System of recurrence sequences. Part I

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System of recurrence sequences
Form I

Let $(x_n)_{n\geq 0}$ and $(y_n)_{n\geq 0}$ be two sequences given by the following relation:

$$\begin{cases} x_{n+1} = a_{11}x_n + a_{12}y_n \\ y_{n+1} = a_{21}x_n + a_{22}y_n \end{cases} ; \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} a_0 \\ b_0 \end{pmatrix}. \tag{1}$$

In the matrix form, we get

$$\underbrace{\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix}}_{X_{n+1}} = \underbrace{\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x_n \\ y_n \end{pmatrix}}_{X_n}.$$

Or, equivalently, we write (1) in the form

$$X_{n+1} = A \cdot X_n \text{ , where } X_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}.$$

Consequently,

$$X_n = A \cdot X_{n-1} = A\left(AX_{n-2}\right) = A^2 \cdot X_{n-2} = ... = A^n \cdot X_0. \tag{2}$$

Remark. If we have X_1 instead of X_0, then $X_n = A^{n-1} \cdot X_1$.

In the general case, a system of k linear recurrence sequences $x_n^{(i)}$, $i = 1, 2, ..., k$ is given by

$$\begin{cases} x_{n+1}^{(1)} = a_{11}x_n^{(1)} + a_{12}x_n^{(2)} + ... + a_{1k}x_n^{(k)} \\ x_{n+1}^{(2)} = a_{21}x_n^{(1)} + a_{22}x_n^{(2)} + ... + a_{2k}x_n^{(k)} \\ \vdots \\ x_{n+1}^{(k)} = a_{k1}x_n^{(1)} + a_{k2}x_n^{(2)} + ... + a_{kk}x_n^{(k)} \end{cases} ; \ x_0^{(i)} \in \mathbb{R}, \text{ for } i = 1, 2, ..., k. \quad (3)$$

In the matrix form

$$\underbrace{\begin{pmatrix} x_{n+1}^{(1)} \\ x_{n+1}^{(2)} \\ \vdots \\ x_{n+1}^{(k)} \end{pmatrix}}_{X_{n+1}} = \underbrace{\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1k} \\ a_{21} & a_{22} & \dots & a_{2k} \\ \vdots & \vdots & \dots & \vdots \\ a_{k1} & a_{k2} & \dots & a_{kk} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x_n^{(1)} \\ x_n^{(2)} \\ \vdots \\ x_n^{(k)} \end{pmatrix}}_{X_n},$$

where $X_0 = \begin{pmatrix} x_0^{(1)} \\ x_0^{(2)} \\ \vdots \\ x_0^{(k)} \end{pmatrix}$. As in (2), we get

$$X_n = \begin{pmatrix} x_n^{(1)} \\ x_n^{(2)} \\ \vdots \\ x_n^{(k)} \end{pmatrix} = A \cdot X_{n-1} = A\left(AX_{n-2}\right) = A^2 \cdot X_{n-2} = ... = A^n \cdot X_0.$$

These problems (the solution of (1) or (3)) reduce to the computation of A^n. So, we must compute the powers of A. That is , we must compute the explicit formula of A^n in terms of n.

Consider the following example:

Example

Solve the system of linear recurrence sequences

$$\begin{cases} x_{n+1} = 2x_n - y_n \\ \\ y_{n+1} = -x_n + 2y_n \end{cases} ; \quad (x_0, y_0) = (0, -1). \tag{4}$$

Or, find x_{100}.

Solution. First, we write the system (4) according to the equivalent matrix form

$$\underbrace{\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix}}_{X_{n+1}} = \underbrace{\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x_n \\ y_n \end{pmatrix}}_{X_n} ; X_0 = \begin{pmatrix} 0 \\ -1 \end{pmatrix}.$$

From (2), we have $X_n = A^n X_0$. Moreover, by our previous computations, the explicit formula of A^n in terms of n is given by

$$A^n = \begin{pmatrix} \frac{1+3^n}{2} & \frac{1-3^n}{2} \\ \frac{1-3^n}{2} & \frac{1+3^n}{2} \end{pmatrix} ; n \geq 0. \tag{5}$$

