Similar Matrices

By

Bellaouar Djamel

University 08 Mai 1945 Guelma

October 2024

We will now introduce the notion of similarity. This is a relation between two matrices. In fact, **similarity is an equivalence relation.**

Definition

Let A and B be two n-by-n matrices. We say that A is **similar to** B if there exists an invertible matrix P such that

$$A = PBP^{-1}.$$

In linear algebra, two n-by-n matrices A and B are called **similar** if $A = PBP^{-1}$ for some invertible matrix P.

Notation. The notation $A \sim B$ means that the matrix A is similar to the matrix B. Also, $A \nsim B$ means that A is not similar to B.

Next, we give an example.

Example

Let A and B be two matrices given by $A = \begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 2 \\ -2 & 5 \end{pmatrix}$.
Then A is similar to B because for the matrix $P = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$, we have after few computation

$$PBP^{-1} = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 2 \\ -2 & 5 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix} = A.$$

But, the question we ask here: **How to find the invertible matrix P so that $A = PBP^{-1}$? Moreover, for any two matrices A and B, does always such matrix P exist for which $A = PBP^{-1}$?**

Problem (Homework)

Let A and B be two matrices given by

$$A = \begin{pmatrix} -4 & 7 \\ 3 & 0 \end{pmatrix}, B = \begin{pmatrix} 13 & -8 \\ 25 & -17 \end{pmatrix}.$$

Show that A is similar to B.

- When you read the next courses, you find this problem **trivial**.

We will see later that in general two n by n matrices A and B are not similar to each other. That is, for some matrices we cannot find an invertible matrix P suct that $A = PBP^{-1}$.
Let us start with the following properties:

Theorem

Let A and B be two n-by-n similar matrices; i.e., there exists an invertible matrix P such that $A = PBP^{-1}$. Then

1. *For each positive integer k, $A^k = PB^k P^{-1}$.*
2. *$p_A(x) = p_B(x)$, that is A and B have the same characteristic polynomial.*

Proof.

We prove the theorem as follows:

1. Assume that A and B are two similar matrices, and let P be an invertible matrix such that $A = PBP^{-1}$. For each integer $k \geq 0$ we have

$$\begin{aligned} A^k &= \underbrace{\left(PBP^{-1}\right)\left(PBP^{-1}\right)...\left(PBP^{-1}\right)}_{k-\text{times}} \\ &= P\underbrace{BB...B}_{k-\text{times}}P^{-1} \\ &= PB^kP^{-1}. \end{aligned}$$

2. We prove the following implication

$$A \sim B \Rightarrow p_A(x) = p_B(x). \tag{1}$$

□

Proof.

That is, if the matrices A and B are similar to each other, then A and B have the same characteristic equation, and hence have the same eigenvalues. In fact, we have

$$\begin{aligned} p_A(x) &= \det(A - xI) \\ &= \det\left(PBP^{-1} - xPP^{-1}\right), \text{ since } PP^{-1} = I \\ &= \det\left(P(B - xI)P^{-1}\right), \text{ since } x \in \mathbb{R} \\ &= \det(P)\det(B - xI)\det\left(P^{-1}\right) \qquad (2) \\ &= \det(B - xI) \qquad (3) \\ &= p_B(x). \end{aligned}$$

Note that the passage from (2) to (3) because $\det\left(P^{-1}\right) = \dfrac{1}{\det(P)}$.

The proof is finished. □

Remark. The converse of (1) is false. For example, for

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2$$

We see that $p_A(x) = p_B(x)$. Therefore, $Sp(A) = Sp(B) = \{1\}$ and $\det(A) = \det(B)$. Further, if A is similar to B then there exists an invertible matrix P such that

$$A = PBP^{-1} = PI_2P^{-1} = I_2.$$

A contradiction since $A \neq I_2$. Thus, A is not similar to B.
Conclusion: We can also write

$$\begin{cases} Sp(A) = Sp(B) \nRightarrow A \sim B, \\ p_A(x) = p_B(x) \nRightarrow A \sim B. \end{cases}$$

Remark

By applying the following rule:

$$\det(A) = 0 \Leftrightarrow 0 \in Sp(A). \tag{4}$$

Let A and B be two similar matrices, i.e., there exists an invertible matrix P such that $A = PBP^{-1}$. We can also prove that $Sp(A) = Sp(B)$. Let $\lambda \in Sp(A)$, there exists a nonzero vector x tel que $Ax = \lambda x$. That is,

$$(A - \lambda I)x = 0 = 0.x$$

Which gives $0 \in Sp(A - \lambda I)$. On the other hand, we have

$$A - \lambda I = P(B - \lambda I)P^{-1}. \tag{5}$$

Remark

Asssume that $0 \notin Sp\left(B-\lambda I\right)$. *By* (4) *and* (5) *we have* $B-\lambda I \in \mathbb{GL}_n\left(\mathbb{R}\right)$. *Consequently,* $A-\lambda I \in \mathbb{GL}_n\left(\mathbb{R}\right)$. *From* (4), $0 \notin Sp\left(A-\lambda I\right)$. *A contradiction. Finally, we deduce that* $0 \in Sp\left(B-\lambda I\right)$, *and hence* $\lambda \in Sp\left(B\right)$. *Thus,* $Sp\left(A\right) \subset Sp\left(B\right)$.

