May 8, 1945 Guelma University **Average time : four weeks** **Department of Mathematics**

2^{nd} year : licence Mathematics **Sheet.1 :Calculus, continuity and differentiability**

EXERCICE .1.

(A): Find the point set M in each case and explain why $f:\mathbb{R}^2\backslash M\to\mathbb{R}$ is continuous. Finally check whether the function has a continuous extension to either $\mathbb{R}^2$ or to $\mathbb{R}^2\backslash L$, where $L\subset M$.

$$1)\ f(x,y)=\frac{x^2-y^2}{x^2+y^2},\quad 2)\ f(x,y)=\frac{3x-2y}{2x-3y},\quad (*)3)\ f(x,y)=\frac{x^3-y^3}{x-y},\quad (*)4)\ f(x,y)=\frac{1-e^{xy}}{xy}$$

(B): Sketch in each of the cases below the domain of the given function or victor function. Then examine whether the (victor) function has limit for $(x,y)\to(0,0)$, and indicate this when it exists.

$$1)\ f(x,y)=\frac{\sin(xy)}{x},\ (*)2)\ f(x,y)=\frac{1}{x}\sin y,\ (*)3)\ f(x,y)=\left(\frac{x\sin y}{\sqrt{x^2+y^2}},\frac{x^2+y^2+x^2y^2}{x^2+3y^2}\right),\ 4)\ f(x,y)=\left(\frac{x}{x+y},\sqrt{x+y}\right)$$

EXERCICE 2.

(1) Let $f:\mathbb{R}^2\backslash\{(0,0)\}\to\mathbb{R}$ be given by $f(x,y)=\dfrac{x^2y^2}{x^2y^2+(x-y)^2}$. Show that

$$\lim_{x\to0}\left(\lim_{y\to0}f(x,y)\right)=\lim_{y\to0}\left(\lim_{x\to0}f(x,y)\right)=0,$$

and that f nevertheless does not have a limit for $(x,y)\to(0,0)$.

(2) $(*)$Let $f:\mathbb{R}^2\backslash\{(0,0)\}\to\mathbb{R}$ be given by

$$f(x,y)=\begin{cases}\sin\left(\frac{1}{x}\right)\sin y, & x\neq0,\\ 0, & x=0.\end{cases}$$

Prove that $f(x,y)\to0$ for $(x,y)\to(0,0)$ and that we nevertheless do not have

$$\lim_{x\to0}\left(\lim_{y\to0}f(x,y)\right)=\lim_{y\to0}\left(\lim_{x\to0}f(x,y)\right).$$

EXERCICE 3.

(A): Find in each of the following cases the gradient of the given functions

$$1)\ f(x,y)=\arctan(x/y),\text{ for }y\neq0,\ (*)2)\ f(x,y)=\ln\left(\sqrt{x^2+y^2}\right),\text{ for }(x,y)\neq(0,0)$$

$$3)\ f(x,y,z)=xe^{y+xz},\text{ for }(x,y,z)\in\mathbb{R}^3,\ (*)4)\ f(x,y,z)=\exp(x^2-y+z),\text{ for }(x,y,z)\in\mathbb{R}^3.$$

(B): Use the chain rule to calculate the derivative of the function $F(u)=f(X(u))$, i.e. without finding $F(u)$ explicitly in the following cases

$$1)\ f(x,y)=xy,\ X(u)=(e^u,\cos u),\ u\in\mathbb{R},\quad (*)2)\ f(x,y)=e^{xy},\ X(u)=(3u^2,u^3),\ u\in\mathbb{R}.$$

(C): Calculate the partial derivatives of the function $F(u,v)=f(X(u,v))$ by means of the chain rule, i.e. without finding $F(u,v)$ explicitly, in the following cases

$$1)\ f(x,y)=x^2y,\ X(u,v)=(u+v,uv),\ (u,v)\in\mathbb{R}^2,\quad (*)2)\ f(x,y)=\frac{x}{x+y},\ X(u,v)=(u^2+v^2,2uv),\ (u,v)\in\mathbb{R}^2.$$

