System of linear differential equations. Part II

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Consider the system of differential equations:

$$\begin{cases} x_1' = a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \\ x_2' = a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \\ \vdots \\ x_n' = a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n, \end{cases} \tag{1}$$

which is written by the following equivalent vector-matrix system:

$$X' = A \cdot X,$$

where the matrix A is **non-diagonalizable**. In this case, the general solution of (1) can be given by:

$$X(t) = e^{tA}c,$$

where $c = \begin{pmatrix} c_1 & c_2 & \ldots & c_n \end{pmatrix}^t$ is a constant.

In this Algebra III, we only consider certain cases. For example, $A \in \mathcal{M}_n(\mathbb{R})$ but has a unique eigenvalue or when $A \in \mathcal{M}_n(\mathbb{R})$ with $n \leq 4$. The situation is particularly simple whenever $A \in \mathcal{M}_2(\mathbb{R})$.

Corollary

Let $A \in \mathcal{M}_n(\mathbb{R})$ be a square matrix having a unique eigenvalue, say λ. Then

$$e^{tA} = e^{\lambda t} \sum_{k=0}^{n-1} (A - \lambda I_n)^k \frac{t^k}{k!}.$$

Proof.

We first have $p_A(x) = (x-\lambda)^n$ since A has a unique eigenvalue λ. We have

$$
\begin{aligned}
e^{tA} &= e^{\lambda t I_n + t(A-\lambda I_n)} && (2)\\
&= e^{\lambda t I_n} e^{t(A-\lambda I_n)} \quad \text{(because } \lambda t I_n \text{ and } t(A-\lambda I_n) \text{ commute)}\\
&= e^{\lambda t} e^{t(A-\lambda I_n)} \quad \text{(because } e^{\alpha I_n} B = e^{\alpha} B \text{ for any } B \in \mathcal{M}_n(\mathbb{R}) \text{ and } \alpha \in \mathbb{R}\text{)}\\
&= e^{\lambda t} \sum_{k=0}^{+\infty} (A-\lambda I_n)^k \frac{t^k}{k!} && (3)\\
&= e^{\lambda t} \sum_{k=0}^{n-1} (A-\lambda I_n)^k \frac{t^k}{k!},
\end{aligned}
$$

where $\sum_{k=n}^{+\infty} (A-\lambda I_n)^k = 0$; this is obtained by Cayley-Hamilton theorem since $p_A(A) = (A-\lambda I_n)^n = 0$. □

Remark. In particular, by Corollary 1, if $A \in \mathcal{M}_2(\mathbb{R})$ with $Sp(A) = \{\lambda\}$ then

$$e^{tA} = e^{\lambda t}\left\{I_2 + (A - \lambda I_2)\, t\right\}. \tag{4}$$

If $A \in \mathcal{M}_3(\mathbb{R})$ with $Sp(A) = \{\lambda\}$ then

$$e^{tA} = e^{\lambda t}\left\{I_3 + (A - \lambda I_3)\, t + \frac{1}{2}(A - \lambda I_3)^2\, t^2\right\}. \tag{5}$$

Example

Solve the system of différentiel equations

$$\begin{cases} x' = 2x + y \\ \\ y' = 2y \end{cases} \tag{6}$$

Solution. Let A be the matrix of (6), i.e.,

$$A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}.$$

From (4), we have

$$\begin{aligned} e^{tA} &= e^{2t}\left\{I_2 + (A - 2I_2)\,t\right\} \\ &= e^{2t}\left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \left(\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} - 2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) t\right\} \\ &= \begin{pmatrix} e^{2t} & te^{2t} \\ 0 & e^{2t} \end{pmatrix}. \end{aligned}$$

Thus, the solution of (6) is given by

$$X(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} e^{2t} & te^{2t} \\ 0 & e^{2t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} c_1 e^{2t} + tc_2 e^{2t} \\ c_2 e^{2t} \end{pmatrix},$$

where c_1, c_2 are constant.

Example

Solve the system of differential equations:

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} -4 & 1 & 1 \\ 1 & -1 & -2 \\ -2 & 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Solution: The characteristic polynomil of A is given by

$$p_A(x) = (x+2)^3.$$

This means that A has a unique eigenvalu, $\lambda = -2$. From (5), we obtain

$$e^{tA} = e^{-2t}\left\{ I_3 + (A+2I_3)\,t + \frac{1}{2}(A+2I_3)^2\,t^2 \right\},$$

where

$$A+2I_3 = \begin{pmatrix} -2 & 1 & 1 \\ 1 & 1 & -2 \\ -2 & 1 & 1 \end{pmatrix} \text{ and } A+2I_3 = \begin{pmatrix} 3 & 0 & -3 \\ 3 & 0 & -3 \\ 3 & 0 & -3 \end{pmatrix}.$$