Corollary

Two similar matrices A and B have the same determinant.

Proof.

Let P be an invertible matrix P such that $A = PBP^{-1}$. It follows that

$$\det(A) = \det\left(PBP^{-1}\right) = \det(P)\det(B)\det\left(P^{-1}\right) = \det(B),$$

and so $\det(A) = \det(B)$. This completes the proof.

Fact

We have

$$\det\left(\prod_{i=1}^{k} A_i\right) = \prod_{i=1}^{k} \det(A_i).$$

□

Example

Consider the following two matrices:

$$A = \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 5 & 2 \\ 4 & 1 \end{pmatrix}.$$

How can we tell (rather quickly) that the matrices A and B are not similar to each other?
In fact, $A \nsim B$ because $\det(A) = -1 \neq \det(B) = -3$. Thus, we have applied the result:

$$\det(A) \neq \det(B) \Rightarrow A \nsim B.$$

- As we have seen before: $\det(A) = \det(B) \nRightarrow A \sim B$.

Theorem

The relation " $\sim$ " similarity is an **equivalence relation**.

Proof.

This relation is what we call an **equivalence relation**, because we have the following three properties:

1. The relation " $\sim$ " is reflexive, because for each matrix $A \in \mathcal{M}_n(\mathbb{R})$ we have
$$A = I_n A I_n^{-1}.$$
Then $A \sim A$.
2. The relation " $\sim$ " is symmetric, because for all matrices $A, B \in \mathcal{M}_n(\mathbb{R})$ we have
$$A \sim B \Rightarrow \exists\ P \in \mathbb{GL}_n(\mathbb{R}) \text{ such that } A = PBP^{-1}.$$

□

Proof.

Thus, $B \sim A$ (i.e., we can just say that A and B are similar to each other). For the matrices A, B, and P of Example 2, verify by direct computation that $A = PBP^{-1}$ and that $B = P^{-1}AP$.

3. The relation " $\sim$ " is transitive, because for all matrices $A, B, C \in \mathcal{M}_n(\mathbb{R})$ we have

$$\left.\begin{array}{c} A \sim B \\ B \sim C \end{array}\right\} \Rightarrow \left\{\begin{array}{l} \exists\ P \in \mathbb{GL}_n(\mathbb{R}) \text{ such that } A = PBP^{-1}, \\ \exists\ Q \in \mathbb{GL}_n(\mathbb{R}) \text{ such that } B = QCQ^{-1}. \end{array}\right.$$

Which gives

$$A = P\left(QCQ^{-1}\right)P^{-1} = \underbrace{(PQ)}_{R}\,C\,(PQ)^{-1} = RCR^{-1} \text{ with } R \in \mathbb{GL}_n(\mathbb{R}).$$

Hence, $A \sim C$.

□

Proposition

Let $P \in \mathbb{GL}_n(\mathbb{R})$. *Define the mapping* T_P *by:* $T_P : \mathcal{M}_n(\mathbb{R}) \rightarrow \mathcal{M}_n(\mathbb{R})$, $A \mapsto T_P(A) = P^{-1}AP$. *Then the following statements hold:*

1. $T_P(I_n) = I_n$
2. $T_P(A+B) = T_P(A) + T_P(B)$
3. $T_P(AB) = T_P(A)\, T_P(B)$
4. $T_P(rA) = rT_P(A)$
5. $T_P\left(A^k\right) = (T_P(A))^k$
6. $T_P\left(A^{-1}\right) = (T_P(A))^{-1}$
7. $T_P\left(e^A\right) = e^{T_P(A)}$
8. $T_Q(T_P(A)) = T_{PQ}(A)$.

Proof.

