

# Eigenvalues and Eigenvectors

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# Eigenvalues and Eigenvectors

## Definitin and Examples

Throughout this chapter  $\mathbb{K}$  denotes the field  $\mathbb{R}$  or  $\mathbb{C}$ , and  $\mathcal{M}_n(\mathbb{K})$  denotes the vector space of  $n$  by  $n$  matrices over  $\mathbb{K}$ .

### Definition

Let  $A$  be an  $n \times n$  square matrix. When  $Ax = \lambda x$  has a non-zero vector solution  $x$ , then

- $\lambda$  is called an **eigenvalue** of  $A$ .
- $x$  is called an **eigenvector** of  $A$  corresponding to  $\lambda$ .
- The couple  $(\lambda, x)$  is called an **eigenpair** of  $A$ .

**Notes:** (i) eigenvector must be non-zero. (ii) But, eigenvalue  $\lambda$  can be zero, can be non-zero.

A vector  $x \in E$  is an eigenvector of  $A$  if

- $x$  is non-zero,
- there exists  $\lambda \in \mathbb{K}$ ,  $Ax = \lambda x$ .

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The **eigenspace** of  $A$  corresponding to  $\lambda$  is the subspace:

$$E_\lambda = \{v \in \mathbb{K}^n ; Av = \lambda v\}.$$

Note that  $E_\lambda$  is a vector subspace of  $\mathbb{K}^n$ . This is the **kernel** of the matrix  $A - \lambda I_n$ . So  $E_\lambda$  consists of all solutions  $v$  of the equation  $Av = \lambda v$ . In other words,  $E_\lambda$  consists of all eigenvectors with eigenvalue  $\lambda$ , together with the zero vector.

### Example

Let  $A = I_2$ . Then any non-zero vector  $v$  of  $\mathbb{R}^2$  will be an eigenvector of  $A$  corresponding to eigenvalue  $\lambda = 1$ .

### Example

Consider the matrix

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}.$$

Calculate the eigenvalues and eigenvectors of  $A$ .

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### Solution.

- First, we find the eigenvalues of  $A$ .

We start with calculating the characteristic polynomial of  $A$ . From definition, we obtain

$$\begin{aligned} p_A(x) &= \begin{vmatrix} 2-x & 1 \\ 1 & 2-x \end{vmatrix} \begin{array}{c} c_1 \\ \downarrow \\ c_1 + c_2 \end{array} \quad (\text{the first column } c_1 \text{ becomes } c_1 + c_2) \\ &= \begin{vmatrix} (3-x) & 1 \\ (3-x) & 2-x \end{vmatrix} = (3-x) \begin{vmatrix} 1 & 1 \\ 1 & 2-x \end{vmatrix} = (3-x)(2-x-1) \\ &= (3-x)(1-x). \end{aligned}$$

Hence,  $p_A(x) = (1-x)(3-x)$ , and so the eigenvalues are  $\lambda_1 = 1$  and  $\lambda_2 = 3$ .

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- Second, we find the eigenvectors. By definition, the eigenspace  $E_{\lambda_1}$  is given by

$$\begin{aligned} E_{\lambda_1} &= \left\{ (x, y) \in \mathbb{R}^2; \begin{array}{l} x + 2y = x \\ 2x + y = y \end{array} \right\} \\ &= \left\{ (x, y) \in \mathbb{R}^2; y = -x \right\} \\ &= \text{Vect} \{ (1, -1) \}. \end{aligned}$$

Thus,  $v_1 = (1, -1)$ .

- Using the same manner, the eigenspace  $E_{\lambda_2}$  is given by

$$\begin{aligned} E_{\lambda_2} &= \left\{ (x, y) \in \mathbb{R}^2; \begin{array}{l} x + 2y = 3x \\ 2x + y = 3y \end{array} \right\} \\ &= \left\{ (x, y) \in \mathbb{R}^2; y = x \right\} \\ &= \text{Vect} \{ (1, 1) \}. \end{aligned}$$

That is,  $v_2 = (1, 1)$ .

