

Characteristic Polynomial, Eigenvalues and Eigenvectors
of a square matrix

Ex 01 : a) Compute the characteristic polynomial of the matrix

$$A_3 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

Ans: $x^2(3-x)$

b) Deduce the characteristic polynomial of the general matrix:

$$A_n = \begin{pmatrix} 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 1 \end{pmatrix} \in M_n(\mathbb{R})$$

i.e., extend to n -dimensions.

Ex 02 : Find the characteristic polynomial of the matrix:

$$A = \begin{pmatrix} 7 & -6 & -2 \\ 2 & 0 & -1 \\ 2 & -3 & 2 \end{pmatrix}, p_A(x) = (x-3)^3$$

Ex 03 : Let $A \in M_n(\mathbb{R})$ be a square matrix. Prove the following equality

$$p_{rA}(x) = r^n p_A\left(\frac{x}{r}\right); r \neq 0$$

Ex 04 : (i) Let A and B be two matrices such that

$$A^2 = B^2 = (AB)^2 = I_n$$

Prove that $AB = BA$.

(ii) Let $A, B \in M_n(\mathbb{R})$. Assume that A is invertible. Show that

$$p_{AB}(\lambda) = p_{BA}(\lambda)$$

Ex 05 : Consider the following matrix

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$$

Find all eigenvalues and corresponding eigenvectors for the matrix A .

Ex 06 : Determine the eigenvalues and eigenvectors of the following matrices

$$1) A = \begin{pmatrix} 1 & 2 \\ 0 & 5 \end{pmatrix}$$

Ans:

eigenvalues	eigenvectors
1	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$
5	$\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$$2) A = \begin{pmatrix} 2 & 6 \\ 0 & 2 \end{pmatrix}$$

Ans:

eigenvalues	eigenvector
2 (double)	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$3) A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & -5 \end{pmatrix}$$

Ans:

eigenvalues	eigenvectors
1	(1, 0, 0)
2	(2, 1, 0)
-5	(5, 6, -14)

$$4) A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}$$

Ans :

eigenvectors: $\left\{ \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \right\} \leftrightarrow 1, \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\} \leftrightarrow 2 \text{ (double)}$

$$5) A = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Ans: $\lambda = 0$ is the only eigenvalue of A (the algebraic multiplicity is 3)
 $(1, 0, 0), (0, 1, -1)$ are corresponding eigenvectors of λ .

$$6) A = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 3 & 2 \end{pmatrix}$$

Ans: $\lambda = 2$ and $E_\lambda = Vect \{(0, 0, 1)\}$

$$7) A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

Ans:

eigenvectors: $\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \right\} \leftrightarrow 0, \left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\} \leftrightarrow 2$

$$8) A = \begin{pmatrix} a & 2 & 3 \\ 0 & 2a & 8 \\ 0 & 0 & 3a \end{pmatrix}, a \in \mathbb{R}$$

Ans :

eigenvalues	eigenvectors
a	$(1, 0, 0)$
$2a$	$(2, a, 0)$
$3a$	$(3a + 16, 16a, 2a^2)$

Ex 07 : Determine all eigenvalues and their algebraic multiplicities of the matrix

$$A = \begin{pmatrix} 1 & a & 1 \\ a & 1 & a \\ 1 & a & 1 \end{pmatrix},$$

where a is a real number.

Similar Matrices, Diagonalizable Matrices

Ex 01 : (a) State the definition of two similar matrices $A, B \in M_n(\mathbb{R})$ (in this case, we denote $A \sim B$).

(b) Prove that : $A - \lambda I_n \sim B \Rightarrow A \sim \lambda I_n + B$.

(c) Let A and B be two similar matrices, i.e.,

$$A = PBP^{-1}$$

for some invertible matrix P , and let (λ, x) be an eigenpair of A . Show that $(\lambda, P^{-1}x)$ is an eigenpair of B .

(d) Let $f_r(x) = a_0 + a_1x + \dots + a_rx^r$ be a polynomial of degree r . If $B = P^{-1}AP$, then prove that

$$f_r(B) = P^{-1}f_r(A)P$$

Ex 02 : Using two methods, prove that similar matrices have the same eigenvalues.

Ex 03 : Prove the following implication (see *The exponential of a Matrix*)

$$A \sim B \Rightarrow e^A \sim e^B$$

Ex 04 : Let A and B be the matrices

$$\begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 2 \\ -2 & 5 \end{pmatrix}$$

Prove that A is similar to B .

