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Algebra 3

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- The ring of polynomials
- Square matrices
- Eigenvalues and Eigenvectors, eigenspaces,...
- Similar matrices
- Similarity and diagonalizability
- System of linear recurrence sequences
- System of linear differential equations
  - * Finding the power of square matrices
  - * Finding the exponential of square matrices
- Non-diagonalizable matrices
- Minimal polynomial
- Nilpotent matrices
- trigonalization of square matrices
- Jordan Canonical form
- ...

Basic expressions in Algebra 3.

- Consider the matrix A given by
- the matrix A is nonsingular since $\det(A) \neq 0$.
- $\det(A) \equiv$ the determinant of A.
- We find the characteristic polynomial of the matrix A :
- Then by the hypothesis, we have
- According to the Cayley-Hamilton's theorem we have
- Let $A = (a_{ij})$ be an $n \times n$ matrix.
- Let A be a square matrix.
- From definition, we obtain
- By definition, we have
- By using Cayley-Hamilton theorem, we get
- Hence, therefore, Thus, it follows that
- which gives, and so, and hence, and therefore
- where

- We Compute the characteristic polynomial of the matrix A :
- In fact, we have
- Thus, we have
- Consider the following Counterexample :
- Let A be an $n \times n$ matrix and λ is an eigenvalue of A.
- The Subspace E_λ is Called the eigenspace of λ.
- Let A be a Square matrix of size n.
- A Scalar λ is an eigenvalue of A if and only if $\det(A - \lambda I) = 0$, where I denotes the identity matrix.
- By definition, λ is an eigenvalue of A if and only if, for some nonzero x, we have

$$Ax = \lambda x = \lambda \cdot I \cdot x$$

$$\Leftrightarrow (A - \lambda I) \cdot x = 0$$

$$\Leftrightarrow \det(A - \lambda I) = 0.$$

- there are two Cases, we distinguish two Cases

- Let A be a square matrix of size n. Then the equation:

$$\det(A - x I) = 0$$

is called the characteristic equation of A.

- So, the characteristic equation is

$$(\lambda + 1)^2 \cdot (\lambda - 3) = 0.$$

- $+$: plus $-$ minus, $\times$ times
- $\frac{a}{b}$: a over b
- Solving the system $X' = AX$:
- Solving the system of linear differential equations $X' = AX$:
- Solving the system of linear recurrence sequences $X_n = A X_{n-1}$:
- In particular,
- We find the dimension of the eigenspace corresponding to the eigenvalue $\lambda = 1$.

Definition: Suppose A, B are two square matrices of size $n \times n$. We say A, B are similar, if $A = PBP^{-1}$ for some invertible matrix P.

Definition: Suppose A is a square matrix of size $n \times n$. We say that A is diagonalizable, if there exists an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix.

Example: Let

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 3 \end{bmatrix}, \quad P = \begin{bmatrix} 1 & 1 & 5 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

Verify that A is diagonalizable, by computing $P^{-1}AP$.

Solution: After computation, we obtain

$$P^{-1}AP = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

- A^2 : A squared
- A^3 : A cubed
- $A^{4,5,\dots,k}$: A to the power of $4, 5, \dots, k$
- A^k : A to the k (to the power of k)
- P^{-1} : P inverse
- $\times$: times, multiplied by
- Therefore A is not diagonalizable.
- Next, we show that A is not diagonalizable. To do this, we have find and count the dimensions of all the eigenspaces E_λ :
- the dimension of E_λ is called the geometric multiplicity of λ. That is,

$$\dim E_\lambda = G_m(\lambda).$$

- So, $\lambda = -1, 2$ are the only eigenvalues of A.
- Now, we compute the dimension $\dim E_\lambda$:
- First, we find all the eigenvalues:

To do this, we solve $\det(A - x I) = 0$.

- Therefore, by theorem in the course, A is diagonalizable.

- Exponential — π : pi $\rightarrow \pi$
- polynomial — Sine $\rightarrow \cos x$
- differential — Cosine $\rightarrow \sin x$

- We can write by definition,
- On the other hand = d'autre part,
- That is,
- We find the eigenvalues and associated eigenvectors of the matrix A:
- We compute $P_A(x)$:
- We compute $\dim E_\lambda$:

- Hence the set of eigenvectors associated with $h = 4$ is spanned by $v_1 = (1,1)$.

Spanned = equipped ≡ engendré. i.e

$$E_h = \text{Vect}\{(1,1)\}.$$

- $x \in A$: x belongs to A
- $x, y \in A$: x, y belong to A.
- $\notin$: does not belong to
- $A \cap B$: A intersection B
- $A \cup B$: A union B
- ϕ ≡ the empty set
- $\mathbb{N}$: the Set of positive integers
  = the Set of natural numbers
- $\mathbb{Z}$: the Set of integers
- $\mathbb{Q}$: the Set of rational numbers
- $\mathbb{R}$: the Set of real numbers
- $\mathbb{R}^+$: the Set of nonnegative real numbers
- $\mathbb{C}$: the Set of Complex numbers
- $\mathbb{R}^n$: n-dimensional Euclidean space

$\mathbb{C}^n$: n- dimensional Complex linear space

$Re(z)$ = real part of the Complex number z

$Im(z)$ = Imaginary part of the Complex number z.

$|z|$ = modulus of the complex number z

i.e $|x+iy| = \sqrt{x^2+y^2}$, $x, y \in \mathbb{R}$.

