

System of recurrence sequences. Part II

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Linear recurrence sequences of order k

Definition

Let $(a_0, a_1, \dots, a_{k-1})$ be a system of k real numbers not all zero. A **linear recurrence sequence of order k** is defined as follows:

$$\begin{cases} x_{n+k} = a_0x_n + a_1x_{n+1} + \dots + a_{k-1}x_{n+k-1}, \\ x_0, x_1, \dots, x_{k-1} \in \mathbb{R} \text{ are given.} \end{cases}$$

Thus, a sequence defined by a **linear recurrence relation** is uniquely determined by its first k terms: $x_0, x_1, \dots, x_{k-1}$.

As an example, for $k = 2$:

$$\begin{cases} x_{n+2} = a_0 x_n + a_1 x_{n+1}, \\ x_0, x_1 \in \mathbb{R} \text{ are given.} \end{cases} \quad (S)$$

In the equivalent vector-matrix system, we obtain

$$\begin{pmatrix} x_{n+2} \\ x_{n+1} \end{pmatrix} = \begin{pmatrix} a_1 & a_0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_{n+1} \\ x_n \end{pmatrix},$$

or equivalently

$$\begin{pmatrix} x_{n+1} \\ x_{n+2} \end{pmatrix}_{X_{n+2}} = \begin{pmatrix} 0 & 1 \\ a_0 & a_1 \end{pmatrix}_A \begin{pmatrix} x_n \\ x_{n+1} \end{pmatrix}_{X_{n+1}}, \quad (S_1)$$

from which it follows that

$$X_n = AX_{n-1} = A^2X_{n-2} = \dots = A^{n-1}X_1, \quad (1)$$

where $X_1 = \begin{pmatrix} x_0 \\ x_1 \end{pmatrix}$. Thus, we must compute A^n for $n \geq 0$.

Application. Consider the following example:

Example

Let (x_n) be the sequence given by

$$x_{n+2} = \frac{2}{\frac{1}{x_n} + \frac{1}{x_{n+1}}} ; x_0, x_1 \in \mathbb{R}_+^*. \quad (2)$$

Find the formula of x_n in terms of n , then calculate $\lim_{n \rightarrow +\infty} x_n$.

Solution. In fact, we write (2) in the form

$$\frac{2}{x_n} = \frac{1}{x_{n-2}} + \frac{1}{x_{n-1}}.$$

Setting $\frac{2}{x_n} = y_n$, we get

$$2y_n = y_{n-1} + y_{n-2}, \text{ that is, } y_n = \frac{1}{2}y_{n-1} + \frac{1}{2}y_{n-2}.$$

In the equivalent vector-matrix system, we have

$$\begin{pmatrix} y_n \\ y_{n-1} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} y_{n-1} \\ y_{n-2} \end{pmatrix}; \quad \begin{cases} y_0 = \frac{1}{x_0} \\ y_1 = \frac{1}{x_1} \end{cases}$$

Therefore,

$$\begin{pmatrix} y_n \\ y_{n-1} \end{pmatrix} = A^{n-1} \begin{pmatrix} y_1 \\ y_0 \end{pmatrix}, \text{ where } A = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & 0 \end{pmatrix}.$$

From the computation (the matrix diagonalizable), we obtain

$$A^{n-1} = \begin{pmatrix} \frac{1}{3} \left[2 + \left(\frac{-1}{2} \right)^{n-1} \right] & \frac{1}{3} \left[1 - \left(\frac{-1}{2} \right)^{n-1} \right] \\ \frac{1}{3} \left[2 - 2 \left(\frac{-1}{2} \right)^{n-1} \right] & \frac{1}{3} \left[1 + 2 \left(\frac{-1}{2} \right)^{n-1} \right] \end{pmatrix},$$

and so

$$y_n = \frac{1}{3} \left[2 + \left(\frac{-1}{2} \right)^{n-1} \right] y_1 + \frac{1}{3} \left[1 - \left(\frac{-1}{2} \right)^{n-1} \right] y_0.$$

Since $x_n = \frac{1}{y_n}$, it follows that

$$x_n = \frac{3}{\left[2 + \left(\frac{-1}{2}\right)^{n-1}\right] \frac{1}{x_1} + \left[1 - \left(\frac{-1}{2}\right)^{n-1}\right] \frac{1}{x_0}}.$$

Passing to the limit as n tends to infinity, we get

$$\lim_{n \rightarrow +\infty} x_n = \frac{3}{\frac{2}{x_1} + \frac{1}{x_0}}.$$

Problem

Let (F_n) be the **Fibonacci sequence** defined by

$$\begin{cases} F_n = F_{n-1} + F_{n-2} \\ F_0 = F_1 = 1 \end{cases}, \quad n \geq 2.$$

Complete the following table:

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
F_n	1	1	...	...	...	...	...	...	...	...	...	...	...	...	...

Compute F_{1000} .