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LECTURE NOTES: ANALYSIS 3

License Mathematics

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L I C E N S E

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CHAPTER 1

Infinite Series

1.1 An introduction to series

A series is the result of adding a sequence of numbers together. While you may never have thought of it this way, we deal with series all the time when we write expressions like $\frac{1}{3} = 0.333.....$, since this means that

$$\frac{1}{3} = \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + ...$$

In general we are concerned with infinite series such as

$$\sum_{n\geq 1} a_n = a_1 + a_2 + a_3 + ...a_n + ...$$

First though, we need to decide what it means to add an infinite sequence of numbers together. Clearly we can't just add the numbers together until we reach the end (like we do with finite sums), because we won't ever get to the end.

For any three numbers a, b and c, the following holds: $a + (b + c) = (a + b) + c$. This property has the important theoretical consequence that you can add any three numbers by choosing two, adding them, and then adding their sum to the third number. In other words, **finite sums can be rearranged and regrouped arbitrarily without changing the sum because of the associative property.** What's just as bad is that the associative property doesn't work for infinite series, as we see in the following example: $1 - 1 + 1 - 1 + 1 - 1 + ...$

If you group the terms into pairs, this series can be rewritten as

$$1-1+1-1+1-1+..... = (1-1)+(1-1)+(1-1)+..... = 0+0+0+0+0+..... = 0.$$

But, if you group the terms differently, you can also obtain

$$1-1+1-1+1-1+..... = 1+(-1+1)+(-1+1)+.... = 1+0+0+0+0+0+.... = 1.$$

We do not yet know if either of these calculations are valid. But what we do know is that they cannot both be valid (because that would imply $0 = 1$). Therefore **it cannot be the case than one can legally regroup or rearrange terms in an infinite series.** In other words, the associative property is invalid for infinite series in general.
Instead, we adopt the following limit-based definition.

Definition 1.1.1. If the sequence $\{s_n\}$ of partial sums defined by

$$s_n = a_1 + a_2 + a_3 + ...a_n = \sum_{k=1}^{k=n} a_k$$

has a unique and finite limit as $n \to \infty$, then we say that $\sum\limits_{n\geq 1} a_n = \lim\limits_{n\to\infty} s_n$, and in this case we say that $\sum\limits_{n\geq 1} a_n$ converges. Otherwise, $\sum\limits_{n\geq 1} a_n$ diverges.

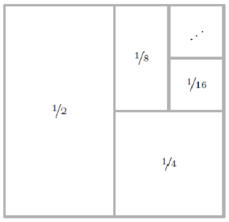
We begin with a particularly simple example.

Example 1.1.1. (Powers of 2). The series $\sum\limits_{n=1}^{\infty} \frac{1}{2^n}$ converges to 1.

We begin by computing a few partial sums:

$$\begin{aligned}
s_1 &= \frac{1}{2} && = \frac{1}{2} && = 1 - \frac{1}{2} \\
s_2 &= \frac{1}{2} + \frac{1}{4} && = \frac{3}{4} && = 1 - \frac{1}{4} = 1 - \frac{1}{2^2} \\
s_3 &= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} && = \frac{7}{8} && = 1 - \frac{1}{8} = 1 - \frac{1}{2^3} \\
s_4 &= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} && = \frac{15}{16} && = 1 - \frac{1}{16} = 1 - \frac{1}{2^3}......,
\end{aligned}$$

then, $s_n = 1 - \frac{1}{2^n}$ and then, $s_{n+1} = s_n + \frac{1}{2^{n+1}} = 1 - \frac{1}{2^n} + \frac{1}{2^{n+1}} = 1 - \frac{1}{2^{n+1}}$, so the formula is correct for all values of n (this technique of proof is known as mathematical induction). With this formula, we see that $\lim_{n\to\infty} s_n = 1$, so $\sum_{n=1}^{\infty} \frac{1}{2^n} = s = 1$.



Remark 1.1.1. There is an alternative, more geometrical, way to see that this series converges to 1. Divide the unit square in half, giving two squares of area $1/2$. Now divide one of these squares in half, giving two squares of area $1/4$. Now divide one of these in half, giving two squares of area $1/16$. If we continue forever, we will subdivide the unit square (which has area 1) into squares of area $1/2, 1/4, 1/8, \ldots.$, verifying that $\sum_{n=1}^{\infty} \frac{1}{2^n} = s = 1$.

