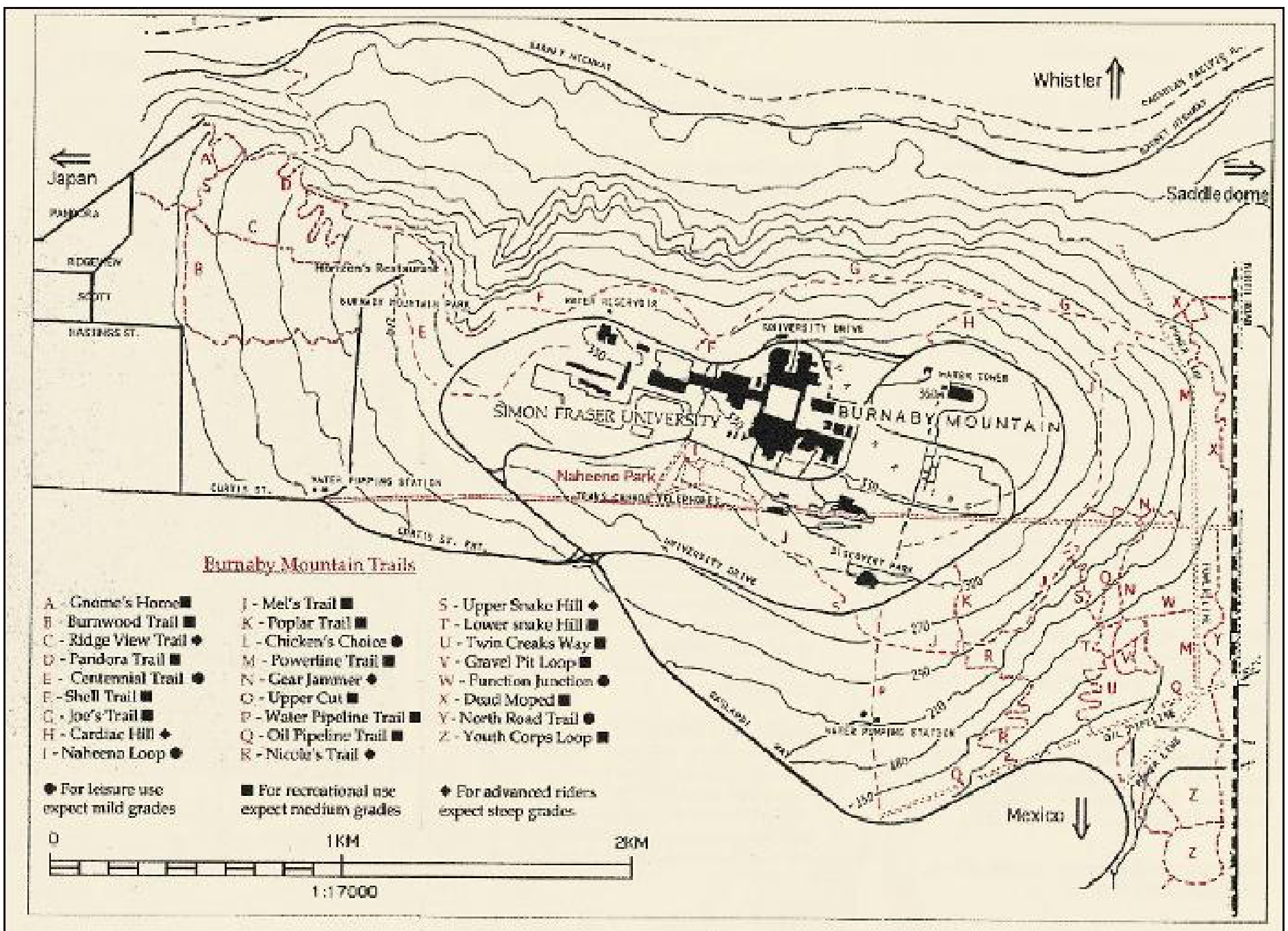
9. Additional Notes

14.6 Directional Derivatives and the Gradient Vector

1. **Quote.** ”A mathematician is a blind man in a dark room looking for a black cat which isn’t there.” (Charles Darwin, English naturalist, 1809 - 1882)

2. **Problem.** The **partial derivatives** give the rates of change in the directions of the coordinate axis. How can we calculate the rate of change in an **arbitrary** direction?

3. **Problem.** Standing at the point P on Burnaby Mountain, in which direction is the steepest ascent?



Courtesy of the Burnaby Mountain Biking Club

4. **Directional Derivative:**
Recall that if $z = f(x, y)$ then the partial derivatives f_x and f_y, which are defined by

$$f_x(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}$$
$$f_y(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h},$$

represent the rates of change of z in the $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ directions. Suppose we now want to find the rate change of z in the direction of some arbitrary unit vector $\vec{\mathbf{u}} = a\hat{\mathbf{i}} + b\hat{\mathbf{j}}$.

Definition. The **directional derivative** of f at (x_0, y_0) in the direction of a unit vector $\vec{\mathbf{u}} = \langle a, b \rangle$ is

$$D_{\vec{\mathbf{u}}} f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

if this limit exists.

5. **Theorem.** If f is a differentiable function of x and y, then f has a directional derivative in the direction of any unit vector $\vec{\mathbf{u}} = \langle a, b \rangle$ and

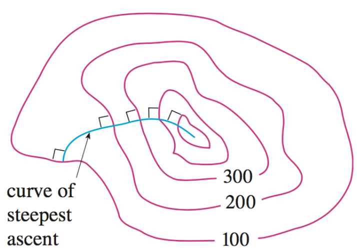
$$D_{\vec{\mathbf{u}}} f(x_0, y_0) = f_x(x_0, y_0)a + f_y(x_0, y_0)b.$$

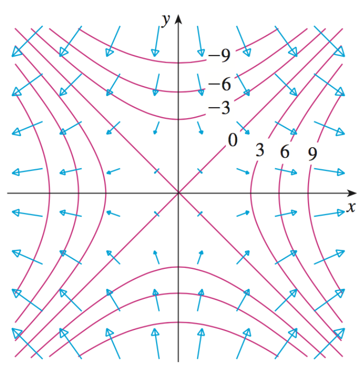
6. **Gradient Vector.**
If f is a function of two variables x and y, then the **gradient** of f is the vector function $\vec{\nabla} f$ defined by

$$\vec{\nabla} f(x,y) = \langle f_x(x_0,y_0), f_y(x_0,y_0)\rangle = \frac{\partial f}{\partial x}\hat{\mathbf{i}} + \frac{\partial f}{\partial y}\hat{\mathbf{j}}$$

7. **Example:** If $f(x,y) = x^2 - y^2$ (a) find the gradient of f and (b) find the directional derivative of f at $(2,1)$ in the direction of $\vec{\mathbf{v}} = \langle 1, 1\rangle$.

8. **Geometry of the Gradient:**
   - $\vec{\nabla} f$ points in the direction of the steepest change (ascent)
   - $\vec{\nabla} f$ is orthogonal to the level curves of f





9. **Maximal value of the directional derivative:**
Suppose f is a differentiable function of two or three variables. The maximum value of the directional derivative $D_{\vec{\mathbf{u}}} f(\vec{\mathbf{x}})$ is $|\vec{\nabla} f(\vec{\mathbf{x}})|$ and it occurs when $\vec{\mathbf{u}}$ has the same direction as the gradient vector $\vec{\nabla} f(\vec{\mathbf{x}})$.

10. **Directional Derivative and Gradient in $\mathbb{R}^3$.**
For functions of three variables we define the gradient and directional derivative in a similar manner. Let $w = f(x, y, z)$, where f is a differentiable function.

The **gradient** of f is the vector function $\vec{\nabla} f$ defined by

$$\vec{\nabla} f(x, y, z) = \langle f_x(x_0, y_0, z_0), f_y(x_0, y_0, z_0), f_z(x_0, y_0, z_0)\rangle = \frac{\partial f}{\partial x}\hat{\mathbf{i}} + \frac{\partial f}{\partial y}\hat{\mathbf{j}} + \frac{\partial f}{\partial z}\hat{\mathbf{k}}$$

The **directional derivative** of f at (x_0, y_0, z_0) in the direction of a unit vector $\vec{\mathbf{u}} = \langle a, b, c\rangle$ is

$$D_{\vec{\mathbf{u}}} f(x_0, y_0, z_0) = \lim_{h\to 0} \frac{f(x_0 + ha, y_0 + hb, z_0 + hc) - f(x_0, y_0, z_0)}{h}$$

if this limit exists.

We can write this directional derivative in a more compact form using vector notation:

$$D_{\vec{\mathbf{u}}} f(\vec{\mathbf{x}}_0) = \lim_{h\to 0} \frac{f(\vec{\mathbf{x}}_0 + h\vec{\mathbf{u}}) - f(\vec{\mathbf{x}}_0)}{h}$$

where $\vec{\mathbf{x}}_0 = \langle x_0, y_0\rangle$ if $n = 2$ or $\vec{\mathbf{x}}_0 = \langle x_0, y_0, z_0\rangle$ if $n = 3$.

11. **Computing the Directional Derivative for functions of 2 or 3 variables.**

$$D_{\vec{\mathbf{u}}} f(\vec{\mathbf{x}}_0) = \vec{\nabla} f(\vec{\mathbf{x}}_0) \cdot \vec{\mathbf{u}}$$

12. **Example:** If $f(x, y, z) = xe^z + yz$ (a) find the gradient of f and (b) find the directional derivative of f at $(1, 1, 0)$ in the direction of $\vec{\mathbf{v}} = \langle 1, 2, -1\rangle$.

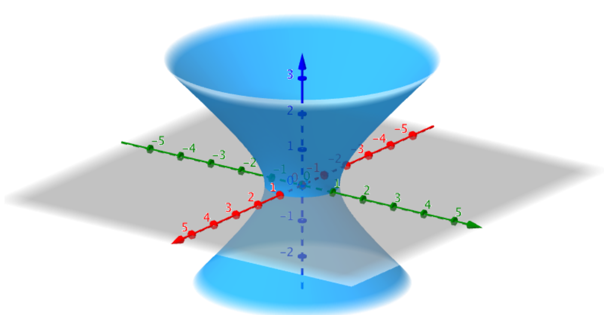
13. **Tangent Plane to Level Surfaces.**
The tangent plane to the level surface $F(x, y, z) = k$ at the point $P(x_0, y_0, z_0)$ is:

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0$$

This can be written in a compact form using vector notation:

$$\vec{\nabla} F(\vec{\mathbf{x}}_0) \cdot (\vec{\mathbf{x}} - \vec{\mathbf{x}}_0) = 0.$$

14. **Example:** Find the tangent plane to the hyperboloid $x^2 + y^2 - z^2 = 1$ at the point $(1, 1, 1)$.



15. **Another look at the tangent plane to the surface $z = f(x, y)$:**
Think of a surface $z = f(x, y)$ as the zero level surface of the function $F(x, y, z) = f(x, y) - z$. What does the tangent plane formula above reduce to in this case?

16. **Example.** Let C be the curve given as the intersection of $xyz = 1$ and $x^2 + 2y^2 + 3z^2 = 6$. Find the tangent line equation to C at the point $(1, 1, 1)$.

17. Additional Notes

14.7 Maximum and Minimum Values

1. **Puzzle.** Which letter should replace the question mark?

A	E	J
D	?	M
H	L	Q

2. **Vocabulary.**

   (a) A function of two variables has a **local maximum** at (a, b) if

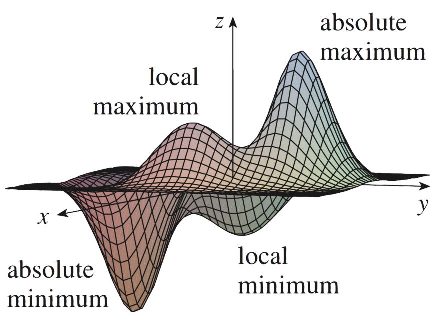
   $$f(x, y) \leq f(a, b)$$

   when (x, y) is near (a, b). The number $f(a, b)$ is called a **local maximum value**.

   (b) If $f(x, y) \geq f(a, b)$ when (x, y) is near (a, b), then $f(a, b)$ is a **local minimum value**.

   (c) If $f(x, y) \leq f(a, b)$ for *all* points (x, y) in the domain of f, then f has an **absolute maximum** at (a, b).

