The sauqre root of a diagonalizable matrix

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Lemma

Let

$$D = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}, \textit{ where } \lambda_i > 0 \ (1 \leq i \leq n).$$

Then

$$\sqrt{D} = \begin{pmatrix} \sqrt{\lambda_1} & & & \\ & \sqrt{\lambda_2} & & \\ & & \ddots & \\ & & & \sqrt{\lambda_n} \end{pmatrix}.$$

Proof.

It is trivial by computation that $\sqrt{D}\sqrt{D} = D$. □

Problem

Let

$$D = \begin{pmatrix} \mathbf{1} & 0 & 0 & 0 \\ 0 & \mathbf{2} & 0 & 0 \\ 0 & 0 & \mathbf{3} & 0 \\ 0 & 0 & 0 & \mathbf{4} \end{pmatrix}$$

Compute $\ln D$, $\cos D$, e^D, D^k *and* $\sqrt{D}$.

Remark. If $D = diag\left\{\lambda_1, \lambda_2, ..., \lambda_n\right\}$ with some $\lambda_i < 0$, then $\sqrt{D} \in \mathcal{M}_n(\mathbb{C})$.

Proposition

Let $A \in \mathcal{M}_n(\mathbb{R})$ be a diagonalizable matrix with $Sp\left(A\right) \subset \mathbb{R}_+$. Then $\sqrt{A} \in \mathcal{M}_n(\mathbb{R})$.

Proof.

Assume that $A = PDP^{-1}$, where $Sp\left(D\right) \subset \mathbb{R}_+$. We put

$$H = P\sqrt{D}P^{-1} \in \mathcal{M}_n(\mathbb{R}).$$

Since $\sqrt{D}\sqrt{D} = D$, it follows that

$$H^2 = \left(P\sqrt{D}P^{-1}\right)\left(P\sqrt{D}P^{-1}\right) = PDP^{-1} = A.$$

Thus, $\sqrt{A} = H$. □

Example

Consider the matrix

$$A = \begin{pmatrix} 11 & -5 & 5 \\ -5 & 3 & -3 \\ 5 & -3 & 3 \end{pmatrix}.$$

Calculate $\sqrt{A}$.

Solution. After simple computation, the eigenpairs of A are:

$$\begin{cases} \lambda_1 = 0, \; E_{\lambda_1} = Vect\left\{(0,1,1)\right\}, \\ \lambda_2 = 1, \; E_{\lambda_2} = Vect\left\{(-1,-1,1)\right\}, \\ \lambda_3 = 16, \; E_{\lambda_3} = Vect\left\{(2,-1,1)\right\}. \end{cases}$$

Further, we see that

$$P = \begin{pmatrix} 0 & -1 & 2 \\ 1 & -1 & -1 \\ 1 & 1 & 1 \end{pmatrix}, \; D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 16 \end{pmatrix} \text{ and } P^{-1} = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{6} & \frac{1}{6} \end{pmatrix}.$$

Which gives

$$\begin{aligned}
\sqrt{A} &= P\sqrt{D}P^{-1} \\
&= \begin{pmatrix} 0 & -1 & 2 \\ 1 & -1 & -1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{0} & 0 & 0 \\ 0 & \sqrt{1} & 0 \\ 0 & 0 & \sqrt{16} \end{pmatrix} \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{6} & \frac{1}{6} \end{pmatrix} \\
&= \begin{pmatrix} 3 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}.
\end{aligned}$$

Definition

Let $A = PDP^{-1}$ be a diagonalizable matrix whose eigenvalues are given by the diagonal matrix

$$D = diag\left\{\lambda_1, \lambda_2, ..., \lambda_n\right\}.$$

For any function $f(x)$ defined at the points $(\lambda_i)_{1\leq i\leq n}$, we have

$$f(A) = P \cdot f(D) \cdot P^{-1} = P \begin{pmatrix} f(\lambda_1) & & & \\ & f(\lambda_2) & & \\ & & \ddots & \\ & & & f(\lambda_n) \end{pmatrix} P^{-1}.$$

Example

Assume that Let $A = PDP^{-1}$. We have

$$\begin{cases} \text{For } f(x) = x^k \Rightarrow f(A) = A^k = P \cdot D^k \cdot P^{-1} \quad \text{for } k \geq 0 \\ \text{For } f(x) = \sqrt{x} \Rightarrow f(A) = \sqrt{A} = P \cdot \sqrt{D} \cdot P^{-1} \\ \text{For } f(x) = \cos x \Rightarrow f(A) = \cos A = P \cdot \cos D \cdot P^{-1} \\ \text{For } f(x) = e^x \Rightarrow f(A) = e^A = P \cdot e^D \cdot P^{-1} \\ \text{For } f(x) = \ln x \Rightarrow f(A) = \ln A = P \cdot \ln D \cdot P^{-1} \\ \ldots \\ \text{and so on.} \end{cases}$$

Problems.

Ex 01. Let M be a real n by n matrix. We denote by $\cos M$ the real part of e^{iM} and $\sin M$ its imaginary part. Show that $\cos M$ and $\sin M$ commute and that

$$(\cos M)^2 + (\sin M)^2 = I_n.$$

Let θ be a real number. Calculate

$$\cos\begin{pmatrix} \theta & 1 \\ 0 & \theta \end{pmatrix} \text{ and } \sin\begin{pmatrix} \theta & 1 \\ 0 & \theta \end{pmatrix}.$$

Ex 02. Let

$$A = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \in \mathcal{M}_2(\mathbb{C}).$$

Calculate $\sqrt{A}$.