System of recurrence sequences. Part II

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System of differential equations. Part I

Definition

Define the **linear system of differential equations** $(x_1'(t), x_2'(t), ..., x_n'(t))$ by

$$\begin{cases} x_1'(t) = a_{11}x_1(t) + a_{12}x_2(t) + ... + a_{1n}x_n(t) + f_1(t) \\ x_2'(t) = a_{21}x_1(t) + a_{22}x_2(t) + ... + a_{2n}x_n(t) + f_2(t) \\ \vdots \\ x_n'(t) = a_{n1}x_1(t) + a_{n2}x_2(t) + ... + a_{nn}x_n(t) + f_n(t), \end{cases} \quad (1)$$

where $a_{ij} \in \mathbb{R}$. The unknowns are the functions $x_1(t), x_2(t), ..., x_n(t)$ which are derivable and $f_i(t)$ are some given functions.

The system is called **homogeneous** if all $f_i = 0$, otherwise it is called **non-homogeneous.**

Matrix Notation

A non-homogeneous system of linear equations (1) is written as the equivalent vector-matrix system

$$X'(t) = A \cdot X(t) + f(t),$$

where

$$X(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_2(t) \end{pmatrix}, \; A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}, f = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_2 \end{pmatrix}$$

In this section, we consider only homogeneous systems: We wish to solve the system

$$X' = AX. \tag{2}$$

There are two cases:

Case 1. Assume that A is diagonalizable. Then there exists an invertible matrix P such that $A = PDP^{-1}$, where D is diagonal. Thus,

$$\begin{cases} X' = PDP^{-1}X = PY' \\ Y' = DY \\ Y = P^{-1}X. \end{cases}$$

The system (2) becomes

$$Y' = DY,$$

which is easier to solve since D is diagonal. Then after, we solve the equation $Y = P^{-1}X$, that is, $X = PY$.

Example

Solve the system of differential equations:

$$X' = AX,\ A = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}, \text{ where } X(0) = \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$

Solution. At first, the eigenvalues of A are $\lambda_1 = -1$ and $\lambda_2 = 4$. The corresponding eigenvectors are $v_1 = (1, -1)$ and $v_2 = (2, 3)$. Thus, we have

$$D = \begin{pmatrix} -1 & 0 \\ 0 & 4 \end{pmatrix}, P = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}.$$

We put $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $Y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$. It follows that

$$Y' = DY \Leftrightarrow \begin{cases} y_1' = -y_1 \\ y_2' = 4y_2 \end{cases} \Leftrightarrow \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} c_1 e^{-t} \\ c_2 e^{4t} \end{pmatrix},$$

and hence

$$X = PY = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} c_1 e^{-t} \\ c_2 e^{4t} \end{pmatrix} = \begin{pmatrix} c_1 e^{-t} + 2c_2 e^{4t} \\ -c_1 e^{-t} + 3c_2 e^{4t} \end{pmatrix}.$$

Since $X(0) = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$, then

$$\begin{cases} c_1 + 2c_2 = 3 \\ -c_1 + 3c_2 = 2 \end{cases} \Rightarrow c_1 = c_2 = 1.$$

Thus is,

$$\begin{cases} x_1 = e^{-t} + 2e^{4t} \\ x_2 = -e^{-t} + 3e^{4t}. \end{cases}$$

We present another method to solve the system $X' = AX$, where A is diagonalizable.

Theorem

Let $A \in \mathcal{M}_n(\mathbb{R})$ be a diagonalizable matrix and let

$$P = \begin{bmatrix} X_1 & X_n & ... & X_n \end{bmatrix}$$

be the invertible matrix formed by n eigenvectors $X_1, X_2, ..., X_n$ of A. Then the system $X' = AX$ has a unique solution given by

$$X(t) = c_1 e^{\lambda_1 t} X_1 + c_2 e^{\lambda_2 t} X_2 + ... + c_n e^{\lambda_n t} X_n, \tag{3}$$

where $c_1, c_2, ..., c_n \in \mathbb{R}^n$ and $\lambda_1, \lambda_2, ..., \lambda_n$ are the eigenvalues of A.

Proof.

It is clear that $X' = AX$ implies

$$X(t) = e^{At}.\xi, \text{ where } \xi \in \mathcal{M}_{n,1}(\mathbb{R}).$$

Since A is diagonalizable, then

$$X(t) = Pe^{Dt}P^{-1} = P \begin{pmatrix} e^{\lambda_1 t} & & & \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ & & & e^{\lambda_n t} \end{pmatrix} P^{-1}.\xi \tag{4}$$

Setting

$$P^{-1}.\xi = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = C.$$

□

Proof.