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### Definition

- The **geometric multiplicity** for a given eigenvalue  $\lambda$ , denoted by  $G_m(\lambda)$ , is the dimension of the eigenspace  $E_\lambda$ . That is,

$$G_m(\lambda) = \dim E_\lambda.$$

- The **algebraic multiplicity** for a given eigenvalue  $\lambda$ , denoted by  $A_m(\lambda)$ , is the number of times the eigenvalue is repeated. For example, if the characteristic polynomial is  $(x - 1)^2(x - 5)^3$  then for  $\lambda = 1$  the algebraic multiplicity is 2 and for  $\lambda = 5$  the algebraic multiplicity is 3.

Note that the algebraic multiplicity is greater than or equal to the geometric multiplicity. That is, we always have  $A_m(\lambda) \geq G_m(\lambda)$ .

# Eigenvalues and Eigenvectors

## Definitin and Examples

**Examples.** Calculate eigenvalues and eigenvectors of the following matrices. Deduce the algebraic multiplicity and the geometric multiplicity of each eigenvalue of  $A$ .

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}.$$

**Ans.** We have  $\lambda_1 = 4$ ,  $v_1 = (2, 3)$  and  $\lambda_2 = -1$ ,  $v_2 = (1, -1)$ .

$$A = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}.$$

**Ans.** We have  $\lambda_1 = e^{i\theta}$ ,  $v_1 = (-i, 1)$  and  $\lambda_2 = e^{-i\theta}$ ,  $v_2 = (i, 1)$ .

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$$A = \begin{pmatrix} 1 & 2 \\ 0 & 5 \end{pmatrix}.$$

**Ans.** We have  $\lambda_1 = 1$ ,  $E_1 = \text{Vect} \{(1, 0)\}$  and  $\lambda_2 = 5$ ,  $E_5 = \text{Vect} \{(1, 2)\}$ .

$$A = \begin{pmatrix} 2 & 6 \\ 0 & 2 \end{pmatrix}.$$

**Ans.** We have  $\lambda = 2$  (double),  $E_\lambda = \text{Vect} \{(1, 0)\}$ .

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & -5 \end{pmatrix}.$$

**Ans.** We have  $\lambda_1 = 1$ ,  $E_1 = \text{Vect} \{(1, 0, 0)\}$ ,  $\lambda_2 = 2$ ,  $E_2 = \text{Vect} \{(2, 1, 0)\}$  and  $\lambda_3 = -5$ ,  $E_{-5} = \text{Vect} \{(5, 6, -14)\}$ .



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## Definitin and Examples

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}.$$

**Ans.** We have  $\lambda_1 = 1$ ,  $E_{\lambda_1} = \text{Vect} \{(-1, 1, 1)\}$ ,  $\lambda_2 = 2$  (the algebraic multiplicity of  $\lambda_2$  is 2),  $E_{\lambda_2} = \text{Vect} \{(0, 1, 0), (0, 0, 1)\}$ . Also, the geometric multiplicity of  $\lambda_2$  is 2.

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

**Ans.** We have  $\lambda = 0$  (triple eigenvalue),  $E_{\lambda} = \text{Vect} \{(1, 0, 0), (0, 1, -1)\}$ . The eigenspace corresponding to  $\lambda = 0$  is of dimension 2.

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 3 & 2 \end{pmatrix}.$$

**Ans.** We have  $\lambda = 2$  (triple eigenvalue),  $E_{\lambda} = \text{Vect} \{(0, 0, 1)\}$ . The eigenspace corresponding to  $\lambda = 2$  is of dimension 1.