   (d) If $f(x, y) \geq f(a, b)$ for *all* points (x, y) in the domain of f, then f has an **absolute minimum** at (a, b).



3. **Theorem.** If f has a local maximum or minimum at (a, b) and the first-order partial derivatives of f exist there, then

$$f_x(a, b) = 0 \text{ and } f_y(a, b) = 0.$$

4. **Critical Point.** A point (a, b) is called a **critical point** of f if

$$f_x(a, b) = f_y(a, b) = 0,$$

or if one of these partials does not exist.

5. **BIG Question.** How do we decide if a critical point is a local maximum or a local minimum?

6. **Second Derivative Test.** Suppose the second partial derivatives of f are continuous on a disk with centre (a,b), and suppose that $f_x(a,b)=0$ and $f_y(a,b)=0$ (that is, (a,b) is a critical point of f). Let

$$D = f_{xx}(a,b)f_{yy}(a,b) - [f_{xy}(a,b)]^2.$$

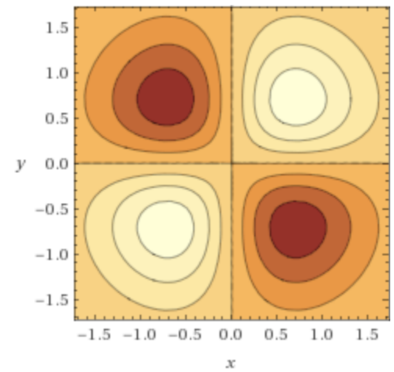
Then

(a)	if $D>0$ and $f_{xx}(a,b)>0$	$\Longrightarrow$	f has a local minimum at (a,b)
(b)	if $D>0$ and $f_{xx}(a,b)<0$	$\Longrightarrow$	f has a local maximum at (a,b)
(c)	if $D<0$	$\Longrightarrow$	f has neither a local minimum nor a local maximum at (a,b). Instead, f has a **saddle point** at (a,b).
(d)	if $D=0$	$\Longrightarrow$	Test is inconclusive.

7. **Example.** Find and classify the critical points of the function

$$f(x, y) = xye^{-x^2-y^2}.$$

Contour plot:



critical point	value of f	f_{xx}	D	classification

8. **Example.** A wooden box without a lid is to have a volume of $32\mathrm{m}^3$. Find the dimensions that minimize the the amount of wood used.

9. **Vocabulary.**
   (a) A **boundary point** point of $D \subseteq \mathbb{R}^2$ is a point (a, b) such that every disk with the center (a, b) contains a point in D and also a point not in D.
   (b) A **closed set** in $\mathbb{R}^2$ is one that contains all its boundary points.
   (c) A **bounded set** in $\mathbb{R}^2$ is one that is contained within some disk.

10. **Extreme Value Theorem.** If f is continuous on a closed, bounded set D in $\mathbb{R}^2$, then f attains an absolute maximum value $f(x_1, y_1)$ and an absolute minimum value $f(x_2, y_2)$ at some points (x_1, y_1) and (x_2, y_2) in D.

11. Find the absolute maximum and minimum values of

$$f(x, y) = 6x^2 - x^4 + 4x^2y - 4y^2 - y^3$$

on the set

$$D = \{(x, y) \mid 0 \leq x \leq 4, 0 \leq y \leq 3\}.$$

12. Additional Notes

14.8 Lagrange Multipliers

1. **Time Machine:** The base of an aquarium with given volume V is made of slate and the sides are made of glass. If slate costs five times as much (per unit area) as glass, find the dimensions of the aquarium that minimize the cost of materials.

2. **Lagrange Multipliers.** Let $f(x, y, z)$ and $g(x, y, z)$ be functions with continuous first-order partial derivatives. If the maximum (or minimum) value of f subject to the condition

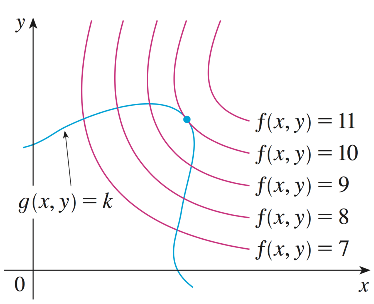
$$g(x, y, z) = k$$

occurs at a point P where $\nabla g(P) \neq \mathbf{0}$, then

$$\nabla f(P) = \lambda \nabla g(P)$$

for some constant λ.

The associated value(s) of λ are called **Lagrange multiplier(s)**.



3. The Method. The idea is to solve the following system of equations with unknowns x, y, and λ.

$$\begin{aligned} g(x, y, z) &= k \\ f_x(x, y, z) &= \lambda\, g_x(x, y, z) \\ f_y(x, y, z) &= \lambda\, g_y(x, y, z) \\ f_z(x, y, z) &= \lambda\, g_z(x, y, z) \end{aligned}$$

The points (x, y, z) that we find are the only possible locations for the extrema.

To find the maximum (or minimum) calculate and compare the values of f at the solutions of the system above.

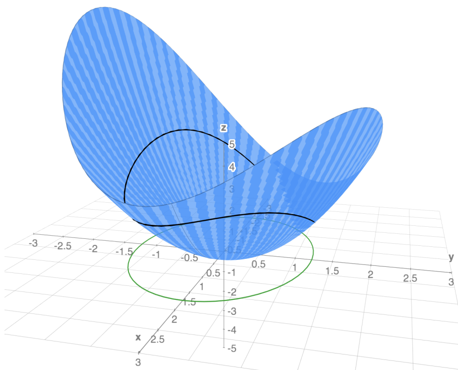
3. **Again Using Method of Lagrange Multipliers:** The base of an aquarium with given volume V is made of slate and the sides are made of glass. If slate costs five times as much (per unit area) as glass, find the dimensions of the aquarium that minimize the cost of materials.

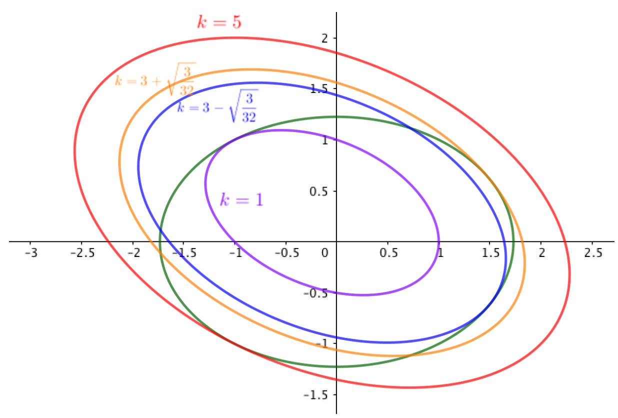
4. **Example.** Find the maximum and minimum values of

$$p(x, y) = x^2 + xy + 2y^2 - y$$

subject to the constraint

$$x^2 + 2y^2 = 3.$$

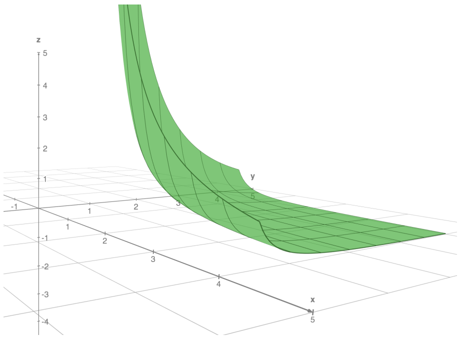




5. **Problem.** Find the first-octant point $P(x, y, z)$ on the surface

$$x^2y^2z = 4$$

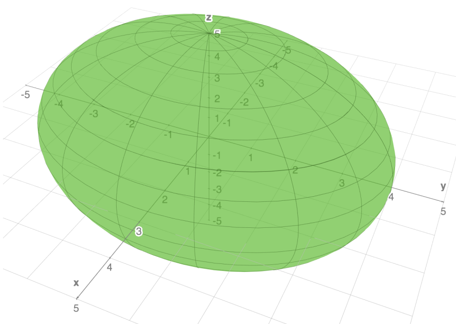
closest to the origin.



6. **Problem** Find the maximum volume of a rectangular box inscribed inside the surface

$$\frac{x^2}{3^2} + \frac{y^2}{4^2} + \frac{z^2}{5^2} = 1$$

with its faces parallel to the coordinate planes.



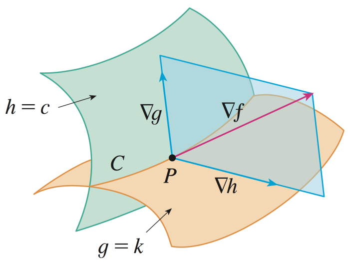
7. **Lagrange Multipliers (Two Constraints).** Let $f(x, y, z)$, $g(x, y, z)$, and $h(x, y, z)$ be functions with continuous first-order partial derivatives. Suppose f has an extreme value (maximum or minimum) at a point P subject to the two conditions

$$g(x, y, z) = 0 \text{ and } h(x, y, z) = 0.$$

If the vectors $\nabla g(P)$ and $\nabla h(P)$ are nonzero and nonparallel, then

$$\nabla f(P) = \lambda \nabla g(P) + \mu \nabla h(P)$$

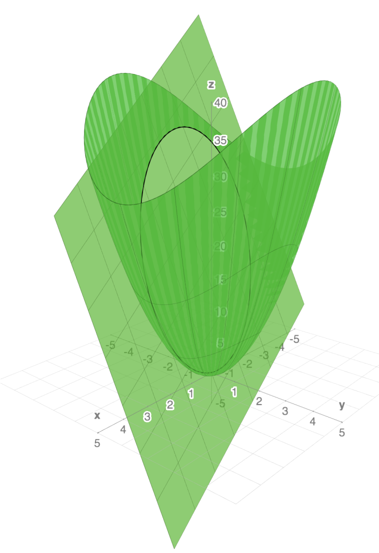
for some constants λ and μ. In other words, $\nabla f(P)$ is in the plane determined by $\nabla g(P)$ and $\nabla h(P)$.



8. **The Method.** Solve the following system of equations with unknowns x, y, z, λ and μ.

$$\begin{aligned} g(x, y, z) &= 0 \\ h(x, y, z) &= 0 \\ f_x(x, y, z) &= \lambda\, g_x(x, y, z) + \mu\, h_x(x, y, z) \\ f_y(x, y, z) &= \lambda\, g_y(x, y, z) + \mu\, h_y(x, y, z) \\ f_z(x, y, z) &= \lambda\, g_z(x, y, z) + \mu\, h_z(x, y, z) \end{aligned}$$

9. **Example.** The plane $4x + 9y + z = 0$ intersects the elliptic paraboloid $z = 2x^2 + 3y^2$ in an ellipse. Find the highest and lowest points on this ellipse.



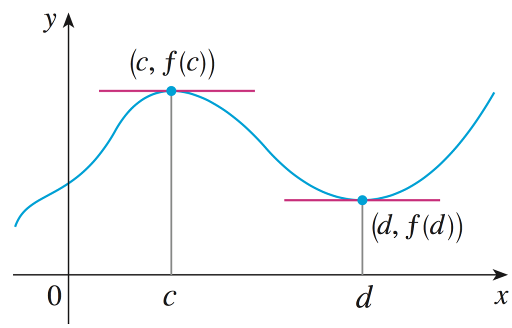
10. **Time Machine: Max/Min in 1D Calculus**
In Calculus I we looked at the question "Find the max/min of $y = f(x)$." In Calculus III we can rewrite this questions as:
Find the max/min of $p(x, y) = y$ under the constraint

$$q(x, y) = y - f(x) = 0.$$

The Method of Lagrange Multipliers implies the max/min values occur at the points (x, y) such that $\vec{\nabla}p = \lambda\vec{\nabla}q$, where

$$\vec{\nabla}p = \langle 0, 1\rangle \qquad \text{and} \qquad \vec{\nabla}q = \langle -f'(x), 1\rangle.$$

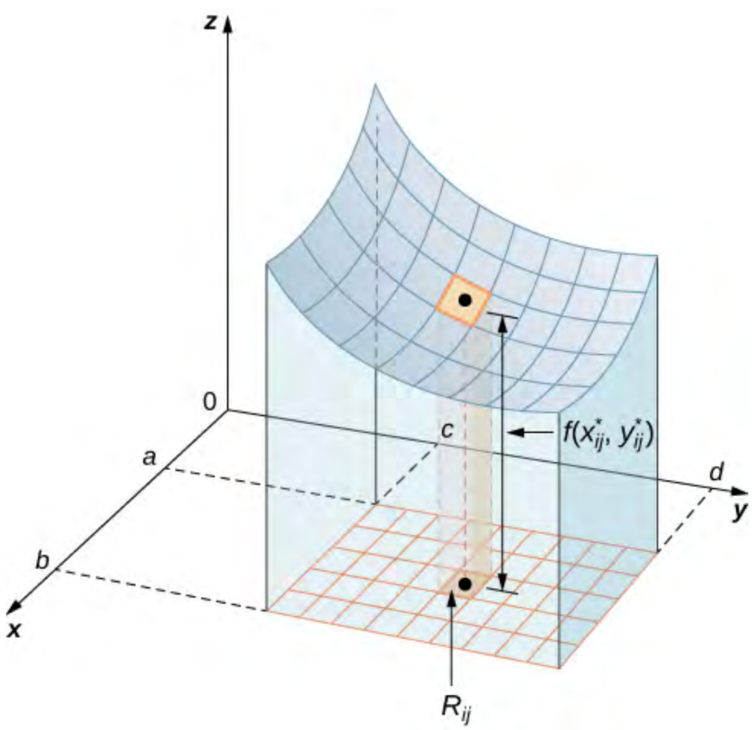
These are only parallel when $f'(x) = 0$, therefore the max/min occur at points where $f'(x) = 0$.