It follows that

$$X_n = A^n X_0 = \begin{pmatrix} \frac{1+3^n}{2} & \frac{1-3^n}{2} \\ \frac{1-3^n}{2} & \frac{1+3^n}{2} \end{pmatrix} \begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} \frac{3^n-1}{2} \\ \frac{-3^n-1}{2} \end{pmatrix}.$$

Thus,

$$\begin{cases} x_n = \frac{3^n-1}{2} \\ y_n = \frac{-3^n-1}{2} \end{cases}, \ n \geq 0.$$

Here we can compute x_{100}, x_{1000}, ...

$$x_{100} = \frac{3^{100}-1}{2} = 257688760366005665518230564882810636351053761000.$$

Also,

$$x_{1000} = \frac{3^{1000}-1}{2} = 661\,035\,409\,740\,403\,318\,445\,227\,629\,876\,072\,182\,982$$
$$711\,016\,376\,074\,083\,832\,460\,184\,113\,414298\,673\,352\,449\,770\,389\,1$$
$$56\,925\,304\,030\,981\,954\,888\,848\,436\,291\,177\,975\,477\,291\,050\,309\,45$$
$$5\,932\,671\,362628\,976\,837\,013\,810\,112\,599\,160\,401\,939\,007\,387\,114$$
$$482\,420\,637\,195\,200\,058\,794\,309\,020\,564\,473\,907\,811\,547\,219\,030\,7$$
$$83\,086\,527\,043\,337\,245\,253\,089\,062\,740\,172\,202\,773\,527\,198\,519\,44$$
$$7\,908\,732\,684\,127\,458\,068110\,415\,134\,281\,889\,291\,145\,114\,208\,199$$
$$153\,943\,948\,459\,278\,202\,042\,449\,468\,804\,686\,621\,085\,923\,179\,96934$$
$$7\,758\,382\,509\,470\,294\,054\,530\,213\,044\,835\,719\,432\,051\,407\,175\,192$$
$$824\,373\,582\,916\,005\,307\,183\,066\,086551\,384\,451\,427\,610\,000.$$

Consider the system of linear recurrence sequences $x_n^{(i)}$, for $i = 1, 2, ..., k$:

$$\begin{cases} x_{n+1}^{(1)} = a_{11}x_n^{(1)} + a_{12}x_n^{(2)} + ... + a_{1k}x_n^{(k)} + c_1 \\ x_{n+1}^{(2)} = a_{21}x_n^{(1)} + a_{22}x_n^{(2)} + ... + a_{2k}x_n^{(k)} + c_2 \\ \vdots \\ x_{n+1}^{(k)} = a_{k1}x_n^{(1)} + a_{k2}x_n^{(2)} + ... + a_{kk}x_n^{(k)} + c_k \end{cases} ; \ c_i, x_0^{(i)} \in \mathbb{R}, \text{ for } i = 1, 2, ..., k.$$

Here, $c_1, c_2, \ldots, c_k$ are **not all zero**.

In the matrix form

$$\underbrace{\begin{pmatrix} x_{n+1}^{(1)} \\ x_{n+1}^{(2)} \\ \vdots \\ x_{n+1}^{(k)} \end{pmatrix}}_{X_{n+1}} = \underbrace{\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1k} \\ a_{21} & a_{22} & \dots & a_{2k} \\ \vdots & \vdots & \dots & \vdots \\ a_{k1} & a_{k2} & \dots & a_{kk} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x_{n}^{(1)} \\ x_{n}^{(2)} \\ \vdots \\ x_{n}^{(k)} \end{pmatrix}}_{X_n} + \underbrace{\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{pmatrix}}_{C},$$

where $X_0 = \begin{pmatrix} x_0^{(1)} \\ x_0^{(2)} \\ \vdots \\ x_0^{(k)} \end{pmatrix}$.

Hence,

$$X_{n+1} = A \cdot X_n + C.$$

This means that

$$\begin{aligned} X_n &= AX_{n-1} + C = A\left(AX_{n-2} + C\right) + C = A^2 X_{n-2} + \left(A + I\right) C \\ &= \ldots \\ &= A^n X_0 + \left(A^{n-1} + A^{n-2} + \ldots + A + I\right) C. \end{aligned} \tag{6}$$

This problem si reduced to the computation of A^n and $\sum_{i=0}^{n-1} A^i$.