Remark 1.1.2. If s_n does not tend to a unique limit finite or infinite, then series $\sum_{n=1}^{\infty} a_n$ is said to be oscillatory.

Properties 1.1.1. ´

- *If we add or remove finitely many terms in a series, then a convergent series remains convergent and a divergent series remains divergent.*
- *If we multiply each term of the series by a non-zero constant, then a convergent series remains convergent and a divergent series remains divergent.*
- *If* $\sum_{n=1}^{\infty} a_n$ *and* $\sum_{n=1}^{\infty} b_n$ *are convergent, then* $\sum_{n=1}^{\infty} a_n \pm \sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} (a_n \pm b_n)$ *is also convergent.*
- *Unless* a_n *and* b_n *are of the same sign, the divergence of the two series* $\sum_{n=1}^{\infty} a_n$ *and* $\sum_{n=1}^{\infty} b_n$ *does not imply the divergence of the series* $\sum_{n=1}^{\infty} (a_n + b_n)$ *as well shown in the example of* $a_n = (-1)^{n+1}$ *and* $b_n = (-1)^n$.
- *The set of numerical series is a vector space on* $\mathbb{C}$, *that of convergent series is a subspace.*

1.1.1 A necessary condition for convergence

Theorem 1.1.1. *If* $\sum a_n$ *is convergent, then* $\lim_{n\to\infty} a_n = 0$.

Proof. . Since we are assuming that $\sum a_n$ converges, then $a_n = s_n - s_{n-1}$ and $\lim_{n\to\infty} s_n = \lim_{n\to\infty} s_{n-1} = s$. therefore

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} s_n - \lim_{n\to\infty} s_{n-1} = s - s = 0.$$

□

1.1.2 The Test for Divergence

If $\lim_{n\to\infty} a_n \neq 0$ then $\sum a_n$ diverges.

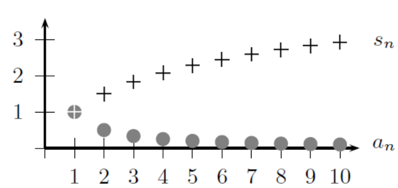
Remark 1.1.3. It is important to remember that the converse to the Test for Divergence is false, i.e., even if the terms of a series tend to 0, the series may still diverge.

Example 1.1.2. (The Harmonic Series). The series $\sum \frac{1}{n}$ diverges. Indeed

$$\begin{aligned}\sum_{n\geq 1}\frac{1}{n} &= 1+\frac{1}{2}+\underbrace{\frac{1}{3}+\frac{1}{4}}_{2\times\frac{1}{4}=\frac{1}{2}}+\underbrace{\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}}_{4\times\frac{1}{8}=\frac{1}{2}}\\ &+\underbrace{\frac{1}{9}+\frac{1}{10}+\frac{1}{11}+\frac{1}{12}+\frac{1}{13}+\frac{1}{14}+\frac{1}{15}+\frac{1}{16}}_{8\times\frac{1}{16}=\frac{1}{2}}+.....\geq 1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+.....\end{aligned}$$

and therefore the series diverges.

Example 1.1.3. (Harmonic series) Indeed, the harmonic series is just such a series: $\frac{1}{n} \to 0$ as $n \to \infty$, but $\sum \frac{1}{n}$ diverges.



Example 1.1.4. If, $a_n = \ln\left(1+\frac{1}{n}\right)$. Then $\lim_{n\to\infty} a_n = 0$, but

$$
\begin{aligned}
\lim_{n\to\infty} s_n &= \lim_{n\to\infty}\sum_{k=1}^{n} a_k = \lim_{n\to\infty}\sum_{k=1}^{n}\left(\ln(k+1)-\ln k\right)\\
&= \lim_{n\to\infty}\left[(\ln 2-\ln 1)+(\ln 3-\ln 2)+(\ln 4-\ln 2)+\ldots+(\ln(n+1)-\ln n)\right]\\
&= \lim_{n\to\infty}\ln(n+1)=\infty.
\end{aligned}
$$

Our goal in this chapter is to develop several tests which we can apply to a wide range of series. Our list of tests will grow to include

1. The Test for Divergence (already given)
2. The Integral Test
3. The p-Series Test
4. The Comparison Test
5. The Limit Comparison Test
6. The Ratio Test(d'Alembert test)
7. Cauchy's nth Root Test(or The root test)
8. Raabe's Test
9. The Absolute Convergence Theorem
10. The Alternating Series Test

Remark 1.1.4. It is important to realize that each test has distinct strengths and weaknesses, so if one test is inconclusive, you need to push onward and try more tests until you find one that can handle the series in question.

