Nilpotent Matrices
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October 2024

Nilpotent matrices

Definition

A **nilpotent** matrix is a square matrix N such that $N^k = 0$ for some positive integer k.

In other words, a square matrix N is said to be **nilpotent** if there exists a positive integer k such that $N^k = 0$. The smallest such k is called the **index** of N.

Example

The matrix

$$N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

is nilpotent with index 2, since $N^2 = 0$.

Theorem

Let N be a nilpotent matrix. Then

- $Sp(N) = \{0\}$,
- $I - N$ *is invertible.*

Proof.

Assume that $N^k = 0$ and $N^{k-1} \neq 0$ for some $k \geq 1$.

- Let (λ, x) be an eigenpair of N, that is, $Nx = \lambda x$ and $x \neq 0$. It follows that $\lambda^k x = N^k x = 0$, and hence $\lambda = 0$.
- Let $x \in \mathbb{R}^n$ such that $(I - N)x = 0$. Therefore, $Nx = x$, form which it follows that $N^k x = N^{k-1} x = 0$. Since $N^{k-1} \neq 0$, then $x = 0$. Thus, $I - N$ is invertible.

The proof is finished. □

Theorem

Let A be a nonzero nilpotent matrix. Then A is non-diagonalizable.

Proof.

Assume, by the way of contradiction that A is diagonalizable, that is, $A = PDP^{-1}$ for some invertible matrix $P = 0$. Since A is nilpotent, there exists a positive integer k such that $A^k = 0$. It follows that $D = P^{-1}AP$, and so

$$D^k = P^{-1}A^kP = 0.$$

Since D is diagonal, then $D = 0$. This means that $A = 0$, a contradiction. □

Theorem

Any strictely triangular matrix is nilpotent.

Proof.

Setting

$$A = \begin{pmatrix} \mathbf{0} & 0 & \cdots & 0 \\ a_{21} & \mathbf{0} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & \mathbf{0} \end{pmatrix}.$$

Since $p_A(x) = x^n$. By Cayley-Hamilton theorem, we have $A^n = 0$. That is, there exists a positive integer k (with $k \leq n$) such that $A^k = 0$, and hence A is nilpotent. □

Example

Determine the index of the following matrix:

$$N = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

It is clear that

$$N^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } N^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Since $N^3 = 0$ and $N^2 \neq 0$, then N is nilpotent of index $k = 3$.

Remark. The product of two non-zero matrices can be zero. Indeed, for a matrix $A \in \mathcal{M}_n(\mathbb{R})$, we have

$$A^2 = 0 \not\Rightarrow A = 0.$$

For example, if $A = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \neq 0$ we see that

$$A^2 = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

But, $A \neq 0$.

Example

Consider the matrix

$$A = \begin{pmatrix} 3 & 9 & -9 \\ 2 & 0 & 0 \\ 3 & 3 & -3 \end{pmatrix}$$

Show that A is nilpotent.

Solution. First, we determine the characteristic polynomial of A.

$$\begin{aligned} p_A(x) &= \begin{vmatrix} 3-x & 9 & -9 \\ 2 & -x & 0 \\ 3 & 3 & -3-x \end{vmatrix} = \begin{vmatrix} 3-x & 0 & -9 \\ 2 & -x & 0 \\ 3 & -x & -3-x \end{vmatrix} \\ &= -x \begin{vmatrix} 3-x & 0 & -9 \\ 2 & 1 & 0 \\ 3 & 1 & -3-x \end{vmatrix} \\ &= -x^3. \end{aligned}$$

By Cayley-Hamilton theorem, $A^3 = 0$. Since $A^2 \neq 0$, then A is nilpotente of index 3.

Theorem

Let N be a nilpotent matrix of index k and let $x \in \mathbb{R}^n$ be a nonzero vector such that $N^{k-1}x \neq 0$. The family $\left\{ Ix, Nx, N^2x, ..., N^{k-1}x \right\}$ is free.

Proof.

Let $(\alpha_i)_{0 \leq i \leq k-1} \in \mathbb{R}$ such that $\sum_{i=0}^{k-1} \alpha_i N^i x = 0$, from which it follows that

$$\begin{cases} \alpha_0 N^{k-1}x + \alpha_1 N^k x + ... + \alpha_{k-1} N^{2k-2}x = 0 \\ \alpha_0 N^{k-2}x + \alpha_1 N^{k-1}x + ... + \alpha_{k-1} N^{2k-3}x = 0 \\ \vdots \\ \alpha_0 Nx + \alpha_1 N^2 x + ... + \alpha_{k-1} N^k x = 0 \\ \alpha_0 Ix + \alpha_1 Nx + ... + \alpha_{k-1} N^{k-1}x = 0 \end{cases} \Rightarrow \begin{cases} \alpha_0 N^{k-1}x = 0 \\ \alpha_1 N^{k-1}x \\ \vdots \\ \alpha_{k-2} N^{k-1}x = 0 \\ \alpha_{k-1} N^{k-1}x = 0 \end{cases}$$

Since $N^{k-1}x \neq 0$, we conclude that $\alpha_0 = \alpha_1 = ... = \alpha_{k-1} = 0$. This completes the proof. □

Problems

Ex 01. Let $A \in \mathcal{M}_n(\mathbb{R})$ be a nilpotent matrix. Prove that

$$\det(A + I_n) = 1.$$

Ex 02. Verify that

$$A = \begin{pmatrix} 1 & -3 & -4 \\ -1 & 3 & 4 \\ 1 & -3 & -4 \end{pmatrix}$$

is nilpotent.

Ex 03. Let

$$A = \begin{pmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{pmatrix}$$

Calculate A^3. Deduce?

Ex 04. Prove the following theorem:

Theorem

If N is nilpotent, then $I + N$ and $I - N$ are both invertible, where I denotes the identity matrix.

Ex 05. Prove the following implication:

$$A \sim 2A \Rightarrow A \text{ is nilpotent over } \mathbb{R}.$$

Ex 06. Why the study of nilpotent matrices in **important**?