We have

1. In fact, $T_P(I_n) = P^{-1} I_n P = P^{-1} P = I_n$.
2. $T_P(A+B) = P^{-1}(A+B)P = P^{-1}AP + P^{-1}BP = T_P(A) + T_P(B)$.
3. $T_P(AB) = P^{-1}ABP = P^{-1}APP^{-1}BP = \left(P^{-1}AP\right)\left(P^{-1}BP\right) = T_P(A)\,T_P(B)$.
4. $T_P(rA) = P^{-1}(rA)P = r\left(P^{-1}AP\right) = rT_P(A)$.
5. $T_P\left(A^k\right) = P^{-1}A^kP = \left(P^{-1}AP\right)^k = (T_P(A))^k$.
6. $T_P\left(A^{-1}\right) = P^{-1}A^{-1}P = \left(P^{-1}AP\right)^{-1} = (T_P(A))^{-1}$.
7. $T_P\left(e^A\right) = P^{-1}e^AP = e^{P^{-1}AP} = e^{T_P(A)}$.
8. It is clear that $T_Q(T_P(A)) = Q^{-1}T_P(A)\,Q$ $= Q^{-1}\left(P^{-1}AP\right)Q = (PQ)^{-1}A(PQ) = T_{PQ}(A)$. This completes the proof.

Remark. Let $A, B \in \mathcal{M}_n(\mathbb{R})$. If $A \sim B$, then

$$A \in \mathbb{GL}_n(\mathbb{R}) \Leftrightarrow B \in \mathbb{GL}_n(\mathbb{R}).$$

In fact, we have $A = PBP^{-1} \Leftrightarrow B = P^{-1}AP$.
Conclusion. Let $A \in \mathcal{M}_n(\mathbb{R})$, and let $B = P^{-1}AP \in \mathcal{M}_n(\mathbb{R})$ be a matrix similar to A. Then A and B have the same characteristic polynomial. Furthermore, $f(A) = P \cdot f(B) \cdot P^{-1}$ for each $f \in \mathbb{K}[X]$, and in particular $A^k = PB^kP^{-1}$for $k \geq 1$.

Corollary

Let $A, B \in \mathcal{M}_n(\mathbb{R})$. If $A \sim B$, then $Tr(A) = Tr(B)$.

- $Tr(A)$ is the **trace** of A.

Proof.

We know that

$$\forall\ M, N \in \mathcal{M}_n(\mathbb{R}) : Tr\,(MN) = Tr\,(NM)\,.$$

Then

$$Tr\,(A) = Tr\left(PBP^{-1}\right) = Tr\left(BPP^{-1}\right) = Tr\,(B)\,.$$

□

Corollary

Two similar matrix have the same rank.

Proof.

Assume that $A = PBP^{-1}$ for some invertible square matrix P. We have $rk\,(B) \geq rk\left(PBP^{-1}\right) = rk\,(A)$. Now note that $B = P^{-1}AP$, so we similarily get $rk\,(A) \geq rk\left(P^{-1}AP\right) = rk\,(B)$.

□

Conclusion. Two similar matrices have the same determinant, same trace, same rank, same characteristic polynomial, same eigenvalues.
Question. *Why the study of similarity between matrices is* **important***?*
Ans. If a matrix A is similar to a diagonal matrix D, then the computations of A^k, e^A, $\cos A$, $\sqrt{A}, \ldots$ are quite **easy**.

Problem

Is always any square matrix similar to a **digonal** *matrix?*

Ex 01. Let A and B be two similar matrices, i.e., there exists an invertible matrix P such that $A = PBP^{-1}$. Prove that

$$(\lambda, x) \text{ is an eigenpair of } A \Rightarrow \left(\lambda, P^{-1}x\right) \text{ is an eigenpair of } B.$$

Ex 02. Let $A, B\ \mathcal{M}_n(\mathbb{R})$ and $f(x) = a_0 + a_1x + ... + a_nx^n \in \mathbb{R}[x]$ be a polynomial of degree n. Prove that

$$A \sim B \Rightarrow f(A) \sim f(B).$$

Ex 03. Consider the two matrices:

$$A = \begin{pmatrix} 1 & 0 & 4 \\ 1 & 1 & 3 \\ 2 & 1 & 7 \end{pmatrix} \text{ et } B = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 3 & 1 & 2 \end{pmatrix}.$$

Prove that $A \nsim B$; i.e., A and B are not similar to each other.

Ex 04. Show that
$$A - \lambda I_n \sim B \Rightarrow A \sim B + \lambda_n I.$$

Ex 05. Using two methods. Prove that similar matrices have the same eigenvalues.

Ex 06. Prove that
$$A \sim B \Rightarrow e^A \sim e^B.$$

Ex 07. Without calculating, neither eigenvalues nor eigenvectors, show that
$$\begin{pmatrix} 1 & -1 \\ 3 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 \\ -1 & 1 \end{pmatrix}.$$

Ex 08. Show by direct computation that the matrices A and B of Example 2 have the same characteristic equation. What are the eigenvalues of A and B?