Special Matrices
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Definition

A matrix with all zero entries is called a **zero matrix** and is denoted by 0. That is,

$$A = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Also, A is called the **null matrix.**

Definition

A square matrix $A = \left(a_{ij}\right)$ is diagonal if $a_{ij} = 0$ for $i \neq j$. In this case, we write $D = diag\left\{\lambda_1, \lambda_2, ..., \lambda_n\right\}$. So, A **diagonal matrix** is given by:

$$D = \begin{pmatrix} \boldsymbol{\lambda_1} & 0 & \cdots & 0 \\ 0 & \boldsymbol{\lambda_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \boldsymbol{\lambda_n} \end{pmatrix}.$$

- Every computations on diagonal matrices are quite **easy**. For example, $\sqrt{D}$, D^k, D^{-1}, e^D, cos$D$, ln D, ...can be easily computed.

Definition

The **unit matrix** or the **identity matrix**:

$$I_n = \begin{pmatrix} \mathbf{1} & 0 & \cdots & 0 \\ 0 & \mathbf{1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{1} \end{pmatrix}$$

This is a diagonal matrix; but, all the diagonal elements are equal to 1.

Fact

For any $A \in \mathcal{M}_n(\mathbb{R})$ *we have*

$$A \cdot I_n = I_n \cdot A = A.$$

- We always write I instead of I_n. So $A \cdot I = I \cdot A = A$.

Definition

A square matrix is **upper triangular** if all entries below the main diagonal are zero. The general form of an upper triangular matrix is given by

$$U = \begin{pmatrix} \mathbf{a_{11}} & a_{12} & \cdots & a_{1n} \\ 0 & \mathbf{a_{22}} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{a_{nn}} \end{pmatrix}.$$

Definition

A square matrix is **lower triangular** if all entries above the main diagonal are 0. The general form of a lower triangular matrix is given by

$$L = \begin{pmatrix} \mathbf{a_{11}} & 0 & \cdots & 0 \\ a_{21} & \mathbf{a_{22}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & \mathbf{a_{nn}} \end{pmatrix}.$$

- The eigenvalues of any triangular matrix are its diagonal entries.

Definition

Strictly triangular matrices are of the form:

$$\begin{pmatrix} \mathbf{0} & a_{12} & \cdots & a_{1n} \\ 0 & \mathbf{0} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{0} \end{pmatrix} \text{ or } \begin{pmatrix} \mathbf{0} & 0 & \cdots & 0 \\ a_{21} & \mathbf{0} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & \mathbf{0} \end{pmatrix}.$$

Properties of transpose:

- $\left(A^t\right)^t = A$.
- $(A+B)^t = A^t + B^t$.
- For scalar α, $(\alpha A)^t = \alpha A^t$.
- $(AB)^t = B^t A^t$.

Example

For the matrix

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} \in \mathcal{M}_{3,2}(\mathbb{R}),$$

we have

$$A^t = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix} \in \mathcal{M}_{2,3}(\mathbb{R}).$$

Theorem

Let $A \in \mathcal{M}_n(\mathbb{R})$. Then A and A^t have the same eigenvalues.

Proof.

Let $x \in \mathbb{R}$. We have

$$\begin{aligned} p_A(x) &= \det(A - xI) = \det\left((A - xI)^t\right) \quad (\text{since } \det B = \det B^t) \\ &= \det\left(A^t - xI\right) \\ &= p_{A^t}(x). \end{aligned}$$

Thus, A and its transpose have the same characteristic polynomial. □

Definition

Let $A = \left(a_{ij}\right)_{1 \leq i,j \leq n}$ be a square matrix. A is said to be **symmetric** if $A^t = A$. That is, $a_{ij} = a_{ji}$ for each $i, j \in \overline{1, n}$. So, an $n \times n$ matrix A is called symmetric if it is equal to its transpose.

Example

The matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 0 & 5 \\ 3 & 5 & 1 \end{pmatrix}$$

is symmetric; since $A^t = A$.