1.1.3 Geometric series

One of the most important types of infinite series are geometric series. A geometric series is simply the sum of a geometric sequence. Geometric series are some of the only series for which we can not only determine convergence and divergence easily, but also find their sums, if they converge:

Definition 1.1.2. Geometric Series. The geometric series

$$a + ar + ar^2 + ar^3 + ... + ar^n + ... = \sum_{n=0}^{\infty} ar^n,$$

converges to $\dfrac{a}{1-r}$ if $|r| < 1$, and diverges otherwise.

An easy way to remember this result is

$$\text{geometric series} \sum = \frac{\text{first term}}{1 - \text{ratio between terms}}.$$

Example 1.1.5. Compute

- $12 + 4 + 4/3 + 4/9 + 4/27.....$
- $\sum_{n=6}^{\infty} (-1)^n \frac{2^{n+3}}{3^n}$
- $\sum_{n=1}^{\infty} \frac{2^{n+1} + 9^{n/2}}{5^n}$
- Use geometric series to approximate the decimal expansion of $1/48$.

Proof.

- The first term is 12 and the ratio between terms is $1/3$, so

$$12 + 4 + 4/3 + 4/9 + 4/27.... = \frac{\text{first term}}{1 - \text{ratio between terms}} = \frac{12}{1 - 1/3} = 18.$$

- This series is geometric with common ratio

$$r = \frac{a_{n+1}}{a_n} = \frac{(-1)^{n+1} \frac{2^{n+4}}{3^{n+1}}}{(-1)^n \frac{2^{n+3}}{3^n}} = -2/3,$$

  and so it converges because $|-2/3| < 1$. Its sum is $\sum_{n=6}^{\infty} (-1)^n \frac{2^{n+3}}{3^n} = \frac{2^9/3^6}{1+2/3} = 512/1215$.

- We break this series into two

$$\sum_{n=1}^{\infty} \frac{2^{n+1} + 9^{n/2}}{5^n} = \sum_{n=1}^{\infty} \frac{2^{n+1}}{5^n} + \sum_{n=1}^{\infty} \frac{9^{n/2}}{5^n}.$$

The first of these series has common ratio $2/5$, so it converges. To analyze the second series, note that $9^{n/2} = 3^n$, so this series has common ratio $3/5$. Since both series converge, we may proceed with the addition:

$$\sum_{n=1}^{\infty} \frac{2^{n+1} + 9^{n/2}}{5^n} = \frac{2^2/5}{1-2/5} + \frac{3/5}{1-3/5} = 4/3 + 3/2 = 17/6.$$

- We express $1/48$ as $1/50$ times a fraction of the form $1/(1-r) : 1/48 = 1/(50-2) = (1/50)\dfrac{1}{1-2/50}$. Now we can expand the fraction on the righthand side as a geometric series,

$$1/48 = (1/50)\left(1 + 2/50 + (2/50)^2 + (2/50)^3 + ...\right)$$

Using the first two terms of this series, we obtain the approximation $1/48 \approx 0.02(1 + 0.02) = 0.0204$.

□

Example 1.1.6. (Repeating Decimals). Write the repeating decimal 3.10454545.... as a fraction in lowest terms.

Notice that we can rewrite this decimal as the sum of infinitely many fractions as follows:

$$\begin{aligned} 3.10454545.... &= 3.10 + 0.0045 + 0.000045 + 0.00000045 + ... \\ &= \frac{31}{10} + \frac{45}{10^4} + \frac{45}{10^6} + \frac{45}{10^8} + ... \\ &= \frac{31}{10} + 45 \sum \frac{1}{10^{2n}} \\ &= \frac{31}{10} + \frac{45}{100^2} \frac{1}{1 - 1/100} = 683/220. \end{aligned}$$

1.1.4 A necessary and sufficient condition for convergence

Theorem 1.1.2. *Let (a_n) be a sequence in $\mathbb{R}$. The series $\sum a_n$ is convergent if and only if*

$$\forall \varepsilon > 0,\ \exists n_0 \in \mathbb{N} \quad \text{such that } p, q \in \mathbb{N} \text{ and } p > q > n_0, \text{ then } \left| \sum_{n=q+1}^{n=p} a_n \right| \leq \varepsilon.$$

Remark 1.1.5. This theorem is not very simple to handle, nevertheless its importance comes from the fact that it constitutes the only known necessary and sufficient condition which applies generally to any convergent series. It is the last method, when no simpler method is not applicable.