11. Additional Notes

Part 15

Multiple Integrals



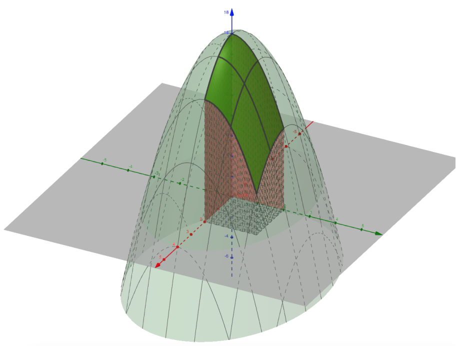
15.1 Double Integrals over Rectangles

1. **Yes We have No Apples**

   A girl with blue eyes
   Went out to view the skies
   She saw an apple tree with apples on it
   She neither took apples nor left apples.
   How many apples were on the tree?

2. **Reminder.** Calculate the area of the region that lies above the segment $[0, 2]$ and below the parabola $y = 16 - x^2$.

3. **Problem.** Calculate the volume of the solid that lies above the square $R = [0, 2] \times [0, 2]$ and below the elliptic paraboloid $z = 16 - x^2 - 2y^2$.



4. **Double Integral.**

   Let $f(x, y)$ be continuous function defined over the rectangle

$$\begin{aligned} R &= [a, b] \times [c, d] \\ &= \{(x, y) : a \leq x \leq b,\ c \leq y \leq d\} \end{aligned}$$

   A **partition** $\mathcal{P}$ of R into subrectangles is a set of rectangles $R_{1,1}, R_{1,2}, \ldots R_{m,n}$ so that there are partitions of $[a, b]$ and $[c, d]$

$$\begin{aligned} a = x_0 < x_1 < \ldots < x_m = b \\ c = y_0 < y_1 < \ldots < y_n = d \end{aligned}$$

   with

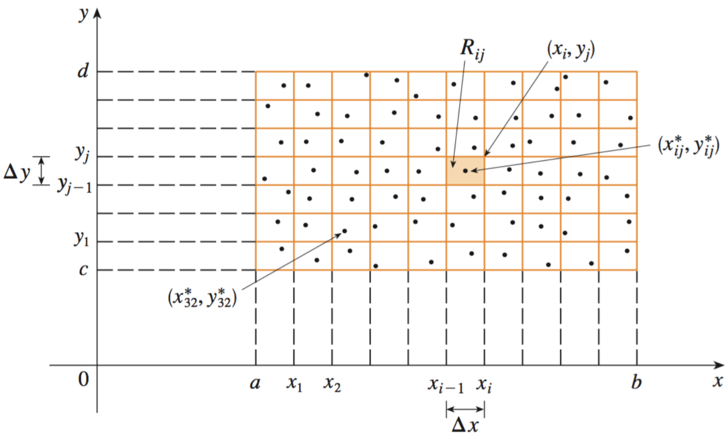
   (a) $\Delta x = (b - a)/m$, $\Delta y = (d - c)/n$

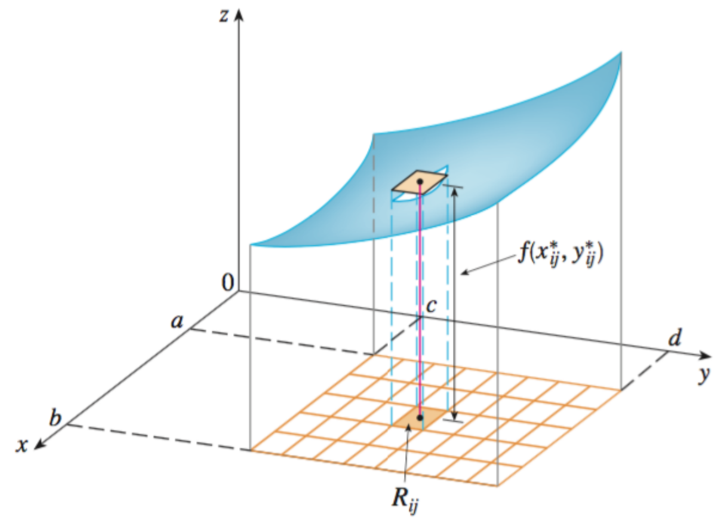
   (b) $x_i = a + i\Delta x$, $y_j = c + j\Delta y$

   (c) $R_{i,j} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$ for some i and j.

   We choose a **collection** of **sample points**

$$S = \{(x^*_{ij}, y^*_{ij}) \in R_{i,j} : 1 \leq i \leq m, 1 \leq j \leq n\}.$$





Definition of the **double integral**: The **double integral** of f over the rectangle R is

$$\int\int_R f(x,y)dA = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A,$$

where $\Delta A = \Delta x \Delta y$ is the area of each R_i, if this limit exists.

5. **Example.** Estimate the volume of the solid that lies above the square $R = [0,2] \times [0,2]$ and below the elliptic paraboloid $z = 16 - x^2 - 2y^2$. use the partition of R into four equal squares and choose (x_{ij}^*, y_{ij}^*) to be the center point of R_i.

6. **Midpoint Rule for Double Integrals.**

$$\int\int_R f(x,y)dA \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(\overline{x}_i, \overline{y}_i)\Delta A$$

where $\overline{x}_i$ is the midpoint of $[x_{i-1}, x_i]$ and $\overline{y}_j$ is the midpoint of $[y_{j-1}, y_j]$.

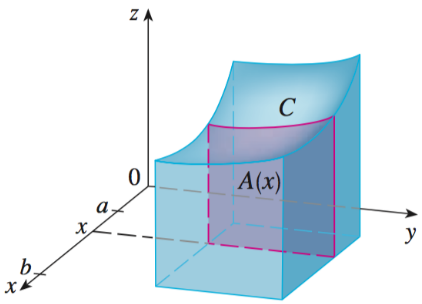
7. **Properties of Double Integrals.**

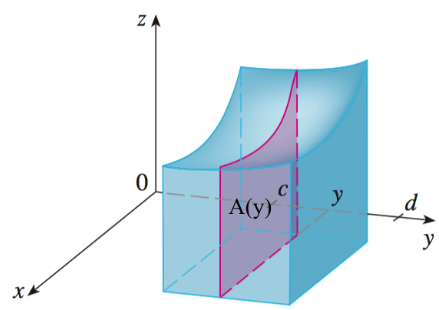
   (a)
   $$\int\int_R [f(x,y)+g(x,y)]dA = \int\int_R f(x,y)dA + \int\int_R g(x,y)dA$$

   (b)
   $$\int\int_R cf(x,y)dA = c\int\int_R f(x,y)dA$$

   (c) If $f(x,y) \geq g(x,y)$ for all (x,y) in R, then
   $$\int\int_R f(x,y)dA \geq \int\int_R g(x,y)dA.$$

8. **How We Calculate Double Integrals?**
   Suppose that $f(x,y)$ is continuous on the rectangle $R=[a,b]\times[c,d]$. Then
   $$\begin{aligned}\int\int_R f(x,y)dA &= \int_a^b \left(\int_c^d f(x,y)dy\right)dx \\ &= \int_c^d \left(\int_a^b f(x,y)dx\right)dy\end{aligned}$$





9. **Examples.** Evaluate

   (a)
   $$\int_2^4\int_{-1}^1 (x^2+y^2)dxdy$$

(b)

$$\int\int_R x\sin(x+y)dA$$

where $R = [0, \pi/6] \times [0, \pi/3]$.

10. **Example.** Evaluate the volume of the solid that lies above the square $R = [0, 2] \times [0, 2]$ and below the elliptic paraboloid $z = 16 - x^2 - 2y^2$.

11. **Example.** Evaluate

$$\int\int_R \frac{xy^2}{x^2+1}dA$$

where $R = [0,1] \times [-3,3]$.

12. **Fact.**

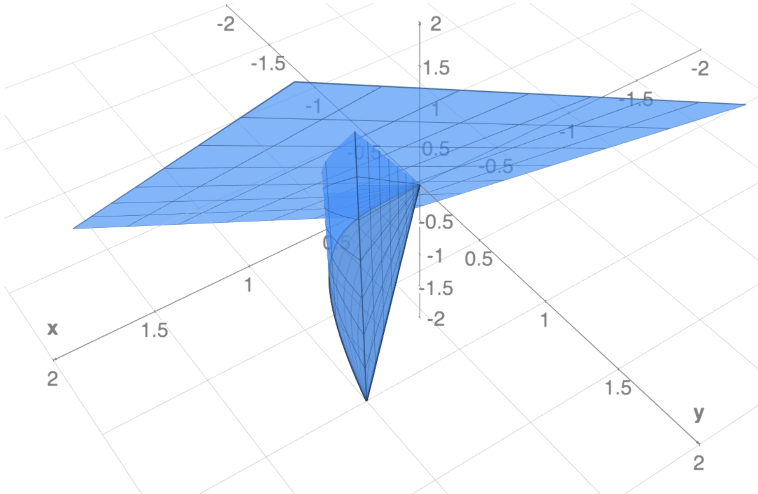
$$\int\int_R f(x)g(y)dA = \int_a^b f(x)dx \int_c^d g(y)dy$$

where $R = [a,b] \times [c,d]$.

13. Additional Notes

15.2 Double Integrals Over General Regions

1. **Problem.** Find the volume of the solid that lies under the plane $x + 2y - z = 0$ and above the region in the xy-plane bounded by $y = x$ and $y = x^4$.



2. **Vertically Simple Region.** We say that the region R is **vertically simple**, or of **type I**, if there are a segment $[a, b]$ and two functions $g_1(x)$ and $g_2(x)$ such that

$$g_1(x) \leq g_2(x) \text{ for all } x \in [a, b]$$

and

$$R = \{(x, y) : x \in [a, b] \text{ and } g_1(x) \leq y \leq g_2(x)\}.$$

3. **Horizontally Simple Region.** We say that the region R is **horizontally simple**, or of **type II**, if there are a segment $[c, d]$ and two functions $h_1(y)$ and $h_2(y)$ such that

$$h_1(y) \leq h_2(y) \text{ for all } y \in [a, b]$$

and

$$R = \{(x, y) : y \in [c, d] \text{ and } h_1(y) \leq x \leq h_2(y)\}.$$

4. **Evaluation.** Suppose that $f(x, y)$ is continuous on the region R.

If R is the vertically simple region (as above), then

$$\int\int_R f(x, y)dA = \int_a^b \int_{g_1(x)}^{g_2(x)} dydx.$$

If R is the horizontally simple region (as above), then

$$\int\int_R f(x, y)dA = \int_c^d \int_{h_1(y)}^{h_2(y)} dxdy.$$

5. **Examples.** Evaluate

(a)

$$\int_0^2 \int_0^{x^3} e^{y/x} dydx$$

(b)

$$\int_0^3 \int_0^y \sqrt{y^2+16}\, dxdy$$

(c) $\displaystyle\int\int_R x\, dA$ where R is bounded by the parabolas $y = x^2$ and $y = 8 - x^2$.

