Characteristic polynomial

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In this section we deal with the characteristic polynomial of an n by n matrix A, which is a polynomial of degree n, from which we give a practical way to find the eigenvalues of A.

Definition

Let $A \in \mathcal{M}_n(\mathbb{R})$ be a square matrix. The characteristic polynomial of A is the polynomial of degree n given by $p_A(x) = \det(A - xI_n)$, where I_n is the identity n by n matrix[a].

[a] In some references the characteristic polynomial of A is the polynomial of degree n given by $p_A(x) = \det(xI_n - A)$.

We simply write

$$p_A(x) = \det(A - xI).$$

Note that if A is an n by n matrix given by

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{pmatrix},$$

then

$$p_A(x) = \begin{vmatrix} \boxed{a_{11} - x} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & \boxed{a_{22} - x} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & \boxed{a_{33} - x} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & \boxed{a_{nn} - x} \end{vmatrix}$$

- Clearly, $p_A(x)$ is always a polynomial of **degree** n and by the Fundamental Theorem of Algebra (**FTA**), this polynomial has n roots.

Definition

The roots of $p_A(x)$ are called **eigenvalues** of A. Also, we have the notation:

$$Sp(A) = \{\lambda \in \mathbb{R} \text{ or } \mathbb{C} \; ; \; \lambda \text{ is an eigenvalue of } A\},$$

which is called the **spectral set** of A. So, $\lambda \in Sp(A) \Leftrightarrow p_A(\lambda) = 0$.

Example

Calculate the characteristic polynomial of the following matrix:

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

From definition, we obtain

$$\begin{aligned} p_A(x) &= \begin{vmatrix} 2-x & 1 \\ 1 & 2-x \end{vmatrix} = (2-x)^2 - 1 \\ &= x^2 - 4x + 3 \end{aligned}$$

We can easily see that $p_A(x) = (1-x)(3-x)$, and so $Sp(A) = \{1, 3\}$.

Example

Calculate the characteristic polynomial of the following matrix:

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}.$$

By definition,

$$\begin{aligned} p_A(x) &= \begin{vmatrix} 2 \overset{+}{-} x & \overset{-}{1} & \overset{+}{1} \\ 1 & 2-x & 1 \\ 1 & 1 & 2-x \end{vmatrix} \\ &= (2-x)\left[(2-x)^2 - 1\right] - (2-x-1) + (1-2+x) \\ &= -x^3 + 6x^2 - 9x + 4. \end{aligned}$$

So, we can take $p_A(x) = x^3 - 6x^2 + 9x - 4$.

For computing the eigenvalues of A, we must factorize $x^3 - 6x^2 + 9x - 4$. So the above method for computing $p_A(x)$ is **not useful**. For this purpose, it is natural to ask the following question:

Problem

How to **compute** *and* **factorize** $p_A(x)$ *at the same time?*

Let us take the matrix $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$. Then

$$\begin{aligned} p_A(x) &= \begin{vmatrix} 2-x & 1 \\ 1 & 2-x \end{vmatrix} \quad \begin{matrix} c_1 \\ \downarrow \\ c_1 + c_2 \end{matrix} \quad \text{(the first column } c_1 \text{ becomes } c_1 + c_2\text{)} \\ &= \begin{vmatrix} (3-x) & 1 \\ (3-x) & 2-x \end{vmatrix} = (3-x) \begin{vmatrix} 1 & 1 \\ 1 & 2-x \end{vmatrix} = (3-x)(2-x-1) \\ &= (3-x)(1-x). \end{aligned}$$

Example

Consider the matrix $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$. In the same manner, we get

$$\begin{aligned} p_A(x) &= \begin{vmatrix} 1-x & 1 & 1 \\ 1 & 1-x & 1 \\ 1 & 1 & 1-x \end{vmatrix} = \begin{vmatrix} -x & 0 & 1 \\ x & -x & 1 \\ 0 & x & 1-x \end{vmatrix} \\ &= x^2 \begin{vmatrix} \overset{+}{-1} & \overset{-}{0} & \overset{+}{1} \\ 1 & -1 & 1 \\ 0 & 1 & 1-x \end{vmatrix} = x^2 \left[-(x-1-1) + (1-0) \right] \\ &= x^2 (3-x) . \end{aligned}$$

Hence, $p_A(x) = x^2 (3-x)$, and so $Sp(A) = \{0, 3\}$.

