Minimal Polynomial

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We introduce here a second polynomial which is extracted from the characteristic polynomial of a square matrix.

Definition

Let A be a square matrix and let $p_A(x)$ be its characteristic polynomial. The **minimal polynomial** of A, denoted by $m_A(x)$, is a polynomial satisfying the following two properties:

1. $m_A(x) | p_A(x)$; i.e., $m_A(x)$ divides the characteristic polynomial $p_A(x)$.

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1. $m_A(x)|p_A(x)$; i.e., $m_A(x)$ divides the characteristic polynomial $p_A(x)$.
2. $m_A(A) = p_A(A) = 0$ (the zero matrix). That is, $m_A(x)$ satisfies Cayley-Hamilton Theorem as does $p_A(x)$.

Theorem

The eigenvalues of a matrix A are the roots of $m_A(x)$.

Proof.

Let λ be an eigenvalue of A and let x be its eigenvector. From the Euclidean division of $m_A(x)$ by $x - \lambda$, we obtain

$$m_A(x) = Q(x)(x - \lambda) + c,\ c \in \mathbb{R} \text{ and } Q \in \mathbb{R}[X].$$

It follows that

$$0 = m_A(A) = Q(A)(A - \lambda I) + cI.$$

If we apply this to the vector x, we get

$$0 = Q(A)(Ax - \lambda x) + cx.$$

Hence $cx = 0$. Since x is not zero, we get $c = 0$, and so $m_A(x) = Q(x)(x - \lambda)$. This means that λ is a root of $m_A(x)$. The proof is finished. □

Remark. The minimal polynomial of A is a polynomial satisfying the following three properties:

1. $m_A(x) | p_A(x)$,
2. $m_A(A) = p_A(A) = 0$ (the zero matrix),
3. For any $\lambda \in Sp(A) : m_A(\lambda) = 0$. This means that $p_A(x)$ and $m_A(x)$ have the same roots.

Example

Calculate the minimal polynomial of the matrices:

1. $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$,
2. $B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

Solution.

1. We can easily prove that $p_A(x) = (1-x)(3-x)$, and so $m_A(x) = p_A(x)$.
2. First, the characteristic polynomial is $p_A(x) = (x-1)^2$. Hence,

$$m_A(x) = (x-1) \text{ or } m_A(x) = (x-1)^2,$$

and since $A - I_2 \neq 0$, then $m_A(x) = p_A(x) = (x-1)^2$.

Example

Determine the minimal polynomials of the following matrices:

$$1)\ A = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$2)\ B = \begin{bmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 4 \end{bmatrix},$$

$$3)\ C = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}.$$

1) It is clear that $p_A(x) = x^3$. Then, $m_A(x) = x^3$ or x^2 or x. On the other hand, we have $m_A(x) = x^2$; since $A \neq 0$ and $A^2 = 0$.

2) Note that after computation, $p_B(x) = (x-3)^2(x-6)$. Since $p_B(x)$ and $m_B(x)$ having the same roots and $m_B(x)$ divides $p_B(x)$, then $m_B(x) = (x-3)(x-6)$ or $m_B(x) = (x-3)^2(x-6)$. But,

$$
\begin{aligned}
(B-3I_3)(B-6I_3) &= \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.
\end{aligned}
$$

It follows that $m_B(x) = (x-3)(x-6)$.

3) From simple computation, we get $p_C(x) = (x-1)^2$. Since $A - I_2 \neq 0$, then

$$m_C(x) = (x-1)^2 = p_C(x).$$

Corollary

Let $A \in \mathcal{M}_n(\mathbb{R})$ with $m_A(x) = (x-a)(x-b)$; $a, b \in \mathbb{R}$. Then A^n can be written in terms of A and I.

Proof.

The proof is by induction on n. Indeed, for $n = 1$, we have

$$A^1 = 1.A + 0.I.$$

Moreover, for $n = 2$, $A^2 = (a+b)A - abI$, since $m_A(A) = 0$. Assume that A^n can be written in terms of A and I, i.e., $A^n = a_n A + b_n I$. Therefore,

$$\begin{aligned} A^{n+1} &= AA^n = A(a_n A + b_n I) = a_n A^2 + b_n A \\ &= a_n((a+b)A - abI) + b_n A \\ &= ((a+b)a_n + b_n)A - aba_n I = f(A, I). \end{aligned}$$

This means that A^{n+1} can be written in terms of A and I. □

We will accept the following corollary without proof.

Corollary

The matrix A is diagonalizable if and only if the roots of $m_A(x)$ are simple.

Example

Let

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

Verify that A is diagonalizable.