Corollary

For every matrix $A \in \mathcal{M}_n(\mathbb{R})$, $A^t A$ and AA^t are always symmetric.

Proof.

It is clear that

$$\left(A^t A\right)^t = A^t \left(A^t\right)^t = A^t A.$$

That is, for each $A \in \mathcal{M}_n(\mathbb{R})$, $A^t A$ is symmetric. □

Proposition

The eigenvalues of a real symmetric matrix are real numbers.

Proof.

See Theorem 26. □

Corollary

Let $A \in \mathcal{M}_n(\mathbb{R})$ *be a symmetric matrix and let* $\alpha_0, \alpha_1, ..., \alpha_m \in \mathbb{R}$ *with* $m \geq 1$*. The matrix*

$$\alpha_0 I + \alpha_1 A + ... + \alpha_m A^m$$

is also symmetric.

Proof.

(Easy). □

Definition

Let $A = (a_{ij})_{1 \leq i,j \leq n}$ be a square matrix. A is said to be **skew-symmetric** if $A^t = -A$. That is, $a_{ij} = -a_{ji}$ for each $i, j \in \overline{1, n}$.

For example, the matrix

$$A = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}$$

is skew-symmetric since $A^t = -A$.

Lemma

Every square matrix $M \in \mathcal{M}_n(\mathbb{R})$ can be written as $A + B$, where A is skew-symmetric and B is symmetric.

Proof.

It is clear that for each $M \in \mathcal{M}_n(\mathbb{R})$ we have

$$M = \underbrace{\frac{1}{2}\left(M - M^t\right)}_{\text{skew-symmetric}} + \underbrace{\frac{1}{2}\left(M + M^t\right)}_{\text{symmetric}}.$$

□

Theorem (18)

Let B be a skew-symmetric matrix; i.e., $B^t = -B$. Then the matrix $A = I - B$ is invertible.

Remark

Note that a matrix A is invertible if and only if $(Ax = 0 \Rightarrow x = 0)$.

Proof.

[Proof of Theorem 17]
It suffices to prove that $Ax = 0$ implies $x = 0$. In fact, if $Ax = 0$, it follows that $Bx = x$. Therefore,

$$\langle x, x \rangle = \langle x, Bx \rangle .$$

On the other hand, we have

$$\begin{array}{rcl} x^t x & = & x^t Bx \\ & \Rightarrow & x^t x = x^t B^t x \quad \text{(since } \left(x^t x\right)^t = x^t x \text{ and } \left(x^t Bx\right)^t = x^t B^t x\text{)} \\ & \Rightarrow & x^t x = x^t \left(-B\right) x \quad \text{(since } B \text{ is skew-symmetric)} \\ & \Rightarrow & x^t x = -x^t Bx \\ & \Rightarrow & x^t x = -x^t x \\ & \Rightarrow & x^t x = 0. \end{array}$$

□

Proof.

[Proof of Theorem 17]
Setting $x = \begin{pmatrix} x_1 & x_2 & \dots & x_n \end{pmatrix}^t$, we find

$$x^t x = \begin{pmatrix} x_1 & x_2 & \dots & x_n \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = x_1^2 + x_2^2 + ... + x_n^2 = 0.$$

Thus, $x_i = 0$ for each $i \in \overline{1, n}$, and so $x = 0$. □

1. Let

$$A = \begin{pmatrix} 0 & -2 & 3 \\ 2 & 0 & 4 \\ -3 & -4 & 0 \end{pmatrix}$$

   Verify that A is skew-symmetric.

2. Prove that $\mathcal{M}_n(\mathbb{R}) = \mathcal{S}_n(\mathbb{R}) \oplus \mathcal{A}_n(\mathbb{R})$, where $\mathcal{S}_n(\mathbb{R})$ is the subspace of all symmetric matrices and $\mathcal{A}_n(\mathbb{R})$ is the subspace of all skew-symmetric matrices.