Example

Calculate the characteristic polynomial of the following matrix:

$$A = \begin{pmatrix} 7 & -6 & -2 \\ 2 & 0 & -1 \\ 2 & -3 & 2 \end{pmatrix}.$$

It is clear that

$$\begin{aligned} p_A(x) &= \begin{vmatrix} 7-x & -6 & -2 \\ 2 & -x & -1 \\ 2 & -3 & 2-x \end{vmatrix} \quad \begin{array}{c} c_1 \\ \downarrow \\ 2 \times c_3 + c_1 \end{array} \\ &= \begin{vmatrix} (3-x) & -6 & -2 \\ 0 & -x & -1 \\ 2(3-x) & -3 & 2-x \end{vmatrix} \end{aligned}$$

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Examples

Therefore,

$$\begin{aligned}
p_A(x) &= (3-x)\begin{vmatrix} 1 & -6 & -2 \\ 0 & -x & -1 \\ 2 & -3 & 2-x \end{vmatrix} \quad \begin{matrix} c_2 \\ \downarrow \\ 3\times c_3 - c_2 \end{matrix} \\
&= (3-x)\begin{vmatrix} 1 & 0 & -2 \\ 0 & -(3-x) & -1 \\ 2 & 3(3-x) & 2-x \end{vmatrix} = (3-x)^2 \begin{vmatrix} \overset{+}{1} & \overset{-}{0} & \overset{+}{-2} \\ 0 & -1 & -1 \\ 2 & 3 & 2-x \end{vmatrix} \\
&= (3-x)^2\left(-2+x+3-2(2)\right) \\
&= (x-3)^3 .
\end{aligned}$$

That is, $p_A(x) = (x-3)^3$.

Example

Calculate the characteristic polynomial of each of the following matrices:

$$A_1 = \begin{pmatrix} 4 & 2 & -1 \\ 2 & 7 & -2 \\ -1 & -2 & 4 \end{pmatrix}, \; A_2 = \begin{pmatrix} 13 & -12 & -6 \\ 6 & -5 & -3 \\ 18 & -18 & -8 \end{pmatrix}$$

$$A_3 = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}, \; A_4 = \begin{pmatrix} 4 & 1 & -1 \\ 2 & 5 & -2 \\ 1 & 1 & 2 \end{pmatrix}$$

Example

(i) From the definition of the characteristic polynomial, we get

$$
\begin{aligned}
p_{A_1}(x) &= \det\left(A_1 - xI_3\right) \\
&= \begin{vmatrix} \mathbf{4-x} & 2 & -1 \\ 2 & \mathbf{7-x} & -2 \\ -1 & -2 & \mathbf{4-x} \end{vmatrix} \qquad \begin{array}{c} 1^{st}\text{column} \\ \downarrow \\ 1^{st}+3^{rd} \end{array} \\
&= \begin{vmatrix} (3-x) & 2 & -1 \\ 0 & 7-x & -2 \\ (3-x) & -2 & 4-x \end{vmatrix} \\
&= (3-x)\begin{vmatrix} 1 & 2 & -1 \\ 0 & 7-x & -2 \\ 1 & -2 & 4-x \end{vmatrix} \qquad \begin{array}{c} 2^{nd}\text{column} \\ \downarrow \\ 2\times 3^{rd}+2^{nd} \end{array}
\end{aligned}
$$

Taht is,

$$\begin{aligned} p_{A_1}(x) &= (3-x)\begin{vmatrix} 1 & 0 & -1 \\ 0 & 3-x & -2 \\ 1 & 2(3-x) & 4-x \end{vmatrix} = (3-x)^2 \begin{vmatrix} \overset{+}{1} & \overset{-}{0} & \overset{+}{-1} \\ 0 & 1 & -2 \\ 1 & 2 & 4-x \end{vmatrix} \\ &= (3-x)^2\left[4-x+4-(0-1)\right] \\ &= (3-x)^2(9-x). \end{aligned}$$

That is, $p_{A_1}(x) = (3-x)^2(9-x)$, and so $Sp(A) = \{3, 9\}$.