Example 1.1.7. (The Harmonic Series). The series $\sum \frac{1}{n}$ diverges. We have

$$\sum_{n=p+1}^{2p} \frac{1}{n} = \frac{1}{p+1} + \frac{1}{p+2} + \frac{1}{p+3} + ... + \frac{1}{2p} > p\frac{1}{2p},$$

therefore $\sum_{n=p+1}^{2p} \frac{1}{n} > \frac{1}{2}$ and we cannot realize $\left| \sum_{n=p+1}^{2p} \frac{1}{n} \right| < \epsilon$ if $\epsilon < \frac{1}{2}$. And therefore the series diverges.

1.2 Positive Term Series

Definition 1.2.1. If all the terms after some finitely many terms of an infinite series are positive then such a series is called positive term series. e.g.

$$-7+8-3-5+9-32+\underbrace{2+3+5+34+....}_{\text{positive terms}} \qquad \text{is a positive term series.}$$

Theorem 1.2.1. *Suppose $a_n \geq 0 \quad \forall n$. Then $\sum a_n$ converges if and only if $(s_n)_n$ is bounded above.*

Example 1.2.1. Let $\sum_{n\geq 1} \frac{1}{n^2}$. Since the series is positive and, $\forall n \geq 1$,

$$\begin{aligned} s_n &= \sum_{k=1}^{k=n} \frac{1}{k^2} = 1 + \sum_{k=2}^{k=n} \frac{1}{k^2} \leq 1 + \sum_{k=2}^{k=n} \frac{1}{k(k-1)} = 1 + \sum_{k=1}^{k=n-1} \frac{1}{k(k+1)} \\ &= 1 + \sum_{k=1}^{k=n-1} \left(\frac{1}{k} - \frac{1}{k+1} \right) = 1 + \left(1 - \frac{1}{n}\right) = 2 - \left(1 - \frac{1}{n}\right) \leq 2. \end{aligned}$$

This shows that $(s_n)_n$ is bounded above, so $\sum_{n\geq 1} \frac{1}{n^2}$ is convergent.

1.2.1 The integral test

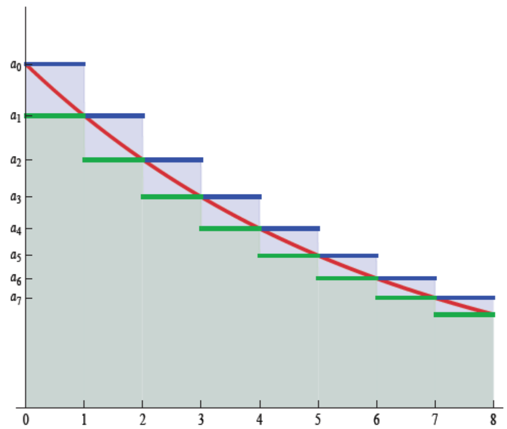
In this subsection we discuss a very simple, but powerful, idea: in order to prove that certain series converge or diverge, we may compare them to integrals.

Definition 1.2.2. In general, if we have a function f defined from $x = a$ to $x = \infty$, we define

$$\int_a^{\infty} f(x)dx = \lim_{b \to \infty} \int_a^b f(x)dx,$$

and we say that this improper integral converges if the limit converges, and that it diverges if the limit diverges.

Theorem 1.2.2. *Suppose that f is a positive, decreasing, and continuous function, and that $a_n = f(n)$. Then $\sum_{n \geq 1} a_n$ converges if and only if the improper integral $\int_1^{\infty} f(x)dx$ converges.*



Proof. Suppose we have a function f which is positive and decreasing, such that $f(n) = a_n$ for $n = 0, 1, 2, 3, ...$ Consider the above picture, which shows the graph of f in red. It is clear from the picture that

$$\text{green area } \leq \int_0^{\infty} f(x)dx \leq \text{green area } + \text{ blue area.} \tag{1.2.1}$$

Now let's figure the area of the green shaded region. This can be subdivided into rectangles of width 1 by drawing vertical line segments from the $x-$axis up to the top of the green area at each integer. If you do this, you will find that (by reading the heights of the rectangles off of the scale on the $y-$axis)

- the area of the first green rectangle is its height times its width, i.e. is $a_1.1 = a_1$,
- the area of the second green rectangle is $a_2.1 = a_2$,

- the area of the third green rectangle is $a_3.1 = a_3$, etc.