EXERCICE 4. Let u and w denote two functions in two variables. We assume that they fulfil the differential equations

$$a\frac{\partial w}{\partial t}=-\frac{\partial u}{\partial z}\quad\text{and}\quad b\frac{\partial u}{\partial t}=-\frac{\partial w}{\partial z},\qquad (z,t)\in\mathbb{R}^2.$$

We also consider two C^1-functions $F,G:\mathbb{R}\to\mathbb{R}$, and we put

$$u(z,t)=F(z+ct)+G(z-ct),\quad w(z,t)=\gamma\left\{F(z+ct)-G(z-ct)\right\}.$$

Prove that one can choose the constants c and γ such that the differential equations are satisfied.

$\mathcal{EXERCICE}$ 5.

(A): Find in each of the following cases the directional derivative of the given function

$$1)\ f(x,y,z) = x + 2xy - 3y^2,\ (x_0, y_0, z_0) = (1,2,1),\ v = (3,4,0),$$
$$2)\ f(x,y,z) = ze^x \cos(\pi y),\ (x_0, y_0, z_0) = (0,-1,1),\ v = (-1,2,1).$$

(B): Given the function $f(x,y,z) = \arctan\left(x + \frac{1}{y}\right) + \sinh(z^2 - 1), \quad y < 0.$

Find the direction in which the directional derivation of f at the point $(1,-1,1)$ is smallest, and indicate this minimum.

$\mathcal{EXERCICE}$ .6.

(A): Let the function $f : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(x,y) = \begin{cases} \frac{xy^3}{x^2+y^2},\ (x,y) \neq (0,0) \\ 0,\ (x,y) = (0,0) \end{cases},$$

(1) Prove that f has partial derivatives of first order at every point of the plane.

(2) Prove that the mixed derivatives f''_{xy} and f''_{yx} both exist at the point $(0,0)$, though $f''_{xy}(0,0) \neq f''_{yx}(0,0)$.

(3) Find f''_{xy} for $(x,y) \neq (0,0)$, and prove that this function does not have any limit for $(x,y) \to (0,0)$.

(B): Prove that

(1) The functions $f(x,y) = \ln\left(\sqrt{x^2+y^2}\right)$, $(*)f(x,y) = e^{\alpha x}\cos(\alpha y)$ and $(*)f(x,y) = \frac{1}{\sqrt{x^2+y^2+z^2}}$ fulfils the differential equation $\triangle f(x,y) = 0$.

(2) $(*)$A C^2-function f in two variables satisfies $\frac{\partial^2 f}{\partial x^2} - \frac{\partial^2 f}{\partial y^2} = 0$. Introduce the new variables $u = x + y$, $v = x - y$ and prove that the function $g(u,v) = f\left(\frac{u+v}{2}, \frac{u-v}{2}\right)$ fulfils the equation $\frac{\partial^2 \partial g}{\partial u \partial v} = 0$.
Furthermore, prove that it follow from $\triangle f(x,y) = 0$ that $\triangle g(u,v) = 0$.

(3) Give a function $f(x,y) = \exp(x + xy - 2y)$, $(x,y) \in \mathbb{R}^2$. Find the approximating polynomial of at most second degree $P(x,y)$ and $Q(x,y)$ from the points $(0,0)$ and $(1,1)$ respectively. Calculate the values $P(1/2,1/2)$ and $Q(1/2,1/2)$, compare these with the value $f(1/2,1/2)$ found on a pocket calculator.

(4) $(*)$A function $f \in C^\infty(\mathbb{R}^2)$ satisfies the equations $f(x,0) = e^x$ and $f'_y(x,y) = 2yf(x,y)$. Find the approximating polynomial of at most second degree for f with $(0,0)$ as the point of expansion.