Example

(*ii*) Compute $p_{A_2}(x)$:

$$\begin{aligned}
p_{A_2}(x) &= \begin{vmatrix} 13-x & -12 & -6 \\ 6 & -5-x & -3 \\ 18 & -18 & -8-x \end{vmatrix} \quad c_1 \to c_1 + c_2 \\
&= \begin{vmatrix} (1-x) & -12 & -6 \\ (1-x) & -5-x & -3 \\ 0 & -18 & -8-x \end{vmatrix} \quad c_2 \to (-2) \times c_3 + c_2 \\
&= \begin{vmatrix} (1-x) & 0 & -6 \\ (1-x) & (1-x) & -3 \\ 0 & (-2)(1-x) & -8-x \end{vmatrix} \\
&= (1-x)^2 \begin{vmatrix} \overset{+}{1} & \overset{-}{0} & \overset{+}{-6} \\ 1 & 1 & -3 \\ 0 & -2 & -8-x \end{vmatrix} = (1-x)^2(-8-x-6+12) \\
&= (1-x)^2(-2-x).
\end{aligned}$$

Example

(iii) Computre $p_{A_3}(x)$:

$$p_{A_3}(x) = \begin{vmatrix} \mathbf{1-x} & -1 & -1 \\ -1 & \mathbf{1-x} & -1 \\ -1 & -1 & \mathbf{1-x} \end{vmatrix} \begin{matrix} c_1 & c_2 \\ \downarrow & \downarrow \\ c_1-c_2 & c_2-c_3 \end{matrix}$$

$$= \begin{vmatrix} (2-x) & 0 & -1 \\ -(2-x) & 2-x & -1 \\ 0 & -(2-x) & 1-x \end{vmatrix}$$

$$= (2-x)^2 \begin{vmatrix} \overset{+}{1} & \overset{-}{0} & \overset{+}{-1} \\ -1 & 1 & -1 \\ 0 & -1 & 1-x \end{vmatrix} = (2-x)^2 [1-x-1-1]$$

$$= -(1+x)(2-x)^2.$$

Thus, $p_{A_3}(x) = -(1+x)(2-x)^2$, and so $Sp(A) = \{-1, 2\}$.

Example

(*iiii*) Compute $p_{A_4}(x)$:

$$p_{A_4}(x) = \begin{vmatrix} 4-x & 1 & -1 \\ 2 & 5-x & -2 \\ 1 & 1 & 2-x \end{vmatrix} \quad \begin{matrix} 1^{\text{st}}\text{column} \\ \downarrow \\ 1^{\text{st}}+3^{\text{rd}} \end{matrix} \quad \begin{matrix} 2^{\text{nd}}\text{column} \\ \downarrow \\ 2^{\text{nd}}+3^{\text{rd}} \end{matrix}$$

$$= \begin{vmatrix} (3-x) & 0 & -1 \\ 0 & 3-x & -2 \\ (3-x) & 3-x & 2-x \end{vmatrix} = (3-x)^2 \begin{vmatrix} \overset{+}{1} & \overset{-}{0} & \overset{+}{-1} \\ 0 & 1 & -2 \\ 1 & 1 & 2-x \end{vmatrix}$$

$$= (3-x)^2 (2-x+2+1)$$

$$= (3-x)^2 (5-x).$$

That is, $p_{A_4}(x) = (3-x)^2 (5-x)$.

Example

Calculate the determinant $\Delta_n = \begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \mathbf{1+x} & 1 & \dots & 1 \\ 1 & 1 & \mathbf{1+x} & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \dots & \mathbf{1+x} \end{vmatrix}$. We compute

Δ_n :

- 1^{st}column $\longrightarrow 1^{\text{st}}$column
- 2^{nd}column $\longrightarrow 2^{\text{nd}}$column - 1^{st}column
- 3^{rd}column $\longrightarrow 3^{\text{rd}}$column - 1^{st}column, ... and so on. We obtain

$$\Delta_n = \begin{vmatrix} 1 & 0 & 0 & \dots & 0 \\ 1 & \mathbf{x} & 0 & \dots & 0 \\ 1 & 0 & \mathbf{x} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \dots & \mathbf{x} \end{vmatrix} = x^{n-1}.$$

Proposition

Let $A \in \mathcal{M}_n(\mathbb{R})$ and $r \in \mathbb{R}^*$. We have

$$p_{rA}(x) = r^n p_A\left(\frac{x}{r}\right).$$

Proof.