Thus the total green area is $a_1 + a_2 + a_3 + ... = \sum_{n=1}^{\infty} a_n$.
Now let's figure the total area of the blue and green regions. As with just the green regions, the combined blue and green regions can be divided into rectangles of width 1. This time, however,

- the area of the left-most rectangle (whose bottom part is green but whose top part is blue) is its height times its width, i.e. is $a_0.1 = a_0$,
- the area of the second rectangle is $a_1.1 = a_1$,
- the area of the third rectangle is $a_2.1 = a_2$, etc.

Thus the total combined blue and green area is $a_0 + a_1 + a_2 + a_3 + ... = \sum_{n=0}^{\infty} a_n$.
Plugging these computations into equation (1.2.1), we see that

$$\sum_{n=1}^{\infty} a_n \leq \int_0^{\infty} f(x)dx \leq \sum_{n=0}^{\infty} a_n.$$

From this inequality we can prove the theorem. First, assume that $\int_0^{\infty} f(x)dx$ converges. This means that the green shaded area, being less than the finite number $\int_0^{\infty} f(x)dx$, is also finite, i.e. $\sum_{n=1}^{\infty} a_n$ converges. Since the starting index of a series is irrelevant to whether or not it converges, $\sum_{n=0}^{\infty} a_n$ converges as well.
Now assume that $\int_0^{\infty} f(x)dx$ diverges, i.e. that the area under the red function is infinite. This means that the combined green and blue shaded area, being greater than the area under the function $f(x)$ (which is $\int_0^{\infty} f(x)dx$), must also be infinite. Therefore $\sum_{n=0}^{\infty} a_n$ diverges. This completes the proof of the Integral Test. □

Example 1.2.2. Does the series $\sum_{n\geq 1} \frac{1}{n^2+1}$ converge or diverge?

We began the section by considering $\sum 1/n$ and $\sum 1/n^2$. What about $\sum 1/n^p$ for other values of p? We can evaluate the integral of $1/x^p$, so the Integral Test can be used to determine which of these series converge. Because series of this form occur so often, we record this fact as its own test.

1.2.2 The p-Series Test

Theorem 1.2.3. *The series* $\sum 1/n^p$ *converges if and only if* $p > 1$.

Proof. When $p = 1$, we already know that the series diverges ($1/n$ is the Harmonic series). For other values of p, we simply integrate the improper integral from the **Integral Test:**

$$\begin{aligned}\int_1^\infty \frac{1}{x^p}dx &= \lim_{b\to\infty}\int_1^b \frac{1}{x^p}dx \\ &= \lim_{b\to\infty}\left(\frac{b^{1-p}}{1-p}\right) - \frac{1}{1-p} \\ &= \begin{cases} \dfrac{1}{1-p}, & \text{if } p > 1 \\ \infty, & \text{if } p \leq 1. \end{cases}\end{aligned}$$

□

The Integral Test Remainder Estimates. Suppose that f is a positive, decreasing, and continuous function, and that $a_n = f(n)$. Then the error in the nth partial sum of $\sum a_n$ is bounded by an improper integral:

$$\left| s_n - \sum_{n=1}^{\infty} a_n \right| \leq \int_n^\infty f(x)dx.$$

The proof of the Integral Test Remainder Estimate is almost identical to the proof of the Integral Test itself, so we content ourselves with an example.

Example 1.2.3. Bound the error in using the fourth partial sum s_4 to approximate $\sum_{n\geq 1}\frac{1}{n^2}$.

The error in this case is the difference between s_n and the true value of the series:

$$Error = \left| s_4 - \sum_{n=1}^{\infty} a_n \right| = \left| s_n - \sum_{n=1}^{\infty} a_n \right| \leq \int_4^\infty \frac{1}{x^2}dx = \lim_{b\to\infty}\int_4^b \frac{1}{x^2}dx = \frac{1}{4}.$$

This is not a very good bound. As we mentioned earlier, EULER approximated the value of this series to within 17 decimal places. How many terms would we need to take to get the upper bound on the error from the Integral Test Remainder Estimates under 10^{-17}?

1.2.3 The Comparison Test

Theorem 1.2.4. *Suppose that* $0 \leq a_n \leq b_n$ *for sufficiently large* n.