Find the matrix C such that

$$A = C.B.C^{-1}$$

Ex 05 : Let A be a diagonalizable matrix which has a unique eigenvalue λ . Prove that $A = \lambda I$, where I is the identity matrix.

Ex 06 : Study the diagonalization of the following matrix:

$$A = \begin{pmatrix} 1 & a & b & c \\ 0 & 1 & d & e \\ 0 & 0 & 2 & f \\ 0 & 0 & 0 & 3 \end{pmatrix} ; a \neq 0, b, c, d, e, f \in \mathbb{R}$$

Ex 07 :

Let $A \in M_n(\mathbb{R})$ be a diagonalizable matrix with $sp(A) = \{-1, 1\}$. Prove that $A^{-1} = A$.

Ex 08 : Let A and B be two diagonalizable matrices with $P^{-1}AP = D_1$ and $P^{-1}BP = D_2$ for some invertible matrix P . Prove that $AB = BA$.

Ex 09 : a) Determine whether the matrix

$$A = \begin{pmatrix} 3 & 0 & 0 \\ 4 & 1 & 2 \\ a & 0 & 3 \end{pmatrix}; a \in \mathbb{R}$$

is diagonalizable or not.

b) Let

$$A_\alpha = \begin{pmatrix} 2 & \alpha & 1 \\ 0 & 2 & 0 \\ 0 & 0 & \alpha \end{pmatrix}$$

Find the values α for which the matrix A_α is diagonalizable. **Ans:** A_α is diago iff $\alpha = 0$.

Ex 10 : For which values of c the matrix

$$A = \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}; c \neq 0$$

is diagonalisable.

Ex 11 : Consider the matrix

$$A(a) = \begin{pmatrix} 1 & 0 & 0 \\ a & -2 & 3 \\ 1 & -1 & 2 \end{pmatrix}, a \in \mathbb{R}$$

Find all the eigenvalues of $A(a)$.

Find the real parameter a for which the matrix $A(a)$ is diagonalizable, and give in this case the diagonalization formula of $A(a)$.

Ex 12 : Let

$$\begin{aligned} f &: \mathbb{P}_3[x] \longrightarrow \mathbb{P}_3[x] \\ p &\longmapsto f(p) = 3xp - (x^2 - 1)p' \end{aligned}$$

We denote by $B = \{1, x, x^2, x^3\}$ the canonical basis of $\mathbb{P}_3[x]$. Calculate $M_f(B)$.

We ask whether f is diagonalizable or not. If the answer is positive, then give the diagonalization formula of f .

Exponential of a matrix, cosine(matrix), sine, log,...

On the powers of a square matrix. How to find A^k ?

Ex 01 :

1. State the definition of the notion « Exponential of a square matrix »

2. Calculate e^O , where O is the zero matrix.

Let $A \in \mathbb{M}_n(\mathbb{R})$ and $\lambda \in \mathbb{R}$. Prove that

$$e^{\lambda I_n} A = e^\lambda A$$

Ex 02 : Let A be the matrix

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$$

Prove that A is diagonalizable, and deduce $\exp(A)$.

Ex 03 : Prove the following theorem : **Theorem** : Let $A \in \mathbb{M}_n(\mathbb{R})$. Then the series

$$\sum_{k=0}^{\infty} \frac{A^k}{k!}$$

is absolutely convergent (then convergent).

Ex 04 : Let A be a square matrix. Prove that

$$\lim_{x \rightarrow 0} \frac{e^{xA} - I}{x} = A.$$

Ex 05 : Let

$$A = \begin{pmatrix} 1 & 3 & -3 \\ 1 & 3 & -1 \\ -2 & 2 & 0 \end{pmatrix}$$

1. Find all the eigenvalues and eigenvectors of A .
2. Deduce that A is diagonalizable.
3. Compute $\exp(A)$.

Ex 06 : Consider the following two matrices

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$$

Verify that

$$e^{A+B} \neq e^A \cdot e^B \neq e^B \cdot e^A$$

Ex 07 : Consider the matrix

$$A = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}$$

Calculate $\log A$ i.e, find a square matrix $B \in M_2(\mathbb{R})$ such that $A = e^B$.