Solution. After computation, we get

$$p_A(x) = (1+x)^2 (x-2).$$

This means that $m_A(x) = (1+x)(x-2)$ or $m_A(x) = (1+x)^2 (x-2)$. But,

$$\begin{aligned}(I+A)(A-2I) &= \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.\end{aligned}$$

Thus, $m_A(x) = (1+x)(x-2)$. It is clear that the roots of $m_A(x)$ are simple, and hence A is diagonalizable.

Example

Study the diagonalization of the matrix

$$A = \begin{pmatrix} a & 0 & 0 \\ 1 & a & 0 \\ 1 & 1 & a \end{pmatrix}, \text{ where } a \in \mathbb{R}.$$

Since A is a lower triangular matrix, then $p_A(x) = (x-a)^3$. Since $(A - aI) \neq 0$, then $m_A(x)$ can not be $(x-a)$. This means that the roots of $m_A(x)$ are not simple, and so A is not diagonalizable.

Example

Consider the matrix

$$A = \begin{pmatrix} a & b & b \\ b & a & b \\ b & b & a \end{pmatrix}.$$

Show that A is diagonalizable.

In fact, we have

$$A = a \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + b \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} = aI_3 + bB.$$

It suffices to prove that B is diagonalizable. After computation we obtain

$$m_B(x) = (x+1)(x-2),$$

and hence B is diagonalizable.

That is, B can be written in the form $B = PDP^{-1}$, from which it follows that

$$\begin{aligned} A &= aI_3 + bPDP^{-1} \\ &= P\left(aI_3 + bD\right)P^{-1} \\ &= PD'P^{-1}, \end{aligned}$$

where $D' = aI_3 + bD$ is diagonal, so A is diagonalizable.

Example

Consider the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

By computation, $m_A(x) = x(x-3)$. This means that A is diagonalizable since the roots of $m_A(x)$ are simple.

Ex 01. Find minimal polynomial of the matrix

$$A = \begin{pmatrix} 2 & 2 & -5 \\ 3 & 7 & -15 \\ 1 & 2 & -4 \end{pmatrix}.$$

Deduce that A is diagonalizable. **Ans.**

$$p_A(x) = (x-3)(x-1)^2 \text{ and } m_A(x) = (x-3)(x-1).$$

Ex 02. Consider the matrix

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Calculate the minimal polynomial of A. **Ans.** $m_A(x) = x(x-2)$.

Ex 03. Calculate the characteristic polynomial of the matrix

$$\begin{pmatrix} 4 & 1 & 1 & 1 \\ 1 & 4 & 1 & 1 \\ 1 & 1 & 4 & 1 \\ 1 & 1 & 1 & 4 \end{pmatrix}.$$

Deduce its minimal polynomial. **Ans.**

$$p_A(x) = (3-x)^3 (7-x) \text{ and } m_A(x) = (3-x)(7-x).$$

Ex 04. Calculate the minimal polynomial of the following matrices

$$\begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix}, \begin{pmatrix} 3 & 0 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix}, \begin{pmatrix} 3 & 0 & 0 \\ 1 & 3 & 0 \\ 0 & 1 & 4 \end{pmatrix}, \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 1 & 0 & 4 \end{pmatrix}.$$

Ex 05. Verify that all matrices of the forn

$$A = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}; \alpha \in \mathbb{R}^*$$

are not diagonalizable.

Ex 06. Calculate the minimal polynomial of the matrix

$$A = \begin{pmatrix} \lambda & & & & \\ 1 & \lambda & & & \\ & \ddots & \ddots & & \\ & & 1 & \lambda & \\ & & & 1 & \lambda \end{pmatrix}, \lambda \in \mathbb{R}.$$

Is it diagonalizable ?

Ex 07. Let $A \in \mathcal{M}_3(\mathbb{R})$ given by

$$A = \begin{pmatrix} 3 & 2 & -2 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

a) Determine the characteristic polynomial of A.
b) Determin the minimal polynomial of A.
c) Is the matrix A diagonalizable?

Ex 08. Find all the matrices $A \in \mathcal{M}_2(\mathbb{C})$ whose minimal polynomial is $x^2 + 1$.

Ex 09. Calculate the minimal polynomial of the matrix:

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

Ans. $m_A(x) = x(x-8)$.

Ex 10. Calculate the characteristic polynomial and its minimal polynomial of the matrix

$$A=\begin{pmatrix} 2 & 5 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 4 & 2 & 0 \\ 0 & 0 & 3 & 5 & 0 \\ 0 & 0 & 0 & 0 & 7 \end{pmatrix}.$$

Ans. $p_A(x) = (x-2)^3 (x-7)^2$ and $m_A(x) = (x-2)^2 (x-7)$.