On the powers of a square matrix
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On the powers of a square matrix. Part I

Here, we propose a method for computing the powers of a strictly triangular matrix. That is, a matrix of the form

$$\begin{pmatrix} \mathbf{0} & \times & \times & \times \\ 0 & \mathbf{0} & \times & \times \\ \vdots & \vdots & \ddots & \times \\ 0 & 0 & \cdots & \mathbf{0} \end{pmatrix} \text{ or } \begin{pmatrix} \mathbf{0} & 0 & \cdots & 0 \\ \times & \mathbf{0} & \cdots & 0 \\ \times & \times & \ddots & \vdots \\ \times & \times & \times & \mathbf{0} \end{pmatrix}.$$

Let us choose the following example:

Example

Let

$$A = \begin{pmatrix} a & b & c \\ 0 & a & b \\ 0 & 0 & a \end{pmatrix}, \text{ where } a, b, c \in \mathbb{R}.$$

Find A^n for $n \geq 0$.

Solution. Setting

$$A = \begin{pmatrix} a & b & c \\ 0 & a & b \\ 0 & 0 & a \end{pmatrix} = \underbrace{\begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix}}_{D} + \underbrace{\begin{pmatrix} 0 & b & c \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix}}_{N}.$$

It is clear that N is nilpotent of index $k = 3$ (that is $N^3 = 0$). Moreover, $DN = ND$. By Binomial formula, we have

$$A^n = (D + N)^n = C_n^0 D^n + C_n^1 D^{n-1} N + C_n^2 D^{n-2} N^2,$$

where

$$C_n^i = \frac{n!}{i!\,(n-i)!}$$

and

$$N^2 = \begin{pmatrix} 0 & 0 & b^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

That is,

$$A^n = D^n + nD^{n-1}N + \frac{n\,(n-1)}{2}D^{n-2}N^2.$$

Problem

Let

$$J_n = \begin{pmatrix} 0 & \mathbf{1} & & & \\ & 0 & \mathbf{1} & & \\ & & \ddots & \ddots & \\ & & & 0 & \mathbf{1} \\ & & & & 0 \end{pmatrix}$$

Prove that $J_n^{n-1} \neq 0$ *and* $J_n^n = 0$. *That is,* J_n *is nilpotent with index* n.

For example, we have

$$J_2 = \begin{pmatrix} 0 & \mathbf{1} \\ 0 & 0 \end{pmatrix}, J_3 = \begin{pmatrix} 0 & \mathbf{1} & 0 \\ 0 & 0 & \mathbf{1} \\ 0 & 0 & 0 \end{pmatrix}, J_4 = \begin{pmatrix} 0 & \mathbf{1} & 0 & 0 \\ 0 & 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 & \mathbf{1} \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

and so on. Here, show that $J_2^2 = 0$, $J_3^3 = 0$ and $J_4^4 = 0$ but $J_3^2 \neq 0$ and $J_4^3 \neq 0$.