(d) Evaluate

$$\int_0^{2\sqrt{\ln 3}} \int_{y/2}^{\sqrt{\ln 3}} e^{x^2}\, dx dy.$$

(e) Find the volume of the solid that lies under the plane $x + 2y - z = 0$ and above the region in the xy-plane bounded by $y = x$ and $y = x^4$.

6. **Properties of Double Integrals.**

   (a) $\int\int_R [f(x,y)+g(x,y)]dA = \int\int_R f(x,y)dA + \int\int_R g(x,y)dA$

   (b) $\int\int_R cf(x,y)dA = c\int\int_R f(x,y)dA$

   (c) If $f(x,y) \geq g(x,y)$ for all (x,y) in R, then
   $$\int\int_R f(x,y)dA \geq \int\int_R g(x,y)dA.$$

   (d) If $D = D_1 \cup D_2$, where D_1 and D_2 do not overlap, except perhaps on their boundaries
   $$\int\int_D f(x,y)dA = \int\int_{D_1} f(x,y)dA + \int\int_{D_2} f(x,y)dA$$

   (e) $\int\int_D dA = A(D) =$ Area of D

   (f) If $m \leq f(x,y) \leq M$ for all (x,y) in D, then
   $$m \cdot A(D) \leq \int\int_D f(x,y)dA \leq M \cdot A(D)$$

7. **Example.** Estimate the value of the integral
   $$\int\int_D e^{x^2+y^2}dxdy$$
   where D is the disk with the center at the origin and the radius $1/2$.

8. Additional Notes

15.3 Double Integrals in Polar Coordinates

1. **Problem.** Find the area bounded by the cardioid

$$r = 1 + \cos\theta.$$

2. **Polar Rectangle.** A **polar rectangle** is a region described in polar coordinates by the inequalities

$$a \le r \le b,\ \alpha \le \theta \le \beta.$$

3. **How We Calculate.**

   (a) If f is a continuous on a polar rectangle R given by then $a \le r \le b$, $\alpha \le \theta \le \beta$, where $0 \le \beta - \alpha \le 2\pi$, then

$$\int\int_R f(x,y)dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)rdrd\theta.$$

   (b) If f is a continuous on a polar region of the form

$$D = \{(r,\theta)|\alpha \le r \le \beta,\ h_1(\theta) \le r \le h_2(\theta)$$

   then

$$\int\int_R f(x,y)dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)rdrd\theta.$$

4. **Example.** Find the volume of the solid that lies below the surface

$$z = x^2 + y^2$$

and above the plane region R bounded by the curve

$$r = 2\cos\theta.$$

5. **Example.** Consider the double integral

$$I = \int_0^{\sqrt{\pi/2}} \int_y^{\sqrt{\pi - y^2}} \sin(x^2 + y^2)\,dx\,dy.$$

(a) Sketch and describe the region of the integration.

(b) Evaluate I by transforming to polar coordinates.

6. **Example.** Prove

$$I = \int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}.$$

7. **Example.** Find the area bounded by the cardioid

$$r = 1 + \cos\theta.$$

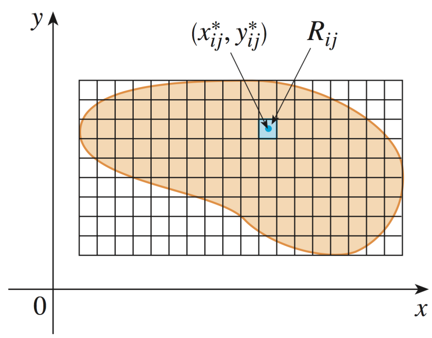
8. **Example.** Find the volume of the solid above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2 + y^2 + z^2 = 1$.

9. Additional Notes

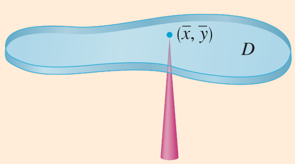
15.4 Applications of Double Integrals

1. **Mass of a Lamina.**
   We define the **mass** m of the lamina that that occupies a bounded region R, with the continuous density function $\rho(x, y)$ by

$$m = \int\int_R \rho(x, y)\, dA.$$



2. **Centroid (Centre of Mass).**
   The coordinates $(\overline{x}, \overline{y})$ of the **centroid** or **center of mass** of a lamina are **defined** by

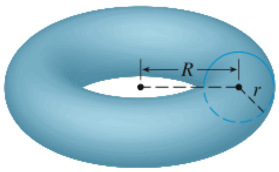


$$\begin{aligned} \overline{x} &= \frac{1}{m}\int\int_R x\rho(x, y) dA, \\ \overline{y} &= \frac{1}{m}\int\int_R y\rho(x, y) dA. \end{aligned}$$

3. **Problem.** Find the mass of the plane lamina that occupies the region bounded by $x = 0$ and $y = 4 - x^2$, with the density $\delta(x, y) = y$. Find the centre of mass of this lamina.

4. **Application of Centre of Mass to Volumes of Solids of Revolution.**
In calculus II, the volume of the solid obtained by revolving the circular disk $R = \{(x,y) : (x-R)^2 + y^2 \leq r^2\}$ about the y-axis can be found by using either the method of cylindrical shells or the washer method. Check for yourself that the volume is:

$$V = 2\pi^2 r^2 R.$$



The First Theorem of Pappus. Suppose that a plane region R is revolved around an axis in its plane generating a solid of revolution with volume V. Assume that the axis does not intersect the interior of R. Then the volume

$$V = A \cdot d$$

of the solid is the product of the area A of R and the distance d traveled by the centroid of R.

5. **Moment of Inertia.**
Let R be a plane lamina and L a straight line that may or may not lie in the xy-plane. The **moment of inertia** I of R **around the axis** L is defined to be

$$I = \int\int_R w^2 dm = \int\int_R w^2 \delta(x,y) dA$$

where $w = w(x,y)$ denotes the perpendicular distance to L from the point (x,y) of R.

6. **Three Special Cases.**
   (a) If L is the x-axis then
   $$I_x = \int\int_R y^2 \rho(x,y) dA.$$
   (b) If L is the y-axis then
   $$I_y = \int\int_R x^2 \rho(x,y) dA.$$
   (c) If L is the z-axis then
   $$I_0 = \int\int_R \left(x^2 + y^2\right) \rho(x,y) dA.$$
   I_0 is called the **polar moment of inertia** (or **moment of inertia about the origin**).

7. **Example.** A uniform rectangular plate with base length a, height b, and mass m is centered at the origin. Show that its polar moment of inertia is
$$I_0 = \frac{1}{2} m(a^2 + b^2).$$

8. **Probability and Expected Values.**
   One random variable:
   The *probability density function* f of a continuous random variable X is a function f such that $f(x) \geq 0$ for all x and $\int_{-\infty}^{\infty} f(x) = 1$. The probability that X lies in an interval $[a, b]$ is found by integrating f from a to b:
$$P(a \leq X \leq b) = \int_a^b f(x)\,dx.$$
   Two random variables:
   The *joint probability density function* f of a pair of continuous random variables X and Y is a function $f(x, y)$ such that $f(x, y) \geq 0$ for all x,y and $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y) = 1$. The probability that (X, Y) lies in a region D is found by integrating f from a to b:
$$P((X, Y) \in D) = \iint_D f(x, y)\,dx.$$
9. **Example: A tragic love story or tragedy?** Xavier and Yolanda both have classes that end at noon and they agree to meet every day after class. They arrive at the coffee shop independently. Xavier's arrival time is X and Yolanda's arrival time is Y, where X and Y are measured in minutes after noon. The individual density functions are
$$f_1(x) = \begin{cases} e^{-x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases} \qquad f_2(x) = \begin{cases} \frac{1}{50}y & \text{if } 0 \leq y \leq 10 \\ 0 & \text{otherwise} \end{cases}$$
   (Xavier arrives sometime after noon and is more likely to arrive promptly than late. Yolanda always arrives by 12:10 PM and is more likely to arrive late than promptly.) After Yolanda arrives, she'll wait for up half and hour for Xavier, but he won't wait for her. Find the probability that they meet.

10. **Expected Values.**
    If X and Y are random variables with joint density function f, we define the X**-mean** and Y**-mean**, also called the **expected values** of X and Y, to be
$$\mu_1 = \iint_{\mathbb{R}^2} xf(x, y)\,dA \qquad \mu_2 = \iint_{\mathbb{R}^2} yf(x, y)\,dA$$

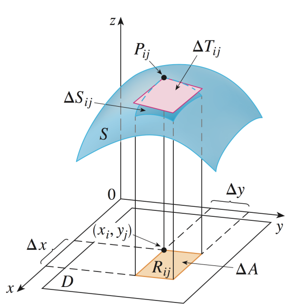
11. Additional Notes

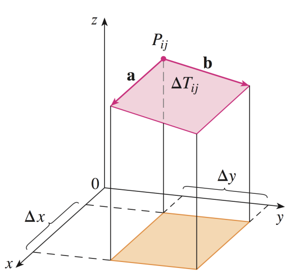
15.5 Surface Area

1. **Math Joke.**
   Student 1: "My math teacher is crazy".
   Student 2: "Why?"
   Student 1: "Yesterday he told us that five is 4+1; today he is telling us that five is 3 + 2."

2. **Reminder - Math 152.** The surface area of the surface obtained by rotating the curve $y = f(x)$, $a \le x \le b$, about the x-axis is
$$S = \int_a^b 2\pi f(x)\sqrt{1 + [f'(x)]^2}dx.$$

3. **Surface Area.** The area of the surface with equation $z = f(x, y)$, $(x, y) \in D$, where f_x and f_y are continuous, is
$$\begin{aligned} A(S) &= \iint_D \sqrt{1 + [f_x(x, y)]^2 + [f_y(x, y)]^2}\, dA \\ &= \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA \end{aligned}$$





4. **Example.** Find the area of the surface

   (a) The part of $z = xy$ that lies within the cylinder $x^2 + y^2 = 1$.

   (b) The part of sphere $x^2 + y^2 + z^2 = 4y$ that lies inside the paraboloid $y = x^2 + z^2$.

5. **Problem.** Show that the area of the part of the plane $z = ax+by+c$ that projects onto a region D in the xy-plane with area $A(D)$ is
$$A(D)\sqrt{a^2+b^2+1}.$$

6. Additional Notes

15.6 Triple Integrals

1. **Quote.** "Let me introduce my selves."
   (A Triple Integral to Math 251 class.)

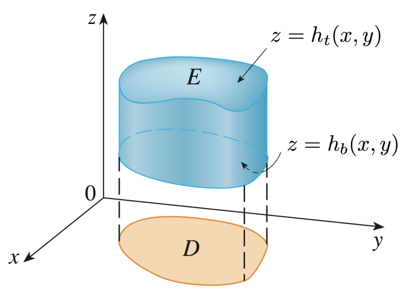
2. **Problem.** Find the volume of the solid bounded by $2x + 3y + z = 6$, $x = 0$, $y = 0$, and $z = 0$.

3. **Triple Integral.** Suppose that the region E with piecewise smooth boundary is z-**simple**, or **type 1**, i.e., each line parallel to the z-axis intersects E (if at all) in a single line segment. This means that E can be described by the inequalities

$$h_b(x, y) \le z \le h_t(x, y), \ (x, y) \in R,$$

where D is the vertical projection of E into xy-plane. Then

$$\iiint_E f(x, y, z)\, dV = \iint_D \left[\int_{h_b(x,y)}^{h_t(x,y)} f(x, y, z)\, dz\right] dA.$$



4. **Four Questions to Ask Yourself When Setting Up a Triple Integral:**

$$\begin{aligned}\iiint_E f\, dV &= \iiint_E f(\vec{x})\, dx_1\, dx_2\, dx_3 \\ &= \iint_D \left(\int_{h_b(x_2,x_3)}^{h_t(x_2,x_3)} f(\vec{x})\, dx_1\right) dA\end{aligned}$$

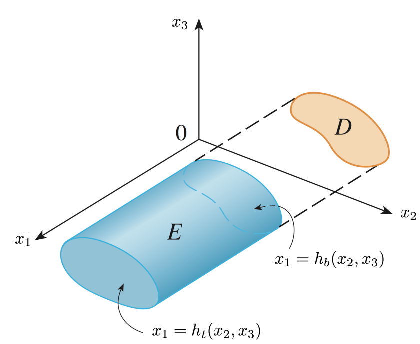
1(a.) What is the footprint/shadow/projection D in x_2x_3?
(this is not necessarily the bottom)

1(b). Are all the sides parallel to the x_1-direction?
(defining equations do not involve x_1, maybe NO sides!)