So, to solve the system stated in Form II, we must compute:

1. A^n,

This means that

$$\begin{aligned} X_n &= AX_{n-1} + C = A\left(AX_{n-2} + C\right) + C = A^2 X_{n-2} + (A + I)\, C \\ &= \ldots \\ &= A^n X_0 + \left(A^{n-1} + A^{n-2} + \ldots + A + I\right) C. \end{aligned} \tag{6}$$

This problem si reduced to the computation of A^n and $\sum_{i=0}^{n-1} A^i$.

So, to solve the system stated in Form II, we must compute:

1. A^n,
2. $\sum_{i=0}^{n-1} A^i$.

Let us take an example:

Example

Solve the system of linear recurrence sequences:

$$\begin{cases} x_{n+1} = 2x_n - y_n - 1 \\ y_{n+1} = -x_n + 2y_n + 2 \end{cases} \quad ; \quad (x_0, y_0) = (0, -1). \qquad (7)$$

Solution. The system (7) can be written in the following matrix form:

$$\underbrace{\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix}}_{X_{n+1}} = \underbrace{\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x_n \\ y_n \end{pmatrix}}_{X_n} + \underbrace{\begin{pmatrix} -1 \\ 2 \end{pmatrix}}_{C},$$

where $C = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$. As before, we have

$$X_n = A^n X_0 + \left(A^{n-1} + A^{n-2} + ... + A + I\right) C.$$

It suffices to compute $A^{n-1} + A^{n-2} + ... + A + I$.

Indeed, in view of (5) we can write

$$A^n = \begin{pmatrix} \frac{1+3^n}{2} & \frac{1-3^n}{2} \\ \frac{1-3^n}{2} & \frac{1+3^n}{2} \end{pmatrix} = \frac{1}{2}U + \frac{3^n}{2}V,$$

where

$$U = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \text{ and } V = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

It follows that

$$\begin{aligned} A^{n-1} + A^{n-2} + ... + A + I &= \frac{n}{2}U + \left(\frac{1+3+...+3^{n-1}}{2}\right)V \\ &= \frac{n}{2}U + \left(\frac{3^n-1}{4}\right)V. \end{aligned}$$

- It is well-known that

$$1 + p + p^2 + \ldots + p^k = \frac{p^{k+1}-1}{p-1}.$$

.

Finally, from (6) we have

$$\begin{aligned} X_n &= \left(\frac{1}{2}U + \frac{3^n}{2}V\right)X_0 + \left[\frac{n}{2}U + \left(\frac{3^n-1}{4}\right)V\right]C \\ &= \left(\frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \frac{3^n}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\right)\begin{pmatrix} 0 \\ -1 \end{pmatrix} + \\ &\quad \left[\frac{n}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \left(\frac{3^n-1}{4}\right)\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\right]\begin{pmatrix} -1 \\ 2 \end{pmatrix} \\ &= \begin{pmatrix} \dfrac{2n-3^n+1}{4} \\ \\ \dfrac{2n+3^n-5}{4} \end{pmatrix};\ n \geq 0. \end{aligned}$$

Thus,

$$\begin{cases} x_n = \dfrac{2n-3^n+1}{4} \\ \\ y_n = \dfrac{2n+3^n-5}{4} \end{cases};\ n \geq 0.$$

Problem (Homework)

Let $A \in \mathcal{M}_n(\mathbb{R})$. *Assume* $(A - I_2)^{-1}$ *exists, prove that*

$$A^{n-1} + A^{n-2} + ... + A + I = (A^n - I_2)(A - I_2)^{-1}.$$

Problem (Homework)

Solve the system of linear recurrence sequences:

$$\begin{cases} x_{n+1} = 3x_n - y_n + 1 \\ y_{n+1} = -x_n + 3y_n + 1 \end{cases} ; \quad (x_0, y_0) = (1, 2). \tag{8}$$

Thank you!