Indeed, we see that

$$p_{rA}(x) = \begin{vmatrix} ra_{11} - x & ra_{12} & \dots & ra_{1n} \\ ra_{21} & ra_{22} - x & \dots & ra_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ ra_{n1} & ra_{n2} & \dots & ra_{nn} - x \end{vmatrix}$$

□

Proof.

$$= \begin{vmatrix} r\left(a_{11} - \frac{x}{r}\right) & ra_{12} & \dots & ra_{1n} \\ ra_{21} & r\left(a_{22} - \frac{x}{r}\right) & \dots & ra_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ ra_{n1} & ra_{n2} & \dots & r\left(a_{nn} - \frac{x}{r}\right) \end{vmatrix}$$

□

Proof.

and so

$$\begin{aligned} p_{rA}(x) &= r^n \begin{vmatrix} a_{11} - \frac{x}{r} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \frac{x}{r} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \frac{x}{r} \end{vmatrix} \\ &= r^n p_A\left(\frac{x}{r}\right). \end{aligned}$$

This completes the proof. □

Exercise

Consider the vendermonde's determinant [a]:

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}.$$

Prove that $\Delta = (b-a)(c-a)(c-b)$*, and give a generalization formula.*

[a] In linear algebra, a Vandermonde matrix is a matrix with a geometric progression in each row. It takes its name from the French mathematician Alexandre-Théophile Vandermonde. It is, in particular, used in numerical analysis for solving a system formed by polynomial interpolation.

Solution

We have

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} \quad \begin{matrix} c_1 \\ \downarrow \\ c_2 - c_1 \end{matrix} \quad \begin{matrix} c_2 \\ \downarrow \\ c_3 - c_2 \end{matrix}$$

$$= \begin{vmatrix} 0 & 0 & 1 \\ b-a & c-b & c \\ b^2-a^2 & c^2-b^2 & c^2 \end{vmatrix} = (b-a)(c-b) \begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & c \\ b+a & c+b & c^2 \end{vmatrix}$$

$$= (b-a)(c-b)(c-a).$$

Fact

In the general case, the **Vendermonde's determinant** *is given by*

$$\Delta_n = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ x_0 & x_1 & \cdots & x_n \\ x_0^2 & x_1^2 & \cdots & x_n^2 \\ \vdots & \vdots & \cdots & \vdots \\ x_0^n & x_1^n & \cdots & x_n^n \end{vmatrix} = \prod_{i>j} \left(x_i - x_j \right).$$

Ex 01. Consider the following two matrices:

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \ B = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ -3 & 2 & 4 \end{pmatrix}.$$

Calculate $p_A(x)$ and $p_B(x)$. **Ans.**

$$p_A(x) = (1+x)^2 (2-x) \text{ and } p_B(x) = -(x-2)^3.$$

Ex 02. Let A be the matrix given by

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & -3 \\ 1 & -1 & 0 \end{pmatrix}.$$

Verify that $p_A(x) = (x+1)(x-1)(x-3)$.

Ex 03. Let

$$A=\begin{pmatrix} -3 & 1 & -1 \\ -7 & 5 & -1 \\ -6 & 6 & -2 \end{pmatrix}.$$

Verify that $p_A(x)=(2+x)^2(4-x)$.

Ex 04. Let $A\in\mathcal{M}_n(\mathbb{R})$ be the tridiagonal matrix given by

$$A=\begin{pmatrix} a & b & & \\ c & a & \ddots & \\ & \ddots & \ddots & b \\ & & c & a \end{pmatrix},\ a,b,c\in\mathbb{R}.$$

Calculate $p_A(x)$.

Problem

Let A be the matrix

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Show that $p_A(x) = x^4 - 1$. Deduce the eigenvalues of A.