- *If* $\sum a_n$ *diverges, then* $\sum b_n$ *also diverges.*
- *If* $\sum b_n$ *converges, then* $\sum a_n$ *also converges.*

Proof. Let s_n denote the nth partial number of a_n and t_n denote the nth partial sum of b_n, so

$$s_n = a_1 + a_2 + ... + a_n, \qquad t_n = b_1 + b_2 + ... + b_n.$$

From our hypotheses (that $0 \leq a_n \leq b_n$ for all n), we know that $s_n \leq t_n$ for all n.

- First suppose that $\sum b_n$ converges, which implies by our definitions that $t_n \to \sum b_n$ as $n \to \infty$. The sequence s_n is nonnegative and monotonically increasing because $s_{n+1} - s_n = a_n \geq 0$ for all n, and

$$0 \leq s_n \leq t_n \leq \sum b_n$$

  so the sequence s_n has a limit by the Monotone Convergence Theorem that the series $\sum a_n$ converges.

- Now suppose that $\sum a_n$ diverges. Because the terms a_n are nonnegative, the only way that $\sum a_n$ can diverge is if $s_n \to \infty$ as $n \to \infty$. Therefore the larger partial sums t_n must also tend to ∞ as $n \to \infty$, so the series $\sum b_n$ diverges as well.

□

Remark 1.2.1. In practice, we will almost always compare with a geometric series or a $p-$series.

Example 1.2.4.

- Show that the series $\sum_{2}^{\infty} \frac{1}{n^2 - 1}$ converges
- Show that the series $\sum_{1}^{\infty} \frac{n}{\sqrt{n^4 + 7}}$ diverges.
- Does the series $\sum_{1}^{\infty} \frac{n \ln n}{\sqrt{(n+3)^5}}$ converge or diverge?

Solution:

- Because $n^2-1 \geq (n-1)^2$, we have that

$$\sum_{2}^{\infty} \frac{1}{n^2-1} = \frac{1}{3} + \frac{1}{8} + \frac{1}{15} + ... \leq \frac{1}{1} + \frac{1}{4} + \frac{1}{9} + ... = \sum_{1}^{\infty} \frac{1}{n^2},$$

  so the series converges by comparison to $\frac{1}{n^2}$.

- We have $\frac{n}{\sqrt{n^4+7}} \geq \frac{n}{\sqrt{n^4+7n^4}} = \frac{1}{\sqrt{8}n}$, so the series we are interested in diverges by comparison to the harmonic series.

- As we know, $\ln n \leq n^{1/4}$ for sufficiently large n and $(n+3)^5 \geq n^5$, we can use the comparison

$$\frac{n \ln n}{\sqrt{(n+3)^5}} \leq \frac{n.n^{1/4}}{n^{5/2}} = \frac{1}{n^{5/4}}.$$

  Because $\sum \frac{1}{n^{5/4}}$ is a convergent $p-$series, $\sum_{1}^{\infty} \frac{n \ln n}{\sqrt{(n+3)^5}}$ converges by the Comparison Test.

If a series converges by the Comparison Test, then we have the following remainder estimate

The Comparison Test Remainder Estimate. Let $\sum a_n$ and $\sum b_n$ be series with positive terms such that $a_n \leq b_n$ for $n \geq N$. Then for $n \geq N$, the error in the nth partial sum of $\sum a_n$, s_n, is given by

$$\left| s_n - \sum_{1}^{\infty} a_n \right| \leq b_{n+1} + b_{n+2} + b_{n+3} + ...$$

Example 1.2.5. How many terms are needed to approximate $\sum_{1}^{\infty} \frac{1}{2^n+n}$ to within $\frac{1}{10}$?

1.2.4 The Limit Comparison Test

Theorem 1.2.5. *Let $\sum a_n$ and $\sum b_n$ be two positive term series such that*

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L,$$

then

- *if L is non-zero and finite, then $\sum a_n$ and $\sum b_n$ converge or diverge together,*

- *if* $L = 0$*, then* $\sum a_n$ *is convergent if* $\sum b_n$ *is convergent,*

- *if* $L = \infty$*, then* $\sum a_n$ *is divergent if* $\sum b_n$ *is divergent.*

Proof. □

Example 1.2.6. Examine the convergence of the series:

$$(a)\ \sum_{n\geq 1} \frac{\sqrt{n^2-1}}{n^4+1}, \quad (b)\ \sum_{n\geq 1} \sqrt{n^2+1}-n, \quad (c)\ 1+1/2^2+2^2/3^3+3^3/4^4+...$$

1.2.5 The Ratio Test (D'ALEMBERT Ratio Test)

There are a great many series for which the above tests are not ideally suited, for example, the series

$$\sum_{1}^{\infty} \frac{4^n}{n!}.$$

Integrating the terms of this series would be difficult. We could try a comparison, but again, the solution is not particular obvious. Instead, the simplest approach to such a series is the following test due to Jean le Rond d'Alembert (1717–1783).