Ex 08 : Let A be a diagonalizable matrix; i.e $A = PDP^{-1}$ for some invertible matrix P , prove that

$$\lim_{k \rightarrow \infty} A^k = P \cdot \lim_{k \rightarrow \infty} D^k \cdot P^{-1}$$

Consider the matrix

$$A = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix}$$

Calculate $\lim_{k \rightarrow \infty} A^k$

Ans :

$$\begin{aligned} \lim_{k \rightarrow \infty} A^k &= P \cdot \lim_{k \rightarrow \infty} D^k \cdot P^{-1} \\ &= \begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix} \lim_{k \rightarrow \infty} \begin{pmatrix} 1^k & 0 \\ 0 & (\frac{1}{4})^k \end{pmatrix} \begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ -\frac{1}{3} & \frac{1}{3} \end{pmatrix} \\ &= \frac{1}{3} \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}. \end{aligned}$$

Ex 09 : Determine the eigenvalues and eigenvectors of the following matrix

$$A = \begin{pmatrix} 3 & 1 \\ -2 & 0 \end{pmatrix}$$

For each positive integer k , calculate A^k .

Ans :

$$\begin{aligned} \text{eigenvectors: } (-1, 2) &\leftrightarrow 1, (-1, 1) \leftrightarrow 2 \\ A^k &= \begin{pmatrix} 2^{k+1} - 1 & 2^k - 1 \\ 2 - 2^{k+1} & 2 - 2^k \end{pmatrix}; k \in \mathbb{N} \end{aligned}$$

Ex 10 : Consider the matrix

$$A = \begin{pmatrix} 9 & 0 & 0 \\ -5 & 4 & 0 \\ -8 & 0 & 1 \end{pmatrix} \quad A^n = \begin{pmatrix} 9^n & 0 & 0 \\ 4^n - 9^n & 4^n & 0 \\ 1 - 9^n & 0 & 1 \end{pmatrix}$$

Compute A^n ; $n \in \mathbb{N}$.

System of linear recurrence sequences,
System of Differential equations, Cayley-Hamilton Theorem

Ex 01 : Solve the following system of linear recurrence sequences

$$\begin{cases} x_{n+1} = 3x_n - y_n \\ y_{n+1} = -x_n + 3y_n \end{cases} ; (x_0, y_0) = (1, 2)$$

Ans :

$$\begin{cases} x_n = 2^{n-1} (3 - 2^n) \\ y_n = 2^{n-1} (3 + 2^n) \end{cases} ; n \geq 0$$

Ex 02 : Consider the determinant

$$D_n = \det \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & & \ddots & \ddots & \ddots \\ & & & -1 & 2 & -1 \\ & & & & -1 & 2 \end{pmatrix}$$

Prove that $D_n = 2D_{n-1} - D_{n-2}$. Calculate the value D_n in terms of n .

Ex 03 : Let (x_n) be the sequence given by

$$x_{n+2} = \frac{2}{\frac{1}{x_n} + \frac{1}{x_{n+1}}} ; x_0, x_1 > 0$$

Evaluate the following limit (in terms of the initial values x_0, x_1):

$$\lim_{n \rightarrow +\infty} x_n$$

Ex 04 : Solve the system of differential equations

$$X' = AX ; A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}$$

Ans:

$$\begin{cases} \lambda_1 = 1, v_1 = (-1, 1, 1) \\ \lambda_2 = 2, v_2 = (0, 1, 0) \text{ et } v_3 = (0, 0, 1) \end{cases}$$

Solve the system of differential equations

$$X'(t) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} X(t)$$

Ans :

$$\begin{aligned} X(t) &= c_1 e^{3t} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + c_2 e^{0t} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + c_3 e^{0t} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} c_1 e^{3t} - c_2 - c_3 \\ c_1 e^{3t} + c_2 \\ c_1 e^{3t} + c_3 \end{pmatrix}, c_i \in \mathbb{R} \end{aligned}$$

Ex 05 : Using Cayley-Hamilton Theorem, calculate the inverse of the matrix

$$M = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & 0 \\ 2 & 0 & -1 \end{pmatrix}$$