1(c). Does E have a bottom & top in the x_1 variable?

$$h_b(x_2, x_3) \le x_1 \le h_t(x_2, x_3)$$

2. What is the integration plan for the 2D region D in x_2x_3?
(double integral)



5. **Example.** Evaluate $\iiint_E \sqrt{y^2+z^2}\,dV$ where E is the region bounded by the paraboloid $x = y^2 + z^2$ and the plane $x = 4$.

6. **Mass, Centroid, and Moments of Inertia.**
If E is a solid body with the density function $\rho(x, y, z)$, then its **mass** m is given by

$$m = \iiint_T \rho(x, y, z)\ dV.$$

The coordinates of its **centroid** are

$$\overline{x} = \frac{1}{m} \iiint_E x\rho(x, y, z)dV, \qquad \overline{y} = \frac{1}{m} \iiint_E y\rho(x, y, z)dV, \qquad \overline{z} = \frac{1}{m} \iiint_E z\rho(x, y, z)dV.$$

The **moments of inertia** of E around the three coordinate axes are

$$I_x = \iiint_E (y^2 + z^2)\rho(x, y, z)dV, \qquad I_y = \iiint_E (x^2 + z^2)\rho(x, y, z)dV, \qquad I_z = \iiint_E (x^2 + y^2)\rho(x, y, z)dV.$$

7. **Example.** Find the mass of the solid T if the density is given by $\rho(x, y, z) = x + y$ and T is the region between the surfaces $z = 2 - x^2$ and $z = x^2$ for $0 \leq y \leq 3$.

8. **Example.** Find the centroid of the first-octant solid of constant density that is interior to the two cylinders $x^2 + z^2 = 1$ and $y^2 + z^2 = 1$.

   Find the moment of inertia around the z-axis of the solid.

9. **Average Value.** The average value $\overline{f}$ of the function $f(x, y, z)$ at points of the space region T is defined to be

$$\overline{f} = \frac{1}{V} \iiint_T f(x, y, z)\, dV$$

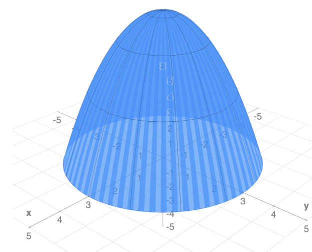
where V is the volume of T.

10. **Example.** Find the average squared distance from the origin of points of the pyramid bounded by $2x+3y+z = 6$, $x = 0$, $y = 0$, and $z = 0$.

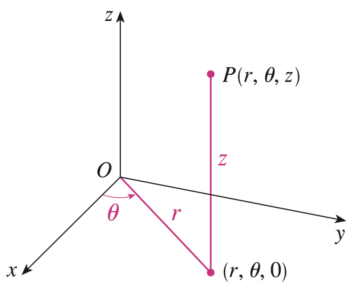
11. Additional Notes

15.7 Triple Integrals in Cylindrical Coordinates

1. **Quote.** "The essence of mathematics is not to make simple things complicated, but to make complicated things simple."
   (Stan Gudder, American Mathematician)
2. **Problem.** Find the centroid of the region bounded by the plane $z = 0$ and the paraboloid $z = 9 - x^2 - y^2$.



3. **Cylindrical Coordinates.** The **cylindrical coordinates** (r, θ, z) of a point P in space are a combination of its polar and rectangular coordinates.



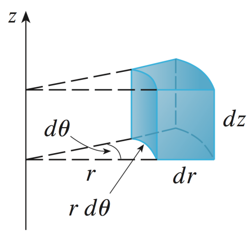
$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = z$$

4. **Example.**
   (a) Find the rectangular coordinates of the point P having cylindrical coordinates $(2, \pi/4, 4)$.
   (b) Find the cylindrical coordinates of the point Q having rectangular coordinates $(-1, 1, 1)$.

5. **Example.** Find the cylindrical-coordinates equation of the paraboloid $z = x^2 + y^2$.

6. **Example.** Describe the surface whose equation in cylindrical coordinates is $r = 2$.

7. **Volume differential in cylindrical coordinates.**



$$dV = r\ dz\ dr\ d\theta$$

8. **Triple Integration in Cylindrical Coordinates.**

$$\iiint_E f(x,y,z)dV = \iiint_U f(r\cos\theta, r\sin\theta, z) r dz dr d\theta.$$

U is a representation of limits on z, r and θ appropriate to describe the space region E in cylindrical coordinates.

9. **Example.** Find the volume of the solid T that lies below the paraboloid $z = x^2 + y^2$ and above one loop of $r = 2\cos 2\theta$.

10. **Example.** Find the mass of ice-cream in the ice-cream cone bounded above by the sphere $x^2 + y^2 + z^2 = 5$ and below by the cone $z = 2\sqrt{x^2 + y^2}$ if the density of ice-cream at the point $P(x, y, z)$ is proportional to the square of the distance of P from the z-axis.

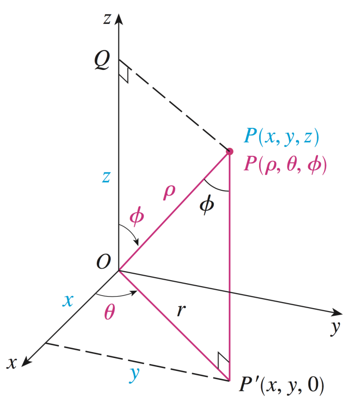
11. Additional Notes

15.8 Triple Integrals in Spherical Coordinates

1. **Problem.** Find the centroid of a homogeneous solid hemisphere of radius 1.
   The coordinates of the **centroid** are given by the triple integrals:

$$\overline{x} = \frac{1}{m}\int\int\int_E x\rho(x,y,z)dV, \qquad \overline{y} = \frac{1}{m}\int\int\int_E y\rho(x,y,z)dV, \qquad \overline{z} = \frac{1}{m}\int\int\int_E z\rho(x,y,z)dV.$$

2. **Spherical Coordinates.** Let the point P in space be given.
   Let ρ be the distance from the origin O to P.
   Let ϕ be the angle between OP and the positive z-axis.
   Let θ be the angular coordinate of the vertical projection Q of P into xy-plane.
   (ρ, θ, ϕ) are **spherical coordinates** of P.



$$x = \rho\cos\theta\sin\phi$$
$$y = \rho\sin\theta\sin\phi$$
$$z = \rho\cos\phi$$

3. **Example.**

   (a) Find the rectangular coordinates of the point P having spherical coordinates $(2, \pi/4, \pi/4)$.

   (b) Find the spherical coordinates of the point Q having rectangular coordinates $(-1, 1, 1)$.

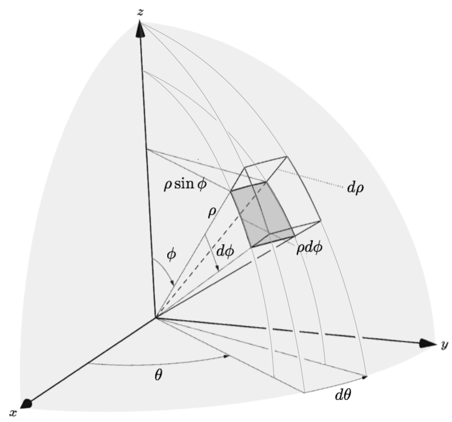
4. **Example.**
   (a) Find the **cylindrical** coordinates equation of the sphere $x^2 + y^2 + z^2 = a^2$.
   (b) Find an equation in **spherical** coordinates of the sphere $x^2 + y^2 + z^2 = a^2$.

5. **Example.** Identify the surfaces.
   (a) $r = 4\sin\theta$
   (b) $\rho^2(\sin^2\phi - 4\cos^2\phi) = 1$

6. **Example.** Sketch the solid described by the given inequalities.
   (a) $0 \leq \theta \leq \pi/2$, $r \leq z \leq 2$ (cylindrical coordinates)
   (b) $0 \leq \phi \leq \pi/3$, $\rho \leq 2$ (spherical coordinates)

7. **Example.** A solid lies above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2 + y^2 + z^2 = z$. Write a description of the solid in terms of inequalities involving spherical coordinates.

8. **Volume Differential in Spherical Coordinates.**



$$dV = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

9. **Triple Integration in Spherical Coordinates.**

$$\iiint_E f(x,y,z)\, dV = \iiint_U f(\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi)\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta.$$

U is a representation of limits on ρ, θ and ϕ appropriate to describe the space region E in spherical coordinates.

10. **Example.** Find the volume and centroid of the uniform solid that lies inside the sphere $\rho = a$ and above the cone $r = z$.

11. **Time Machine.** You wish to drill a hole in a sphere, removing points which lie within a circular cylinder whose axis goes through the center of the sphere. Suppose the sphere has radius $= 1$. What should be the radius of the hole so that exactly half the volume of the sphere is removed?

12. Additional Notes

15.9 Change of Variables in Multiple Integrals

1. **Time Machine.** Evaluate

$$\int_0^1 x\sqrt{1+x^2}dx$$

2. **Reminder.** Evaluate

$$\iiint_E e^{\sqrt{x^2+y^2+z^2}}dV$$

where E is enclosed by the sphere $x^2+y^2+z^2=9$ in the first octant.

3. **Example:** Evaluate the integral $\iint_R e^{\frac{x+y}{x-y}}\, dA$ where R is the trapezoidal region with vertices $(1,0)$, $(2,0)$, $(0,-2)$, and $(0,-1)$.

4. **Problem.** A transformation is defined by

$$x = \frac{1}{2}(u+v),\ y = \frac{1}{2}(u-v).$$

Find the image of the region

$$S = \{(u,v)|1 \le v \le 2, -v \le u \le v\}.$$

5. **Question.** How does a change of variables affect double integral?

6. **Jacobian.** The Jacobian of the transformation T given by $x = g(u, v)$ and $y = h(u, v)$ is

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\[2ex] \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

7. **Change of Variables in a Double Integral.** Suppose that T is C^1 transformation whose Jacobian is nonzero and that maps a region S in the uv-plane onto a region R in the xy-plane. Suppose that f is continuous on R and that R and S are type I or type II plane regions. Suppose also that T is one-to-one, except perhaps on the boundary of S. Then

$$\iint_R f(x,y)\, dA = \iint_S f(x(u,v), y(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| \, du\, dv.$$

8. **Example.** Find the Jacobian of the following transformations:

   (a) $x = \frac{1}{2}(u+v),\ y = \frac{1}{2}(u-v)$.

   (b) $x = r\cos\theta$, $y = r\sin\theta$.

9. **Example:** Evaluate the integral $\iint_R e^{\frac{x+y}{x-y}}\, dA$ where R is the trapezoidal region with vertices $(1,0)$, $(2,0)$, $(0,-2)$, and $(0,-1)$.

10. **Example.** Evaluate

$$\iint_R xy\, dA$$

where R is the region in the first quadrant bounded by the lines $y = x$ and $y = 3x$ and the hyperbolas $xy = 1$ and $xy = 3$. (Use the change of variable: $x = \frac{u}{v},\ y = v$.)

11. **Example:** Evaluate the
$$\iint_R \sin\left(9x^2 + 4y^2\right)\, dA,$$
where R is the region in the first quadrant bounded by the ellipse $9x^2 + 4y^2 = 1$, by making and appropriate change of variables.

12. **Triple Integrals.** Under hypothesis similar to those for the change of variables in a double integral

$$\iiint_R f(x,y,z)\,dA =$$

$$\iiint_S f(x(u,v,w),y(u,v,w),z(u,v,w))\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right|\,du\,dv\,dw$$

where

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} & \dfrac{\partial x}{\partial w} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} & \dfrac{\partial y}{\partial w} \\ \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} & \dfrac{\partial z}{\partial w} \end{vmatrix}$$

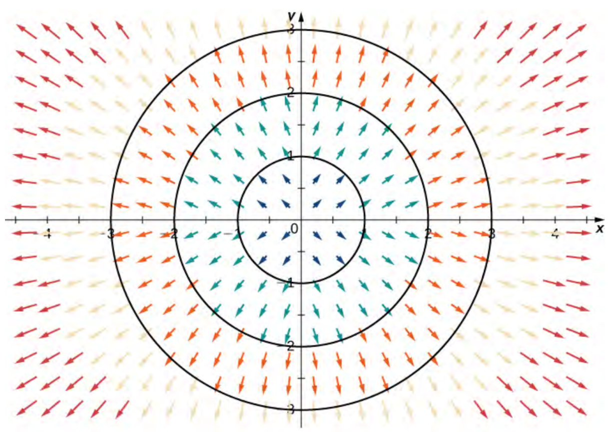
is the **Jacobian** of the transformation T given by $x = x(u,v,w)$, $y = y(u,v,w)$, and $z = z(u,v,w)$.