Then

$$
\begin{aligned}
e^{tA} &= e^{-2t}\left\{I_3 + (A+2I_3)\,t + \frac{1}{2}(A+2I_3)^2\,t^2\right\} \\
&= e^{-2t}\left\{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} -2 & 1 & 1 \\ 1 & 1 & -2 \\ -2 & 1 & 1 \end{pmatrix} t + \frac{1}{2}\begin{pmatrix} 3 & 0 & -3 \\ 3 & 0 & -3 \\ 3 & 0 & -3 \end{pmatrix} t^2\right\} \\
&= e^{-2t}\begin{pmatrix} \frac{3}{2}t^2 - 2t + 1 & t & t - \frac{3}{2}t^2 \\ \frac{3}{2}t^2 + t & t+1 & -\frac{3}{2}t^2 - 2t \\ \frac{3}{2}t^2 - 2t & t & -\frac{3}{2}t^2 + t + 1 \end{pmatrix}.
\end{aligned}
$$

Problem

Solve the system of differential equations

$$X' = A \cdot X, \text{ where } A = \begin{pmatrix} 3 & -18 \\ 2 & -9 \end{pmatrix}.$$

Problem

Solve the system of differential equations

$$\begin{pmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \\ x_4'(t) \end{pmatrix} = \begin{pmatrix} 1 & 2 & 1 & 3 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{pmatrix}.$$

Theorem

Let $A \in \mathcal{M}_3(\mathbb{R})$. If A has two distinct eigenvalues λ and μ (where λ has multiplicity 2), then

$$e^{tA} = e^{\lambda t}\left(I + t\left(A - \lambda I\right)\right) + \frac{e^{\mu t} - e^{\lambda t}}{\left(\mu - \lambda\right)^2}\left(A - \lambda I\right)^2 - \frac{te^{\lambda t}}{\mu - \lambda}\left(A - \lambda I\right)^2. \qquad (7)$$

Proof.

From (2) and (3), we have

$$\begin{aligned} e^{tA} &= e^{\lambda t} \sum_{k=0}^{+\infty} (A-\lambda I)^k \frac{t^k}{k!} \\ &= e^{\lambda t}(I+(A-\lambda I)) + e^{\lambda t} \sum_{k=2}^{+\infty} (A-\lambda I)^k \frac{t^k}{k!} \\ &= e^{\lambda t}(I+(A-\lambda I)) + e^{\lambda t} \sum_{r=0}^{+\infty} (A-\lambda I)^{2+r} \frac{t^{2+r}}{(2+r)!} \qquad (8) \end{aligned}$$

□

Proof.

Now, let $p_A(x) = (x-\lambda)^2 (x-\mu)$ be the characteristic polynomial of A. First, we note that

$$A - \mu I = (A - \lambda I_n) - (\mu - \lambda) I.$$

By Cayley-Hamilton theorem, we get

$$0 = (A - \lambda I)^2 (A - \mu I) = (A - \lambda I)^3 - (\mu - \lambda)(A - \lambda I)^2,$$

from which is follows that

$$(A - \lambda I)^3 = (\mu - \lambda)(A - \lambda I)^2.$$

By induction, for every $r \geq 1$,

$$(A - \lambda I)^{2+r} = (\mu - \lambda)^r (A - \lambda I)^2.$$

□

Proof.

It follows from (8) that

$$\begin{aligned}\sum_{r=0}^{+\infty} (A-\lambda I)^{2+r} \frac{t^{2+r}}{(2+r)!} &= \sum_{r=0}^{+\infty} (\mu-\lambda)^{r} \frac{t^{2+r}}{(2+r)!}{}^{r} (A-\lambda I)^{2} \\ &= \frac{1}{(\mu-\lambda)^{2}} \sum_{r=0}^{+\infty} \frac{t^{k}}{k!} (\mu-\lambda)^{k} (A-\lambda I)^{2}.\end{aligned}$$

Finally, we obtain

$$\begin{aligned}e^{tA} &= e^{\lambda t} (I + (A-\lambda I)) + \frac{e^{\lambda t}}{(\mu-\lambda)^{2}} \left\{ e^{(\mu-\lambda)t} - 1 - (\mu-\lambda)\, t \right\} (A-\lambda I)^{2} \\ &= e^{\lambda t} (I + t\,(A-\lambda I)) + \frac{e^{\mu t}-e^{\lambda t}}{(\mu-\lambda)^{2}} (A-\lambda I)^{2} - \frac{te^{\lambda t}}{\mu-\lambda} (A-\lambda I)^{2}.\end{aligned}$$

This completes the proof. □

Example

Solve the system of differential equations

$$\begin{pmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{pmatrix} = \begin{pmatrix} 2 & -1 & 2 \\ 10 & -5 & 7 \\ 4 & -2 & 2 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix}.$$

Solution. We first find the characteristic polynomial of A. By computation, $p_A(x) = x^2(x+1)$. This means that A has two eigenvalues $\lambda = 0$ (with multiplicity 2) and $\mu = -1$ (simple). It follows from (7) that

$$e^{At} = I_3 + tA + \left(t + e^{-t} - 1\right) A^2.$$

Simple computation we obtain

$$e^{At} = \begin{pmatrix} 4t + \frac{2}{e^t} - 1 & 1 - \frac{1}{e^t} - 2t & 3t + \frac{1}{e^t} - 1 \\ 8t - \frac{2}{e^t} + 2 & \frac{1}{e^t} - 4t & 6t - \frac{1}{e^t} + 1 \\ 4 - \frac{4}{e^t} & \frac{2}{e^t} - 2 & 3 - \frac{2}{e^t} \end{pmatrix}.$$