It follows from (4) that

$$\begin{aligned} X(t) &= \begin{bmatrix} X_1 & X_n & \ldots & X_n \end{bmatrix} \begin{pmatrix} e^{\lambda_1 t} & & & \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ & & & e^{\lambda_n t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} \\ &= \begin{bmatrix} e^{\lambda_1 t} X_1 & e^{\lambda_2 t} X_n & \ldots & e^{\lambda_n t} X_n \end{bmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} \\ &= c_1 e^{\lambda_1 t} X_1 + c_2 e^{\lambda_2 t} X_2 + \ldots + c_n e^{\lambda_n t} X_n. \end{aligned}$$

Therefore,

$$X(t) = c_1 e^{\lambda_1 t} X_1 + c_2 e^{\lambda_2 t} X_2 + \ldots + c_n e^{\lambda_n t} X_n. \tag{5}$$

This completes the proof. □

Example

Solve the system of differential equations:

$$X' = AX,\ A = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}, \text{ where } X(0) = \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$

Solution. After the computation of the eigenvalues and eigenvectors of the matrix A. It follows from (5) that

$$X\left(t\right)=c_1e^{-t}\begin{pmatrix}1\\-1\end{pmatrix}+c_2e^{4t}\begin{pmatrix}2\\3\end{pmatrix}.$$

Hence

$$\begin{cases} x\left(t\right)=c_1e^{-t}+2c_2e^{4t}, \\ y\left(t\right)=-c_1e^{-t}+3c_2e^{4t}. \end{cases}$$

Since $X\left(0\right)=\begin{pmatrix}3\\2\end{pmatrix}$, then

$$\begin{cases} x_1=e^{-t}+2e^{4t} \\ x_2=-e^{-t}+3e^{4t}. \end{cases}$$

Example

Solve the system of differential equations:

$$X' = AX \text{ with } A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}.$$

Solution. Simple computation we get

$$\begin{cases} \lambda_1 = 1, \ v_1 = (-1, 1, 1) \\ \lambda_1 = 2, \ v_2 = (0, 1, 0) \text{ and } v_3 = (0, 0, 1). \end{cases}$$

The matrix A is diagonalizable, and by (3) we obtain

$$X(t) = c_1 e^t \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_3 e^{2t} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

where c_1, c_2, c_3 are constants. That is,

$$\begin{cases} x(t) = -c_1 e^t \\ y(t) = c_1 e^t + c_2 e^{2t} \\ z(t) = c_1 e^t + c_3 e^{2t}. \end{cases}$$

Remark. In another way, which is very long and based on the calculation of P and P^{-1} with $A = PDP^{-1}$. From which it follows that

$$e^{At} = Pe^{Dt}P^{-1}. \tag{6}$$

Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$. The solution of the differential system $X' = AX$ is $X(t) = e^{At}.C$, where C is an arbitrarily constant. Since $X(0) = \begin{pmatrix} 3 & 2 \end{pmatrix}^t$, then $C = \begin{pmatrix} 3 & 2 \end{pmatrix}^t$. Therefore,

$$X(t) = e^{At} \cdot X(0). \tag{7}$$

Simple computation gives $P = \begin{pmatrix} 2 & 1 \\ 3 & -1 \end{pmatrix}$ and $P^{-1} = \begin{pmatrix} \frac{1}{5} & \frac{1}{5} \\ \frac{3}{5} & \frac{-2}{5} \end{pmatrix}$.

Hence

$$\begin{aligned} X(t) &= e^{At} \cdot C_0 = \begin{pmatrix} 2 & 1 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} e^{4t} & 0 \\ 0 & e^{-t} \end{pmatrix} \begin{pmatrix} \frac{1}{5} & \frac{1}{5} \\ \frac{3}{5} & \frac{-2}{5} \end{pmatrix} \begin{pmatrix} 3 \\ 2 \end{pmatrix} \\ &= \begin{pmatrix} 2e^{4t} + e^{-t} \\ 3e^{4t} - e^{-t} \end{pmatrix}. \end{aligned}$$

Problems

Ex 01. Calculate e^{At} for each $t \in \mathbb{R}$, where

$$A = \begin{pmatrix} 0 & -1 & 1 \\ 0 & 0 & 1 \\ -1 & 0 & 1 \end{pmatrix}.$$

Deduce the general solution of the system of differential equations:

$$\begin{cases} p' = -q + r \\ q' = r \\ r' = -p + r \end{cases}$$

Ex 02. Solve the system of differential equations:

$$\begin{cases} x'(t) = y(t) \\ y'(t) = z(t) \\ z'(t) = w(t) \\ w'(t) = x(t) \end{cases}$$

Ex 03. Solve the system of differential equations $X' = A \cdot X$, where

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$