Theorem 1.2.6. *Suppose that* $\sum a_n$ *is a series with positive terms and let* $L = \lim_{n\to\infty} a_{n+1}/a_n$.

- *If* $L < 1$ *then* $\sum a_n$ *converges.*

- *If* $L > 1$ *then* $\sum a_n$ *diverges.*

- *If* $L = 1$ *or the limit does not exist then the Ratio Test is inconclusive.*

Proof. You should think of the Ratio Test as a generalization of the Geometric Series Test. For example, if $(a_n) = (ar^n$ is a geometric sequence then $\lim_{n\to\infty} a_{n+1}/a_n = r$, and we know these series converge if and only if $|r| < 1$.
If $L > 1$ then the sequence a_n is increasing (for sufficiently large n), and therefore the series diverges by the **Test for Divergence.**
Now suppose that $L < 1$. Choose a number r sandwiched between L and $1 : L < r < 1$. Because $a_{n+1}/a_n \to L$, there is some integer N such that $0 \leq a_{n+1}/a_n \leq r$. For all $n \geq N$. Set $a = a_N$. Then we have

$$a_{N+1} \leq ra_N = ar,$$

and

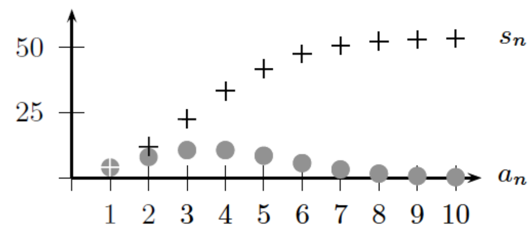
$$a_{N+2} \le r a_{N+1} < ar^2,$$

and in general, $a_{N+k} < ar^k$. Therefore for sufficiently large n (namely, $n \ge N$), the terms of the series $\sum a_n$ are bounded by the terms of a convergent geometric series (since $0 < r < 1$), and so $\sum a_n$ converges by the **Comparison Test.** □

Example 1.2.7. 1. Does the series $\sum_1^\infty \frac{4^n}{n!}$ converge or diverge?

2. Does the series $\sum_1^\infty \frac{n^2}{2^n}$ converge or diverge?

3. Does the series $\sum_1^\infty \frac{10^n}{n2^{2n+1}}$ converge or diverge?



Remark 1.2.2. • It is important to note that the Ratio Test is always inconclusive for series of the form $\sum \frac{\text{polynomial}}{\text{polynomial}}$. As an example, we consider the harmonic series $\sum 1/n$ and $\sum 1/n^2$.

• **When it is a good idea to use the Ratio Test:** The Ratio Test is likely to work well for a series whose terms contain only things that are multiplied and divided, and for series whose terms contain expressions like $2^n, 3^n, c^n, n^n, n!$, etc.

1.2.6 Cauchy's nth Root Test(or The root test)

Theorem 1.2.7. *If $\sum a_n$ is a positive term series such that*

$$\lim_{n\to\infty} (a_n)^{1/n} = L,$$

then

- *If* $L < 1$ *then* $\sum a_n$ *converges.*
- *If* $L > 1$ *then* $\sum a_n$ *diverges.*
- *If* $L = 1$ *or the limit does not exist then the Ratio Test is inconclusive.*

Example 1.2.8. Examine the convergence of the following series:

- $\sum_{n\geq 1} \frac{(n-\ln n)^n}{2^n n^n}$,
- $\sum_{n\geq 1} 3^{(-1)^n - n}$,
- $\sum_{n\geq 1} \left(\frac{n}{n+1}\right)^{n^2}$.

Remark 1.2.3. Let $\sum a_n$ be a positive term series

1. If $\lim_{n\to\infty} a_{n+1}/a_n$ exists, then $\lim_{n\to\infty} a_n^{1/n}$ exists also, and we have
$$\lim_{n\to\infty} a_{n+1}/a_n = \lim_{n\to\infty} a_n^{1/n},$$
2. The reciprocal of (1) is false, in general.

