On the inverse of a square matrix
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Fact

Let $A \in \mathcal{M}_n(\mathbb{R})$*. If* $\det(A) \neq 0$*, then* A^{-1} *exists. Moreover, the formula of* A^{-1} *is given by:*

$$A^{-1} = \frac{1}{\det(A)} (Com(A))^t, \tag{1}$$

where $Com(A)$ *denotes the comatrix of* A*. If* A^{-1} *exists, we say that* A *is* **invertible**. *By French "***inversible***".*

Example

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{M}_2(\mathbb{R})$. We have

$$\det(A) = ad - cb \text{ and } A^{-1} = \frac{1}{ad - cb} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Example

Consider the matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 8 & 8 & 9 \end{pmatrix} \in \mathcal{M}_3(\mathbb{R}).$$

By definition, we obtain

$$\begin{aligned} \det(A) &= \begin{vmatrix} \overset{+}{1} & \overset{-}{2} & \overset{+}{3} \\ 4 & 5 & 6 \\ 8 & 8 & 9 \end{vmatrix} = \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2\begin{vmatrix} 4 & 6 \\ 8 & 9 \end{vmatrix} + 3\begin{vmatrix} 4 & 5 \\ 8 & 8 \end{vmatrix} \\ &= -3 + 24 - 24 \\ &= -3 \neq 0. \end{aligned}$$

From (1), we have

$$
\begin{aligned}
A^{-1} &= \frac{-1}{3}\begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix}^t \\
&= \frac{-1}{3}\begin{pmatrix} \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} & -\begin{vmatrix} 4 & 6 \\ 8 & 9 \end{vmatrix} & \begin{vmatrix} 4 & 5 \\ 8 & 8 \end{vmatrix} \\ -\begin{vmatrix} 2 & 3 \\ 8 & 9 \end{vmatrix} & \begin{vmatrix} 1 & 3 \\ 8 & 9 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 8 & 8 \end{vmatrix} \\ \begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix} & -\begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix} \end{pmatrix}^t \\
&= \frac{-1}{3}\begin{pmatrix} -3 & 12 & -8 \\ 6 & -15 & 8 \\ -3 & 6 & -3 \end{pmatrix}^t = \frac{-1}{3}\begin{pmatrix} -3 & 6 & -3 \\ 12 & -15 & 6 \\ -8 & 8 & -3 \end{pmatrix}.
\end{aligned}
$$

As required.

Thus, we have $A^{-1} = \begin{pmatrix} 1 & -2 & 1 \\ -4 & 5 & -2 \\ \frac{8}{3} & -\frac{8}{3} & 1 \end{pmatrix}$. We can easily check that

$$A \cdot A^{-1} = A^{-1} \cdot A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Ex 01. Consider the matrix

$$A = \begin{pmatrix} 1 & -\alpha & & & \\ & 1 & -\alpha & & \\ & & \ddots & \ddots & \\ & & & 1 & -\alpha \\ & & & & 1 \end{pmatrix} ; \ \alpha \in \mathbb{R}$$

Prove that

$$A^{-1} = \begin{pmatrix} 1 & \alpha & \alpha^2 & \dots & \alpha^{n-1} \\ & 1 & \alpha & \dots & \alpha^{n-2} \\ & & \ddots & \ddots & \vdots \\ & & & 1 & \alpha \\ & & & & 1 \end{pmatrix} .$$

Ex 03 Let $A, B \in \mathcal{M}_2(\mathbb{R})$. Assume that one of the matrices A or B is invertible. Show that AB amd BA have the same characteristic polynomial, i.e., $p_{AB}(x) = p_{BA}(x)$.

Ex 03 When does $A^{-1} = A$?