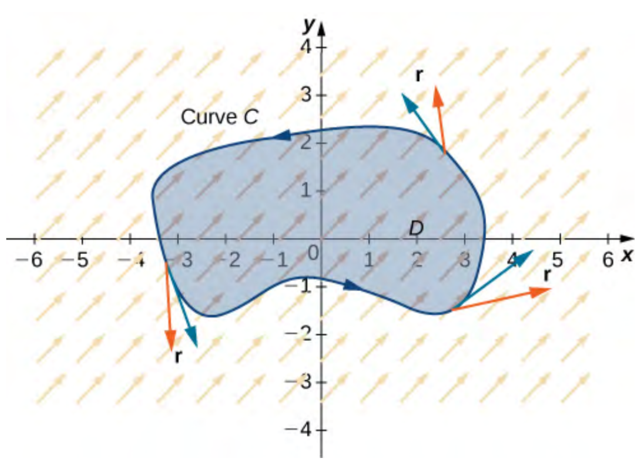
13. **Example.** Derive the formula for triple integration in spherical coordinates.

14. Additional Notes

Part 16

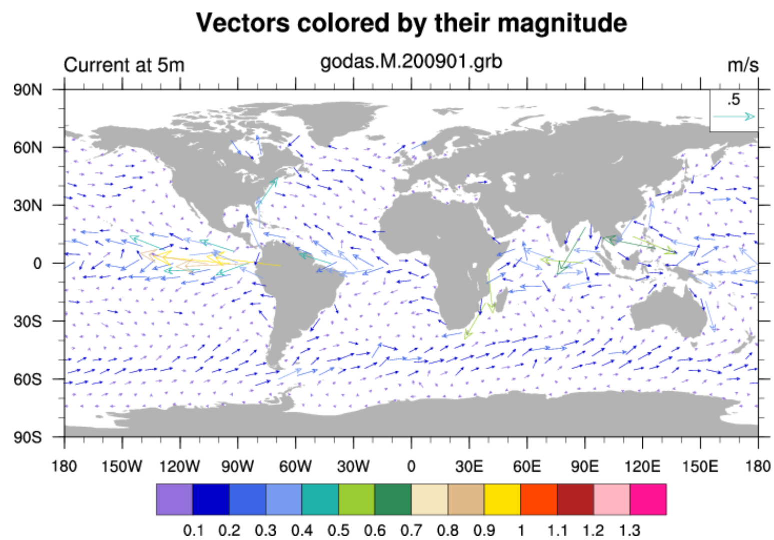
Vector Calculus

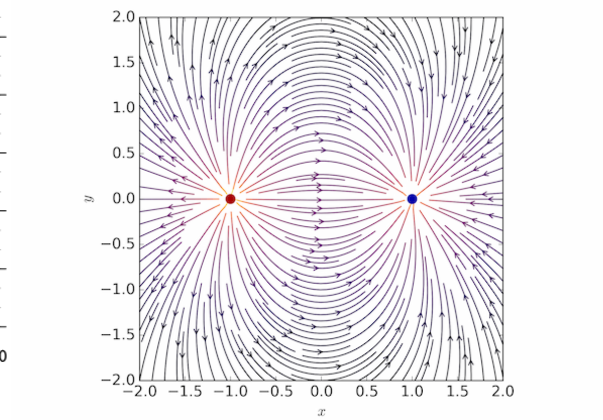




16.1 Vector Fields

1. **Definition.** Let D be a set in $\mathbb{R}^2$ (a plane region). A **vector field on** $\mathbb{R}^2$ is a function $\vec{\mathbf{F}}$ that assigns to each point (x, y) in D a two-dimensional vector $\vec{\mathbf{F}}(x, y)$.

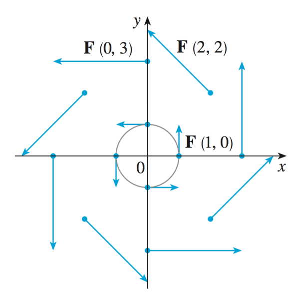




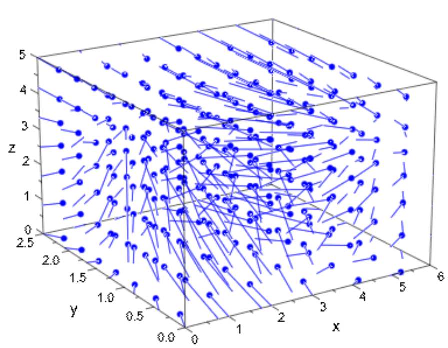
2. **Example.** The vector field on $\mathbb{R}^2$ defined by

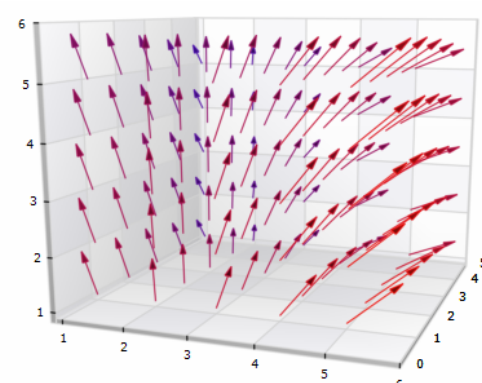
$$\vec{\mathbf{F}}(x, y) = \frac{-y\hat{\mathbf{i}} + x\hat{\mathbf{j}}}{\sqrt{x^2 + y^2}}.$$

is shown in the diagram.



3. **Definition.** Let E be a set in $\mathbb{R}^3$. A **vector field on** $\mathbb{R}^3$ is a function $\vec{\mathbf{F}}$ that assigns to each point (x, y, z) in E a three-dimensional vector $\vec{\mathbf{F}}(x, y, z)$.





4. **Example.** Assume that an object with mass M is located at the origin in $\mathbb{R}^3$. Let an object P with mass m be located at the point (x, y, z). Show that the gravitational force acting on the object P is

$$\vec{\mathbf{F}}(P) = -\frac{mMG}{|\vec{\mathbf{x}}|^3}\vec{\mathbf{x}}$$

where $\vec{\mathbf{x}} = \langle x, y, z\rangle$ is the position vector of the object P and G is the gravitational constant.

5. **Gradient Field.** If f is a scalar function of three variables, its **gradient** is a vector field on $\mathbb{R}^3$ given by

$$\vec{\nabla} f(x, y, z) = f_x(x, y, z)\hat{\mathbf{i}} + f_y(x, y, z)\hat{\mathbf{j}} + f_z(x, y, z)\hat{\mathbf{k}}$$

6. **Example.** Find the gradient field of the scalar function

$$f(x, y, z) = \frac{1}{\sqrt{x^2 + y^2 + z^2}}.$$

7. **Conservative Fields.** A vector filed $\mathbf{F}$ is called a **conservative vector field** if it is the gradient of some scalar function, that is, if there exists a function f such that

$$\mathbf{F} = \vec{\nabla} f.$$

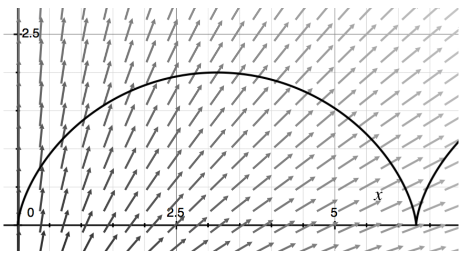
In this situation f is called a **potential function** for $\mathbf{F}$.

8. **Example.** Is the gravitational field a conservative vector field?

9. Additional Notes.

16.2 Line Integrals

1. **Motivating Problem.** Find the work done by the force $\vec{\mathbf{F}}(x, y) = x\,\hat{\mathbf{i}} + (y+2)\,\hat{\mathbf{j}}$ in moving an object along an arch of cycloid
$$\vec{\mathbf{r}}(t) = (t - \sin t)\,\hat{\mathbf{i}} + (1 - \cos t)\,\hat{\mathbf{j}},\ 0 \le t \le 2\pi.$$



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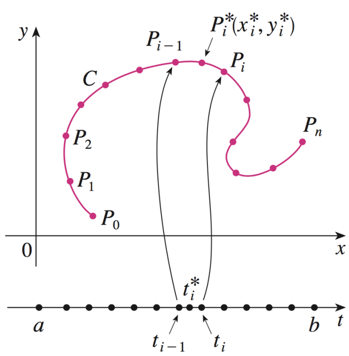
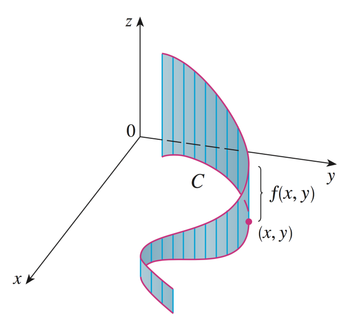
2. Double and triple integrals are based on integrating a function over an area or a volume. A *line integral* is based on integrating along a space curve. There are two types of integrals following a space curve:
Integrating ...
   (a) ... a **scalar field** $f(\vec{x})$ with respect to **arclength** s,
   (b) ... a **vector field** $\vec{\mathbf{F}}(\vec{x})$ with respect to a **space curve** $\vec{\mathbf{r}}(t)$.

3. **Line Integral of a Scalar Field in 2D (with respect to arclength):**

Definition. If f is defined on a smooth curve C given by
$$x = x(t),\ y = y(t),\ a \le t \le b,$$
then the **line integral of f along C** is
$$\int_C f(x, y)\, ds = \int_a^b f(x(t), y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$





4. **Example.** Evaluate $\int_C x\, ds$, where the **piecewise-smooth curve** C is the arc of parabola $y = x^2$ from $(1, 1)$ to $(3, 9)$ followed by the line segment from $(3, 9)$ to $(4, 7)$.

5. **Line Integral of a Scalar Field in 3D (with respect to arclength):**

Definition. Suppose that C is a smooth space given by

$$x = x(t),\ y = y(t),\ z = z(t),\quad a \le t \le b.$$

If f is a function of three variables that is continuous on some region containing C then the **line integral of** f **along** C is

$$\int_C f(x,y,z)ds = \int_a^b f(x(t),y(t),z(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt.$$

Also

$$\int_C f(x,y,z)\ ds = \int_a^b f(\vec{\mathbf{r}}(t))\ |\vec{\mathbf{r}}'(t)|\ dt$$

where

$$\vec{\mathbf{r}}(t) = x(t)\ \hat{\mathbf{i}} + y(t)\ \hat{\mathbf{j}} + z(t)\ \hat{\mathbf{k}},\ a \le t \le b.$$

6. **Example.** Find the mass of the helix

$$\vec{\mathbf{r}}(t) = \cos t\ \hat{\mathbf{i}} + \sin t\ \hat{\mathbf{j}} + t\ \hat{\mathbf{k}},\ 0 \le t \le 2\pi$$

if the density at the point is given by $\rho(x,y,z) = x^2 + y^2 + z^2$.

7. **Line Integral of a Vector Field (along a space curve):**

Definition. Let $\vec{\mathbf{F}}$ be a continuous vector field defined on a smooth curve C given by a vector function $\vec{\mathbf{r}}(t)$, $a \le t \le b$. Then the **line integral of $\vec{\mathbf{F}}$ along** C is defined as

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_a^b \vec{\mathbf{F}}(\vec{\mathbf{r}}(t)) \cdot \vec{\mathbf{r}}\,'(t)\, dt = \int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\, ds.$$

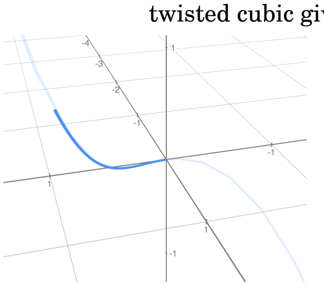
If

$$\vec{\mathbf{F}}(x,y,z) = P(x,y,z)\,\hat{\mathbf{i}} + Q(x,y,z)\,\hat{\mathbf{j}} + R(x,y,z)\,\hat{\mathbf{k}}$$
$$\vec{\mathbf{r}}(t) = x(t)\,\hat{\mathbf{i}} + y(t)\,\hat{\mathbf{j}} + z(t)\,\hat{\mathbf{k}}$$

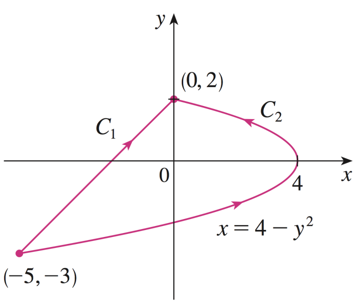
then

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_C P\,dx + Q\,dy + R\,dz.$$

8. **Example.** Determine the work done by the force $\vec{\mathbf{F}}(x,y,z) = xy\,\hat{\mathbf{i}} + yz\,\hat{\mathbf{j}} + xz\,\hat{\mathbf{k}}$ in moving a particle along the twisted cubic given by $\vec{\mathbf{r}}(t) = t\,\hat{\mathbf{i}} + t^2\,\hat{\mathbf{j}} + t^3\,\hat{\mathbf{k}}$, $0 \le t \le 1$.