Definition

A matrix $A \in \mathcal{M}_n(\mathbb{R})$ is called **orthogonal** if $A^t = A^{-1}$ (or if $A^t A = AA^t = I$).

Example

The matrix

$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix};\ \theta \in \mathbb{R}$$

is orthogonal, since

$$\begin{aligned} A^t A &= AA^t = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2. \end{aligned}$$

An orthogonal matrix has the following properties:

1. its column vectors (rows) are orthonormal,
2. $A^t A = AA^t = I_n$,
3. $A^t = A^{-1}$,
4. For every $x \in \mathbb{R}^n : \|Ax\| = \|x\|$,
5. For every $x, y \in \mathbb{R}^n : \langle Ax, Ay \rangle = \langle x, y \rangle$.

Corollary

Let $A \in \mathcal{M}_n(\mathbb{R})$ be an orthogonal matrix. Then $\det(A) = \pm 1$.

Proof.

Since $A^t = A^{-1}$, then $A^t A = I_n$. It follows that

$$\det\left(A^t A\right) = \det\left(A^t\right) \det(A) = (\det(A))^2 = \det(I_n) = 1.$$

Hence $\det(A) = \pm 1$. □

Theorem

Let $A \in \mathcal{M}_n(\mathbb{R})$ be an orthogonal matrix. The following properties are equivalent.

1) *A is orthogonal.*
2) *For every $x \in \mathbb{R}^n : \|Ax\| = \|x\|$.*
3) *For every $x, y \in \mathbb{R}^n : \langle Ax, Ay \rangle = \langle x, y \rangle$.*

Proof.

1)⇒2). Assume that A is orthogonal. Let $x \in \mathbb{R}^n$, we have

$$\|Ax\|^2 = \langle Ax, Ax \rangle = \left\langle x, A^t Ax \right\rangle = \langle x, I_n x \rangle = \langle x, x \rangle = \|x\|^2 .$$

Therefore, $\|Ax\| = \|x\|$.
2)⇒3). Assume that $\forall\ x \in \mathbb{R}^n : \|Ax\| = \|x\|$. Let $x, y \in \mathbb{R}^n$, we have

$$\|A(x+y)\|^2 = \|x+y\|^2 ;$$

Proof.

That is, $\langle Ax+Ay, Ax+Ay\rangle = \langle x+y, x+y\rangle$, and so

$$\langle Ax, Ax\rangle + \langle Ay, Ay\rangle + 2\langle Ax, Ay\rangle = \langle x, x\rangle + \langle y, y\rangle + 2\langle x, y\rangle$$

Thus, $\langle Ax, Ay\rangle = \langle x, y\rangle$.
3)⇒1). Assume that $\forall\ x, y \in \mathbb{R}^n : \langle Ax, Ay\rangle = \langle x, y\rangle$. It follows that

$$\left\langle x, A^t Ay\right\rangle = \langle x, y\rangle$$

i.e., $\langle x, A^t Ay - y\rangle = 0$. In particular, for $x = x^t Ay - y$, we obtain

$$\left\|A^t Ay - y\right\|^2 = 0.$$

Hence $A^t Ay = y$, and therefore $A^t A = I_n$. □

Exercise

Consider the matrix

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

For each $\theta \in \mathbb{R}$, prove that $e^{\theta A}$ is orthogonal[a].

[a] See the chapter of exponential of square matrices.

Exercise

Let A be an orthogonal matrix. Prove the following properties:

1. A^{-1} *is orthogonal.*
2. *For every* $\lambda \in Sp(A) \Rightarrow |\lambda| = 1$.
3. *If* A_1 *and* A_2 *are two orthogonal matrices, then* $A_1 A_2$ *is also orthogonal.*

Definition

Let $A = (a_{ij})_{1 \leq i,j \leq n} \in \mathcal{M}_n(\mathbb{C})$. That is a_{ij} is a complex number for $1 \leq i, j \leq n$. The matrix $(\overline{a_{ij}})_{1 \leq i,j \leq n}$ is called **conjugate** of A, denoted by $\overline{A}$. The **transpose conjugate** matrix of A is called the **adjoint** of A, denoted by A^*. Note that $A^* = \overline{A^t} = \left(\overline{A}\right)^t$.