- $\sqrt{a}$ ≡ the square root of a
- a^2 ≡ a squared.
- x^t ≡ the transpose of x
- $\det(A)$ ≡ determinant of a square matrix A
- $\text{rank}(A)$ ≡ the rank of a matrix A.
- $\text{tr}(A)$ ≡ the trace of a square matrix A.
- A^t ≡ the transpose of A

- $\|A\|$: the norm of A.
- $\bar{A}$: the Conjugate of the matrix A.
- A^* : Conjugate transpose of A, i.e

$$A^* = (\bar{A})^t = \overline{A^t}.$$

- A^{-1} : inverse of square matrix A (if it exists)
- I_n : $n \times n$ unit matrix
  = the n-by-n identity matrix.
- O_n = $n \times n$ zero matrix

- A is symmetric $\Leftrightarrow A = A^t$
- A is skew-symmetric $\Leftrightarrow A = -A^t$

- First, we study the diagonalization of the matrix A_α :

- $$\lim_{x \to 0} \frac{e^{xA} - I}{x} = A$$

the limit of exponential x, A minus I over x (as x tends to zero)
- equals A
- is equal to A.

- $\sum_{k=0}^{\infty} \frac{A^k}{k!}$ the sum for k from zero to infinity of A to the k over k factorial.

- Since A is diagonalizable, then

- From (1), (2) and (3), we have

- Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

a_{ij} are Called the entries of A

entries ≡ elements

- Then (1,1,0), (0,1,-1) are the eigenvectors of A associated with the eigenvalue $\lambda = -1$.

- We find the eigenvectors associated with each of the eigenvalues:

- It is sufficient to prove that ...

- (Here O is the zero matrix).

- Taking the determinant of both sides of the equation, we find

- Similarity and diagonalizability :

- Then A is Similar to B because $A = PBP^{-1}$, where

$$P = \begin{bmatrix} 4 & -3 \\ -1 & -1 \end{bmatrix}$$

- Thus, A is Similar to B.

- Then, multiplying both sides of this equation on the right by P^{-1}, we obtain
- Multiplying both sides of this equation on the left by P^{-1}, we get

- This shows that...

- where Q is the matrix $Q = P^{-1}$, which is invertible.

- If A is Similar to B, then there exists an invertible $n \times n$ matrix P such that $A = PBP^{-1}$.

- which shows that A and B have the same characteristic equation and hence the same eigenvalues.

- If $A = PDP^{-1}$, then $A^2 = PD^2P^{-1}$ and in general,

$$A^k = PD^kP^{-1}.$$

Since D^k is easy to compute, then so is A^k.

- Let us show that the matrix A is nilpotent.

- which can be written as

- or, equivalently,

- We thus see that the eigenvalues of A are ...

- Since $P = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}$ is invertible, then we know that the vectors $v_1, v_2, \ldots, v_n$ form a linearly independent set.

- We show that the matrix $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is not diagonalizable.

- which means that ...

- After simple computation, we get

- We prove that A is idempotent. That is, $A^2 = A$. In fact, we have

- We can use this to compute A^k quickly for large k.
- the matrix D is a diagonal matrix, i.e entries off the main diagonal are all zero.
- We have

$$A = P D P^{-1}.$$

Again,

$$A^2 = P D^2 P^{-1}.$$

In general, we have

$$A^k = P D^k P^{-1}.$$

- Equivalently,
- Or, equivalently,
- $\Longleftrightarrow$ if and only if
- $\Longrightarrow$ implies
- An $n \times n$ matrix with n distinct eigenvalues is diagonalizable.
- then, A has three linearly independent eigenvectors and it is therefore diagonalizable.

Theorem: An $n \times n$ matrix with n distinct eigenvalues is diagonalizable.

- for some invertible matrix P
- for some positive integer k.
- For every $k \geq 0$, we have

$$A^k = P \cdot D^k \cdot P^{-1}$$

- For every $k \geq 0$, we have

$$A^k = P \cdot T^k \cdot P^{-1}.$$

- For each $k \geq 0$, we have
- $]a, b[$: open interval
- $[a, b]$: closed interval
- E, F : Vector spaces
- $\mathcal{P}_n[X]$: Vector space of polynomials of degree $\leq n$.
- $\leqslant$: less or equals
- $\mathcal{M}_{n,m}(\mathbb{R})$: Vector space of $n \times m$ matrices
- α : alpha , β : beta γ, Γ : gamma
- δ, Δ : delta , ε : epsilon
- ξ : zeta . θ : theta
- λ : Lambda , μ : mu

- Π : pi ρ : rho
- τ : tau , χ : chi
- ψ, Ψ : psi
- ω, Ω : omega
- Explain why the diagonal entries of a skew-symmetric matrix must be zero.
- $x^3 - 2x^2 + 7x + 2$

x cubed minus two x squred plus seven x plus two.

- $x \neq a$: x is different from a
- Let $p(x) = a_0 + a_1 x + a_2 x^2$ be a polynomial of degree 2 with real coefficients.

For $A \in \mathcal{M}_n(\mathbb{R})$ we difine

$p(A) = a_0 I + a_1 A + a_2 A^2$,

where I is the $n \times n$ identity matrix.

- It follows from $(*)$ that...

- $\sum_{i=1}^{n} A_i$: the sum for i from one to n of A, i.
  : the sum of A, i for i from one to n.

- We will usually denote the minimal polynomial of A as $m_A(x)$.

- Let $A = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 2 & -1 \\ 1 & 1 & 0 \end{bmatrix}$

the characteristic polynomial is

$$x^3 - 2x^2 + x = x(x-1)^2.$$

- Since the minimal polynomial divides the characteristic polynomial, every root of the minimal polynomial is an eigenvalue.
- The Converse is also true.

- Using the Cayley-Hamilton theorem, we obtain.
- Let $S = \{v_1, v_2, \dots, v_k\}$. We say that S is Linearly independent if whenever $c_1 v_1 + \dots + c_k v_k = 0$ we have $c_1 = c_2 = \dots = c_k = 0$.