9. **Example:** Evaluate $\displaystyle\int_C y^2\,dx + x\,dy$ where
(a) C is the line segment from $(-5,-3)$ to $(0,2)$,
(b) C is the arc of the parabola $x = 4 - y^2$ from $(-5,-3)$ to $(0,2)$.



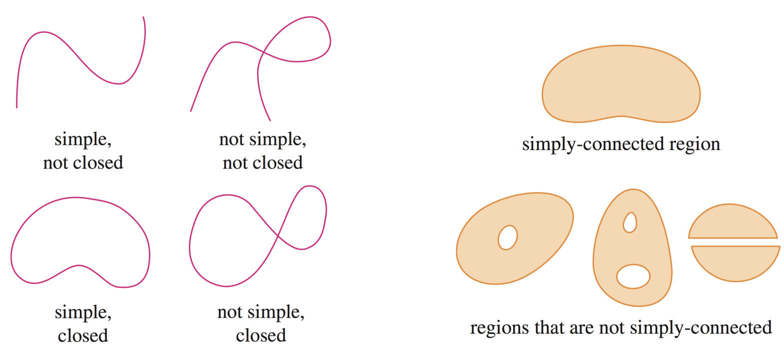
10. Additional Notes

16.3 Fundamental Theorem for Line Integrals

1. **Quote.** "There again, that is a fundamental principle: no two situations are alike."
   (Lakhdar Brahimi, Algerian Diplomat, 1934 -)
2. **Reminder.** The Fundamental Theorem of Calculus: If F' is continuous on $[a, b]$ then
$$\int_a^b F'(x)dx = F(b) - F(a).$$
3. **Theorem 2 (Fundamental Theorem for Line Integrals).** Let C be a smooth curve given by the vector function $\vec{\mathbf{r}}(t)$, $a \le t \le b$. Let f be a differentiable function of two or three variables whose gradient vector ∇f is continuous on C. Then
$$\int_C \nabla f \cdot d\vec{\mathbf{r}} = f(\vec{\mathbf{r}}(b)) - f(\vec{\mathbf{r}}(a)).$$

4. **Example.** Find the work done by the gravitational field
$$\vec{\mathbf{F}}(\vec{\mathbf{x}}) = -\frac{mMG}{|\vec{\mathbf{x}}|^3}\vec{\mathbf{x}}$$
   in moving a particle with mass m from the point $(1, 0, 0)$ to the point $(1, 2, 3)$.

5. **Terminology for Curves and Regions:**
   A curve C is **simple** if it does not intersect itself.
   A curve C is **closed** if its initial point equal to its terminal point.
   A region D is **open** if for every point $P \in D$ there is a disk with centre P that lies entirely in D.
   A region D is **connected** if any two points can be connect by a path that lies in D.
   A region D is **simply-connected** if every simple closed curve in D encloses only points that are in D.



6. **Properties of Conservative Vector Fields:**
   Let $\vec{\mathbf{F}}$ be a vector field over an *open simply-connected* region D. The following are equivalent:
   (a) $\vec{\mathbf{F}}$ is conservative,
   (b) $\displaystyle\int_{\vec{\mathbf{x}}_0}^{\vec{\mathbf{x}}_1} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$ is *independent of path* from $\vec{\mathbf{x}}_0$ to $\vec{\mathbf{x}}_1$,
   (c) $\displaystyle\oint_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = 0$ for all closed paths C in D.

7. **Definition.** If $\vec{\mathbf{F}}$ is a continuous vector field with domain D, we say that the line integral $\displaystyle\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$ is **independent of path** if
$$\int_{C_1} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_{C_2} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$$
for any two paths C_1 and C_2 in D that have the same initial and terminal points.

8. **Theorem 3.** $\displaystyle\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$ is independent of path in D if and only if $\displaystyle\oint_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = 0$ for every closed path C in D.

9. **Theorem 4.** Suppose that $\vec{\mathbf{F}}$ is a vector field that is continuous on an open connected region D. If $\displaystyle\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$ is independent of path in D, then $\vec{\mathbf{F}}$ is a conservative vector field on D; that is, there exists a function F such that $\nabla f = \vec{\mathbf{F}}$.

10. **How can we determine if a vector field is conservative?**

In the following two theorems assume $\vec{\mathbf{F}}(x,y) = P(x,y)\,\hat{\mathbf{i}} + Q(x,y)\,\hat{\mathbf{j}}$ is a vector field, where P and Q have continuous first-order partial derivatives on a domain D.

12. **Theorem 5.** If $\vec{\mathbf{F}}$ is conservative then

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

throughout D,

13. **Theorem 6.** If D is *open* and *simply-connected* and

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

throughout D, then $\vec{\mathbf{F}}$ is conservative.

[Proof of Theorem 6 is delayed until Section 16.4]

14. **Example.** Determine whether or not the field

$$\vec{\mathbf{F}}(x,y) = (2x\cos y - y\cos x)\,\hat{\mathbf{i}} + (-x^2\sin y - \sin x)\,\hat{\mathbf{j}}$$

is conservative. If it is find a function f such that $\vec{\mathbf{F}} = \nabla f$.

15. **Example.** Determine $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$ for

$$\vec{\mathbf{F}}(x, y) = (y^2/x^2)\,\hat{\mathbf{i}} - (2y/x)\,\hat{\mathbf{j}}$$

and C a curve from $P(1, 1)$ to $Q(4, -2)$.

16. **Example.** Let

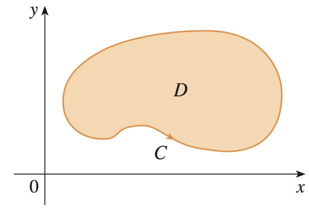
$$\vec{\mathbf{F}}(x, y) = \frac{-y\,\hat{\mathbf{i}} + x\,\hat{\mathbf{j}}}{x^2 + y^2}.$$

(a) Show that $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$.

(b) Show that $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$ is not independent of path.

17. Additional Notes

16.4 Green's Theorem



1. Green's Theorem is considered the double integral counterpart to the Fundamental Theorem of Calculus. It relates the 2D integral of certain type of function (a sort of "derivative") over a region D with the 1D integral of a vector field over the boundary ∂D.

2. **Green's Theorem.** Let C be a positively oriented, piecewise-smooth, simple closed curve in the plane and let D be the region bounded by C. If P and Q have continuous partial derivatives on an open region that contains D, then
$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA.$$

3. **Notation.**
$$\int_C P\,dx + Q\,dy = \oint_C P\,dx + Q\,dy.$$

4. **Example.** Use Green's Theorem to evaluate the line integral

$$\oint_C e^y\,dx + 2xe^y\,dy$$

where C is the square with sides $x = 0$, $x = 1$, $y = 0$, and $y = 1$.

5. **Example.** Use Green's Theorem to evaluate $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$ if $\vec{\mathbf{F}} = \langle e^{-x} + y^2, e^{-y} + x^2 \rangle$ and C consists of the arc of the curve $y = \sin x$ from $(0, 0)$ to $(\pi, 0)$ and the line segment from $(\pi, 0)$ to $(0, 0)$.

6. **Area.** Green's Theorem gives the following formulas for the area of D:

$$A = \oint_C xdy = -\oint_C ydx = \frac{1}{2}\oint_C xdy - ydx.$$

7. **Example.** Find the area under one arch of the cycloid $x = t - \sin t$, $y = 1 - \cos t$.

8. **Example.** Evaluate $\oint_C y^2\ dx + 3xy\ dy$, where C is the boundary of the semiannular region D in the upper half-plane between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

9. **Example.** If $\vec{\mathbf{F}}(x, y) = \dfrac{-y\hat{\mathbf{i}} + x\hat{\mathbf{j}}}{x^2 + y^2}$, show that $\oint_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = 2\pi$ for every positively oriented simple closed path that encloses the origin.

10. Additional Notes

16.5 Curl and Divergence

1. **Differential Operators:**
The Fundamental Theorem of Calculus, the Fundamental Theorem of Line Integrals, and Green's theorem all associate an integral of a "derivative" of a function (scalar or vector) over some domain with an integral of another function over the boundary of the domain. What is meant by "derivative" in the context of scalar and vector fields?

Let $\vec{\nabla} = \left\langle \dfrac{\partial}{\partial x}, \dfrac{\partial}{\partial y}, \dfrac{\partial}{\partial z} \right\rangle$

We call $\vec{\nabla}$ a **differential operator** (an operator is a function whose domain and codomain are sets of functions). With this notation, the gradient can be thought of as the product of the vector $\vec{\nabla}$ with the scalar f:

$$\vec{\nabla} f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle \qquad \text{(scalar product} \to \textbf{gradient}\text{)}$$

We can also form "derivatives" of vector fields $\vec{\mathbf{F}} = P\,\hat{\mathbf{i}} + Q\,\hat{\mathbf{j}} + R\,\hat{\mathbf{k}}$ as follows:

$$\vec{\nabla} \cdot \vec{\mathbf{F}} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \qquad \text{(dot product} \to \textbf{divergence}\text{)}$$

$$\vec{\nabla} \times \vec{\mathbf{F}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \hat{\mathbf{i}} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \hat{\mathbf{j}} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \hat{\mathbf{k}} \qquad \text{(cross product} \to \textbf{curl}\text{)}$$

Another differential operator is the **Laplacian operator** which is the divergence of the gradient ($\vec{\nabla} \cdot \vec{\nabla}$):

$$\vec{\nabla}^2 f = \vec{\nabla} \cdot \vec{\nabla} f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

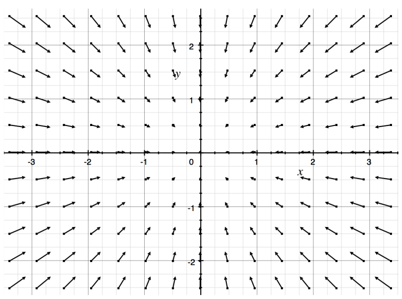
In this section we introduce and explore the notions of *divergence* and *curl* of a vector field.

2. **Definition (curl).** If $\vec{\mathbf{F}} = P\,\hat{\mathbf{i}} + Q\,\hat{\mathbf{j}} + R\,\hat{\mathbf{k}}$ is a vector field on $\mathbb{R}^3$ and the partial derivatives of P, Q, and R all exist, then the **curl** of $\vec{\mathbf{F}}$ is the vector field on $\mathbb{R}^3$ defined by:

$$\operatorname{curl}\vec{\mathbf{F}} = \vec{\nabla} \times \vec{\mathbf{F}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\hat{\mathbf{i}} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\hat{\mathbf{j}} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\hat{\mathbf{k}}$$

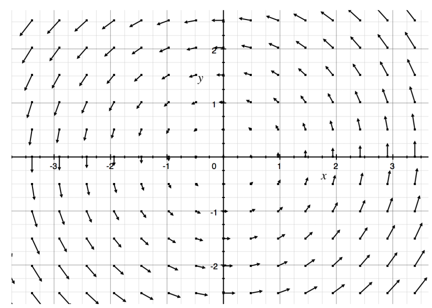
3. **Examples.** Find the curl of each of the following vector fields.