1.2.7 Raabe's Test

Theorem 1.2.8. *If* $\sum a_n$ *is a positive term series such that*

$$\lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}} - 1\right) = L,$$

then

- $\sum a_n$ *is convergent if* $L > 1$,
- $\sum a_n$ *is divergent if* $L < 1$,
- *Test fails if* $L = 1$.

Remark 1.2.4. The Raabe's test is used when D'Alembert's ratio test is failed and the ratio a_n/a_{n+1} does not contains the number e.

Example 1.2.9. Examine the convergence of the series:

$$1+\frac{3}{7}+\frac{3.6}{7.10}+\frac{3.6.9}{7.10.13}+\frac{3.6.9.12}{7.10.13.16}+.....$$

In the case where the limit L of the Raabe's Test is equal to 1, a refinement is still possible:

Theorem 1.2.9. *If $\sum a_n$ is a positive term series such that*

$$\lim_{n\to\infty}\left[n\left(\frac{a_n}{a_{n+1}}-1\right)-1\right]\ln n = L,$$

then

- *$\sum a_n$ is convergent if $L>1$,*
- *$\sum a_n$ is divergent if $L<1$,*
- *Test fails if $L=1$.*

Proposition 1.2.1. ***(Bertrand' series)***[1]
Let α and β two real numbers. The series $\sum_{n\geq 2}\frac{1}{n^{\alpha}(\ln n)^{\beta}}$ converges if and only if $(\alpha>1)$, or $(\alpha=1$ and $\beta>1)$.

Proof.

1. If $\alpha>1$, there exists a real constant such that $1<\gamma<\alpha$. Then

$$n^{\gamma}\frac{1}{n^{\alpha}(\ln n)^{\beta}}=\frac{1}{n^{\alpha-\gamma}(\ln n)^{\beta}}\rightarrow_{n\to\infty} 0,$$

   since $\alpha-\gamma>0$. then, from Riemann's rule the series converges.

2. If $\alpha<1$. With the same manner, we have

$$n\frac{1}{n^{\alpha}(\ln n)^{\beta}}=\frac{1}{n^{\alpha-1}(\ln n)^{\beta}}\rightarrow_{n\to\infty}\infty,$$

   so the series diverges.

3. assume that $\alpha=1$.

[1] Joseph Bertrand (1822-1900), French mathematician

a If $\beta \leq 0$, for all $n \geq 3$ we get

$$\frac{1}{n(\ln n)^{\beta}} \geq \frac{1}{n},$$

thus, from the comparaison test the series $\sum_{n\geq 2} \frac{1}{n(\ln n)^{\beta}}$ diverges.

b If $\beta > 0$. The function $x \mapsto f(x) = \frac{1}{x(\ln x)^{\beta}}$ is positive and decreasing on $[2, +\infty[$, thus by using integral test (see theorem 1.2.2).

(i) If $\beta \neq 1$, we have

$$\int_2^x \frac{dt}{t(\ln t)^{\beta}} = \left[\frac{(\ln t)^{1-\beta}}{1-\beta}\right]_2^x = \frac{1}{1-\beta}\left((\ln x)^{1-\beta} - (\ln 2)^{1-\beta}\right),$$

we deduce that

$$\lim_{x\to\infty} \int_2^x \frac{dt}{t(\ln t)^{\beta}} = \begin{cases} \frac{(\ln 2)^{1-\beta}}{1-\beta}, & \text{if } \beta > 1 \\ +\infty, & \text{if } 0 < \beta < 1. \end{cases}$$

Then, the series converges if $\beta > 1$, and diverges if $0 < \beta < 1$.

(ii) Now, if $\beta = 1$.

$$\int_2^x \frac{dt}{t(\ln t)} = \ln(\ln(x)) - \ln(\ln(2)) \longrightarrow_{x\to\infty} = \infty.$$

Thus, the series $\sum_{n\geq 2} \frac{1}{n(\ln n)}$ diverges.

□

1.3 Alternating Series

We have focused almost exclusively on series with positive terms up to this point. In this short section we begin to delve into series with both positive and negative terms, presenting a test which works for many series whose terms alternate in sign.

Definition 1.3.1. A series with terms alternately positive and negative is called an alternating series. For example, $1 - 1/2 + 1/3 - 1/4 + 1/5 - 1/6 + 1/7 - ....$ The general form