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$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

**Ans.** We have  $\lambda_1 = 0$  (simple eigenvalue),  $E_{\lambda_1} = \text{Vect} \{(-1, 1, 0)\}$  and  $\lambda_2 = 2$  (double eigenvalue),  $E_{\lambda_2} = \text{Vect} \{(0, 0, 1), (1, 1, 0)\}$ . The eigenspace corresponding to  $\lambda_1$  is of dimension 1 and the eigenspace corresponding to  $\lambda_2 = 2$  is of dimension 2.

$$A = \begin{pmatrix} a & 2 & 3 \\ 0 & 2a & 8 \\ 0 & 0 & 3a \end{pmatrix}; a \in \mathbb{R}.$$

**Ans.** We have  $\lambda_1 = a$  and  $E_{\lambda_1} = \text{Vect} \{(1, 0, 0)\}$ ,  $\lambda_2 = 2a$  and  $E_{\lambda_2} = \text{Vect} \left\{ \left( \frac{2}{a}, 1, 0 \right) \right\}$ ,  $\lambda_3 = 3a$  and  $E_{\lambda_3} = \text{Vect} \left\{ \left( \frac{1}{2a^2} (3a + 16), \frac{8}{a}, 1 \right) \right\}$ .

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## Definitin and Examples

### Corollary

*Let  $(\lambda, x)$  be an eigenpair of  $A$ . Then  $(\lambda^k, x)$  is an eigenpair of  $A^k$ .*

### Proof.

In fact, we see that

$$Ax = \lambda x \Rightarrow A^2x = A(\lambda x) = \lambda Ax = \lambda^2 x.$$

Therefore,

$$Ax = \lambda x \Rightarrow \forall k \geq 0 : A^k x = \lambda^k x.$$

The result is proved. □

# Eigenvalues and Eigenvectors

## Definitin and Examples

### Corollary

*Let  $A$  be an invertible matrix and let  $(\lambda, x)$  be an eigenpair of  $A$  with  $\lambda \neq 0$ . Then  $\left(\frac{1}{\lambda}, x\right)$  is an eigenpair of  $A^{-1}$ .*

### Proof.

By definition, we have

$$\begin{aligned} A^{-1}x &= A^{-1}(\lambda x) = A^{-1}\left(\frac{\lambda}{\lambda}x\right) = \frac{1}{\lambda}A^{-1}(\lambda x) \\ &= \frac{1}{\lambda}A^{-1}(Ax) \quad (\text{since } Ax = \lambda x) \\ &= \frac{1}{\lambda}x. \end{aligned}$$

Thus,  $A^{-1}x = \frac{1}{\lambda}x$ . The proof is finished. □

# Eigenvalues and Eigenvectors

## Problems

**Ex 01.** Calculate the eigenvalues and eigenvectors of the following matrix:

$$A = \begin{pmatrix} -3 & 1 & -1 \\ -7 & 5 & -1 \\ -6 & 6 & -2 \end{pmatrix}.$$

**Ans.**  $\lambda_1 = -2$ ,  $v_1 = (1, 1, 0)$  and  $\lambda_2 = 4$ ,  $v_2 = (0, 1, 1)$ .

**Ex 02.** Let  $P \in \text{GL}_n(\mathbb{R})$  and let  $D$  be the following diagonal matrix:

$$D = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}$$

Calculate the eigenpairs of  $D$ , then deduce the eigenpairs of the matrix  $PDP^{-1}$ .

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**Ex 03.** Let  $A \in \mathcal{M}_n(\mathbb{R})$  and  $\alpha \in \mathbb{R}^*$ . Prove that

$v$  is an eigenvector of  $A \Rightarrow \alpha v$  is also an eigenvector of  $A$ .

**Ex 04.** Let  $A \in \mathcal{M}_n(\mathbb{R})$  and  $\lambda_1, \lambda_2$  be two eigenvalues of  $A$  with  $\lambda_1 \neq \lambda_2$ . Prove that

$$E_{\lambda_1} \cap E_{\lambda_2} = \{0_{\mathbb{R}^n}\}.$$

Recall that  $E_{\lambda} = \{x \in \mathbb{R}^n ; Ax = \lambda x\}$ .