(a) $\vec{\mathbf{F}} = -x\,\hat{\mathbf{i}} - y\,\hat{\mathbf{j}}$



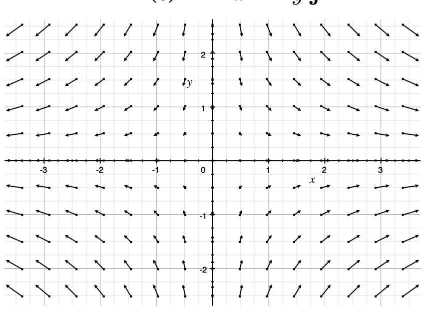
$\vec{\mathbf{F}} = -x\,\vec{\mathbf{i}} - y\,\vec{\mathbf{j}}$

(b) $\vec{\mathbf{F}} = -y\,\hat{\mathbf{i}} + x\,\hat{\mathbf{j}}$



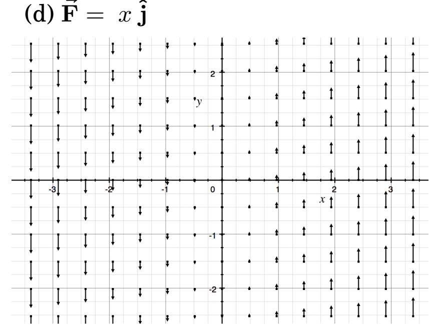
$\vec{\mathbf{F}} = -y\,\vec{\mathbf{i}} + x\,\vec{\mathbf{j}}$

(c) $\vec{\mathbf{F}} = x\,\hat{\mathbf{i}} - y\,\hat{\mathbf{j}}$



$\vec{\mathbf{F}} = x\,\vec{\mathbf{i}} - y\,\vec{\mathbf{j}}$

(d) $\vec{\mathbf{F}} = \ x\,\hat{\mathbf{j}}$



$\vec{\mathbf{F}} = 0\,\vec{\mathbf{i}} - x\,\vec{\mathbf{j}}$

4. **Theorem (curl of conservative field is** 0**).** If f is a function of three variables that has continuous second order partial derivatives, then

$$\operatorname{curl}(\vec{\nabla} f) = \vec{\mathbf{0}}.$$

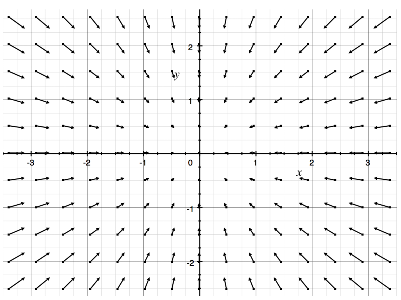
5. **Theorem.** If $\vec{\mathbf{F}}$ is a vector field defined on all of $\mathbb{R}^3$ whose component functions have continuous partial derivatives and $\operatorname{curl}\vec{\mathbf{F}} = \vec{\mathbf{0}}$, then $\vec{\mathbf{F}}$ is conservative.

6. **Definition (divergence).** If $\vec{\mathbf{F}} = P\,\hat{\mathbf{i}} + Q\,\hat{\mathbf{j}} + R\,\hat{\mathbf{k}}$ is a vector field on $\mathbb{R}^3$ and $\dfrac{\partial P}{\partial x}$, $\dfrac{\partial Q}{\partial y}$, and $\dfrac{\partial R}{\partial z}$ all exist, then the **divergence** of $\vec{\mathbf{F}}$ is the function on $\mathbb{R}^3$ defined by:

$$\text{div}\,\vec{\mathbf{F}} = \vec{\nabla} \cdot \vec{\mathbf{F}} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

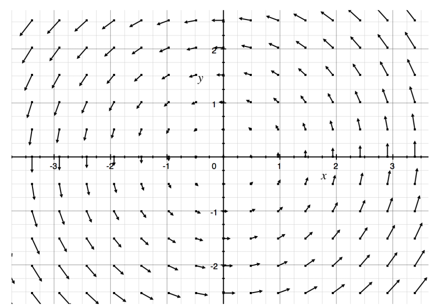
7. **Examples.** Find the divergence of each of the following vector fields.

(a) $\vec{\mathbf{F}} = -x\,\hat{\mathbf{i}} - y\,\hat{\mathbf{j}}$



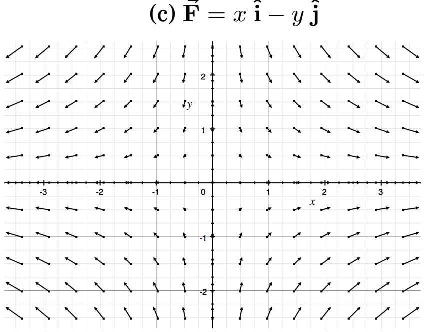
$\vec{\mathbf{F}} = -x\,\vec{\mathbf{i}} - y\,\vec{\mathbf{j}}$

(b) $\vec{\mathbf{F}} = -y\,\hat{\mathbf{i}} + x\,\hat{\mathbf{j}}$



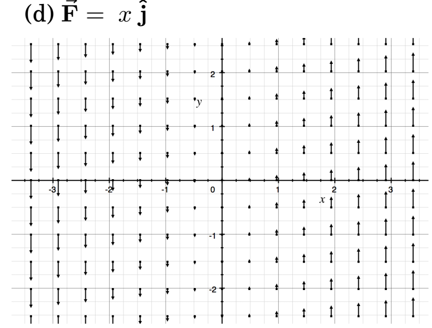
$\vec{\mathbf{F}} = -y\,\vec{\mathbf{i}} + x\,\vec{\mathbf{j}}$

(c) $\vec{\mathbf{F}} = x\,\hat{\mathbf{i}} - y\,\hat{\mathbf{j}}$



$\vec{\mathbf{F}} = x\,\vec{\mathbf{i}} - y\,\vec{\mathbf{j}}$

(d) $\vec{\mathbf{F}} = \ x\,\hat{\mathbf{j}}$



$\vec{\mathbf{F}} = 0\,\vec{\mathbf{i}} - x\,\vec{\mathbf{j}}$

8. **Theorem.** If $\vec{\mathbf{F}} = P\,\hat{\mathbf{i}} + Q\,\hat{\mathbf{j}} + R\,\hat{\mathbf{k}}$ is a vector field on $\mathbb{R}^3$ and P, Q, and R have continuous second-order partial derivatives, then

$$\text{div curl}\,\vec{\mathbf{F}} = 0.$$

In ∇ notation this says $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{\mathbf{F}}) = 0$.

9. **Basic Identities of Vector Analysis.**
    1. $\nabla(f+g) = \nabla f + \nabla g$
    2. $\nabla(cf) = c\nabla f$, for some constant c
    3. $\nabla(fg) = f\,\nabla g + g\,\nabla f$
    4. $\nabla(f/g) = (g\,\nabla f - f\,\nabla g)/g^2$, at points $\vec{\mathbf{x}}$ where $g(\vec{\mathbf{x}}) \neq 0$
    5. $\text{div}(\vec{\mathbf{F}} + \vec{\mathbf{G}}) = \text{div}\,\vec{\mathbf{F}} + \text{div}\,\vec{\mathbf{G}}$
    6. $\text{curl}(\vec{\mathbf{F}} + \vec{\mathbf{G}}) = \text{curl}\,\vec{\mathbf{F}} + \text{curl}\,\vec{\mathbf{G}}$
    7. $\text{div}(f\vec{\mathbf{F}}) = f\,\text{div}\,\vec{\mathbf{F}} + \vec{\mathbf{F}}\cdot\nabla f$
    8. $\text{div}(\vec{\mathbf{F}} \times \vec{\mathbf{G}}) = \vec{\mathbf{G}}\cdot\text{curl}\,\vec{\mathbf{F}} - \vec{\mathbf{F}}\cdot\text{curl}\,\vec{\mathbf{G}}$
    9. $\text{div curl}\,\vec{\mathbf{F}} = 0$
    10. $\text{curl}(f\vec{\mathbf{F}}) = f\,\text{curl}\,\vec{\mathbf{F}} + \vec{\mathbf{F}} \times \nabla f$
    11. $\text{curl}\,\nabla f = \vec{\mathbf{0}}$
    12. $\nabla^2(fg) = f\,\nabla^2 g + g\,\nabla^2 f + 2(\nabla f\cdot\nabla g)$
    13. $\text{div}(\nabla f \times \nabla g) = 0$
    14. $\text{div}(f\,\nabla g - g\,\nabla f) = f\,\nabla^2 g - g\,\nabla^2 f$

You are encouraged to prove these identities yourself.

10. **Vector Forms of Green's Theorem:**
The curl and divergence operators allow us to rewrite Green's Theorem in vector form.
Let C (oriented counter-clockwise) be given by the vector function

$$\vec{\mathbf{r}}(t) = x(t)\,\hat{\mathbf{i}} + y(t)\,\hat{\mathbf{j}} \quad a \le t \le b.$$

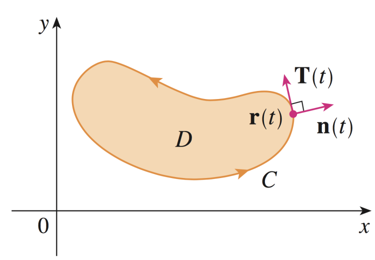
For the 2D vector field $\vec{\mathbf{F}} = P\,\hat{\mathbf{i}} + Q\,\hat{\mathbf{j}}$ since $\text{curl}\,\vec{\mathbf{F}} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\hat{\mathbf{k}}$ we have the following.

Green's Theorem (curl form):.

$$\oint_C \vec{\mathbf{F}}\cdot d\vec{\mathbf{r}} = \iint_D (\text{curl}\,\vec{\mathbf{F}})\cdot\hat{\mathbf{k}}\,dA$$

For the divergence form of Green's Theorem, observe the unit tangent and (outward) unit normal to C are

$$\vec{\mathbf{T}}(t) = \frac{x'(t)}{|\,\vec{\mathbf{r}}\,'(t)|}\,\hat{\mathbf{i}} + \frac{y'(t)}{|\,\vec{\mathbf{r}}\,'(t)|}\,\hat{\mathbf{j}}, \qquad \vec{\mathbf{n}}(t) = \frac{y'(t)}{|\,\vec{\mathbf{r}}\,'(t)|}\,\hat{\mathbf{i}} - \frac{x'(t)}{|\,\vec{\mathbf{r}}\,'(t)|}\,\hat{\mathbf{j}}.$$



Therefore

$$\begin{aligned}\oint_C \vec{\mathbf{F}}\cdot\vec{\mathbf{n}}\,ds &= \int_a^b (\vec{\mathbf{F}}\cdot\vec{\mathbf{n}})|\vec{\mathbf{r}}\,'(t)|\,dt \\ &= \int_a^b P(x(t),y(t))\,y'(t)\,dt - Q(x(t),y(t))\,x'(t)\,dt \\ &= \int_a^b P\,dy - Q\,dx = \iint_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right)\,dA\end{aligned}$$

The integrand in the last integral is the divergence of $\vec{\mathbf{F}}$, which gives the following.

Green's Theorem (divergence form):.

$$\oint_C \vec{\mathbf{F}}\cdot\vec{\mathbf{n}}\,ds = \iint_D \text{div}\,\vec{\mathbf{F}}(x,y)\,dA$$

11. Additional Notes

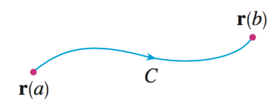
16.6 Integral Theorems Summary, and a Look to Calculus IV

1. **Integral Theorems:**

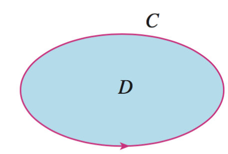
2. The following table list the higher-dimensional versions of the fundamental theorems of calculus. These are referred to as the *integral theorems*. The first three were featured in calculus II and III, the last two will be featured in calculus IV.

Fundamental Theorem of Calculus $\quad \int_a^b F'(x)\,dx = F(b) - F(a)$

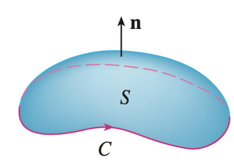
Fundamental Theorem for Line Integrals $\quad \int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$



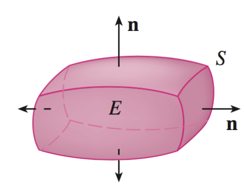
Green's Theorem $\quad \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \int_C P\,dx + Q\,dy$



Stokes' Theorem $\quad \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{r}$



Divergence Theorem $\quad \iiint_E \operatorname{div} \mathbf{F}\,dV = \iint_S \mathbf{F} \cdot d\mathbf{S}$



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