Ex 06 : Let $A \in \mathbb{M}_n(\mathbb{R})$ be a square matrix with

$$p_A(x) = c_0 + c_1 x + \dots + c_{n-1} x^{n-1} + c_n x^n; c_0 \neq 0$$

Prove that

$$A^{-1} = -\frac{1}{c_0} \sum_{k=1}^n c_k A^{k-1}$$

Ex 07 : Let $A \in \mathbb{M}_n(\mathbb{R})$ be a square matrix which has one eigenvalue, say λ . Prove that

$$e^{tA} = e^{\lambda t} \sum_{k=0}^{n-1} (A - \lambda I_n)^k \frac{t^k}{k!}.$$

Solve the system of differential equations

$$X' = \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Solve the system of differential equations

$$\begin{pmatrix} x'_1(t) \\ x'_2(t) \\ x'_3(t) \end{pmatrix} = \begin{pmatrix} -4 & 1 & 1 \\ 1 & -1 & -2 \\ -2 & 1 & -1 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix}$$

Let $A \in \mathbb{M}_3(\mathbb{R})$ be a non diagonalizable matrix. Assume that λ, λ , and μ are the eigenvalues of A with $\lambda \neq \mu$. Prove that

$$e^{tA} = e^{\lambda t} (I + t(A - \lambda I)) + \frac{e^{\mu t} - e^{\lambda t}}{(\mu - \lambda)^2} (A - \lambda I)^2 - \frac{te^{\lambda t}}{\mu - \lambda} (A - \lambda I)^2.$$

Solve the system of differential equations

$$\begin{pmatrix} x'_1(t) \\ x'_2(t) \\ x'_3(t) \end{pmatrix} = \begin{pmatrix} 2 & -1 & 2 \\ 10 & -5 & 7 \\ 4 & -2 & 2 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix}$$

Minimal Polynomial, Trigonalizable Matrices

Ex 01 : Calculate the minimal polynomial of the matrix

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

Verify that all the matrices of the form

$$A = \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} ; c \neq 0$$

are not diagonalisable.

Ex 02 : Determine the minimal polynomial of the following matrices

$$a) A = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, b) A = \begin{pmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 4 \end{pmatrix}, c) \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}$$

Ex 03 : If $m_A(x) = (x - a)(x - b)$. Show that A^n can be only written in terms of A and I .

Ex 04 : Calculate the minimal polynomial of the matrix

$$A = \begin{pmatrix} \lambda & & & & \\ 1 & \lambda & & & \\ & \ddots & \ddots & & \\ & & 1 & \lambda & \\ & & & 1 & \lambda \end{pmatrix} ; \lambda \in \mathbb{R}$$

Is it diagonalizable ?

Ex 05 : Consider the matrix

$$A = \begin{pmatrix} a & b & b \\ b & a & b \\ b & b & a \end{pmatrix} ; a, b \in \mathbb{R}$$

Show that A is diagonalizable.

Ex 06 : Prove that the $n \times n$ matrix

$$A = \begin{pmatrix} 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 1 \end{pmatrix} \in M_n(\mathbb{R})$$

is diagonalizable.

Ex 07 : Trigonalize the matrix

$$A = \begin{pmatrix} 2 & -1 \\ 1 & 4 \end{pmatrix} \quad p_A(x) = (x-3)^2$$

Calculate A^n .

Ex 08 : Trigonalize the matrix

$$A = \begin{pmatrix} 2 & -1 & 2 \\ 10 & -5 & 7 \\ 4 & -2 & 2 \end{pmatrix} \quad p_A(x) = \begin{vmatrix} x-2 & -1 & 2 \\ 10 & x+5 & 7 \\ 4 & -2 & x-2 \end{vmatrix} = x^2(x+1)$$

Calculate A^n for any positive integer n .

Ex 09 : Trigonalize the matrix

$$A = \begin{pmatrix} 7 & -6 & -2 \\ 2 & 0 & -1 \\ 2 & -3 & 2 \end{pmatrix} ; p_A(x) = (x-3)^3$$

Then, compute A^n .

Ex 10 : Consider the matrix

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

- 1) Verify that $A^2 = A + 2I$ and deduce the expression of A^{-1} in terms of A and I , where I is the identity 3 by 3 matrix.
- 2) Using the proof by induction, prove that

$$A^n = a_n \cdot A + b_n \cdot I,$$

where

$$a_n, b_n \in \mathbb{N} \text{ and } \begin{cases} a_{n+1} = a_n + b_n \\ b_{n+1} = 2a_n \end{cases}, n \geq 0 \quad (S)$$

- 3) Write the system (S) in the matrix form.
- 4) Calculate a_n and b_n in terms of n . Then, deduce the expression of A^n .
- 5) Calculate the minimal polynomial of the matrix A . What to deduce?
- 6) Solve the system of differential equations $X' = A \cdot X$