Definition

A matrix $A \in \mathcal{M}_n(\mathbb{C})$ is called **Hermitian**[a] if $A^* = A$. Thta is, if $\overline{A^t} = A$.

[a] On the other hand, a matrix A is said to be skew-Hermitian if $A^* = -A$.

Example

The matrix

$$A = \begin{pmatrix} 1 & 1+i & 2+3i \\ 1-i & -2 & -i \\ 2-3i & i & 0 \end{pmatrix}$$

is Hermitian; because $A^* = A$.

Proposition

The diagonal coefficients of a Hermitian matrix are real.

Proof.

From Definition 22, the result is obvious since $a_{ii} = \overline{a_{ii}}$ for $1 \leq i \leq n$.

Remark

Let $A \in \mathcal{M}_n(\mathbb{C})$ *. We can easily prove that* $A + A^*$, AA^* *and* A^*A *are Hermitian.*

Theorem

The eigenvalues of a Hermitian matrix are real.

Proof. Let (λ, x) be an eigenpair of a Hermitian matrix A (note that $x \neq 0$). We can write

$$\begin{aligned}
\lambda \langle x, x \rangle &= \langle \lambda x, x \rangle \\
&= \langle Ax, x \rangle = (Ax)^t \overline{x} = x^t A^t \overline{x} \\
&= x^t \left(\left(\overline{A} \right)^t \right)^t \overline{x} \quad \left(\text{since } \left(\overline{A} \right)^t = A\right) \\
&= x^t \overline{A} \overline{x} = x^t \overline{Ax} = \langle x, Ax \rangle = \langle x, \lambda x \rangle = \overline{\lambda} \langle x, x \rangle .
\end{aligned}$$

That is, $\lambda = \overline{\lambda}$.

Definition

A matrix $U \in \mathcal{M}_n(\mathbb{C})$ is said to be **unitary** if $U^{-1} = U^*$. In other words, a square matrix U with complex coefficients is said to be unitary if it satisfies the equalities:

$$U^* U = UU^* = I_n.$$

- The unitary matrices with real coefficients are the orthogonal matrices.
- Note that a complex square matrix A is **normal** if it commutes with its conjugate transpose A^*. That is, $A^* A = AA^*$. Thus, unitary, Hermitian and skew-Hermitian matrices are normal.

Example

The matrix

$$A = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

is unitary; since

$$AA^* = A^*A = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2.$$

Any unitary matrix U satisfies the following properties:

a. its determinant has modulus 1;
b. its eigenvectors are orthogonal;
c. U is diagonalizable, i.e.,

$$U = VDV^*,$$

where V is a unitary matrix and D is a unitary diagonal matrix.

d. U can be written as an exponential of a matrix:

$$U = e^{iH},$$

where i is the imaginary unit and H is a Hermitian matrix.

Proposition

Let U be a square matrix of size n with complex coefficients; the following five propositions are equivalent:

1. *U is unitary;*
2. *U^* is unitary;*
3. *U is invertible and its inverse is U^*;*
4. *the columns of U form an orthonormal basis for the canonical Hermitian product over $\mathbb{C}^n$;*
5. *U is normal and its eigenvalues have modulus 1.*

Definition

Let $A \in \mathcal{M}_n(\mathbb{K})$. Then A is called **idempotent** if $A^2 = A$.

Examples of 2×2 idempotent matrices are:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 3 & -6 \\ 1 & -2 \end{pmatrix}$$

Theorem

If A is idempotent, then A is diagonalizable.

Proof.

Since $A^2 = A$, it follows that $m_A(x) = x(x-1)$ which has simple roots, and hence A is diagonalizable. $\square$