a If $\beta \leq 0$, for all $n \geq 3$ we get

$$\frac{1}{n(\ln n)^{\beta}} \geq \frac{1}{n},$$

thus, from the comparaison test the series $\sum_{n\geq 2} \frac{1}{n(\ln n)^{\beta}}$ diverges.

b If $\beta > 0$. The function $x \mapsto f(x) = \frac{1}{x(\ln x)^{\beta}}$ is positive and decreasing on $[2, +\infty[$, thus by using integral test (see theorem 1.2.2).

(i) If $\beta \neq 1$, we have

$$\int_2^x \frac{dt}{t(\ln t)^{\beta}} = \left[\frac{(\ln t)^{1-\beta}}{1-\beta}\right]_2^x = \frac{1}{1-\beta}\left((\ln x)^{1-\beta} - (\ln 2)^{1-\beta}\right),$$

we deduce that

$$\lim_{x\to\infty} \int_2^x \frac{dt}{t(\ln t)^{\beta}} = \begin{cases} \dfrac{(\ln 2)^{1-\beta}}{1-\beta}, & \text{if } \beta > 1 \\ +\infty, & \text{if } 0 < \beta < 1. \end{cases}$$

Then, the series converges if $\beta > 1$, and diverges if $0 < \beta < 1$.

(ii) Now, if $\beta = 1$.

$$\int_2^x \frac{dt}{t(\ln t)} = \ln(\ln(x)) - \ln(\ln(2)) \longrightarrow_{x\to\infty} = \infty.$$

Thus, the series $\sum_{n\geq 2} \frac{1}{n(\ln n)}$ diverges.

□

1.3 Alternating Series

We have focused almost exclusively on series with positive terms up to this point. In this short section we begin to delve into series with both positive and negative terms, presenting a test which works for many series whose terms alternate in sign.

Definition 1.3.1. A series with terms alternately positive and negative is called an alternating series. For example, $1 - 1/2 + 1/3 - 1/4 + 1/5 - 1/6 + 1/7 - ....$ The general form of alternating series is given by

$$a_1 - a_2 + a_3 - a_4 + a_5 - a_6 + a_7 - a_8 + ... = \sum_{n\geq 1} (-1)^{n+1} a_n, \quad (a_n > 0)$$

or

$$-a_1 + a_2 - a_3 + a_4 - a_5 + a_6 - a_7 + a_8 - .... = \sum_{n\geq 1} (-1)^{n} a_n, \quad (a_n > 0)$$

1.3.1 The Alternating Series Test

Theorem 1.3.1. *Suppose that the sequence (a_n) satisfies the three conditions:*

1. $a_n \geq 0$ *for sufficiently large* n*,*
2. $a_{n+1} \leq a_n$ *for sufficiently large* n *(i.e., a_n is monotonically decreasing), and*
3. $a_n \to 0$ *as* $n \to \infty$.

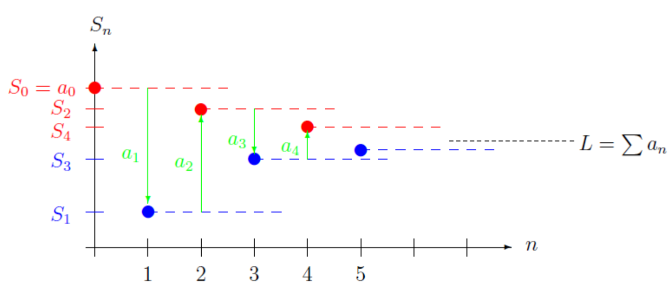
Then the alternating series $\sum_{n\geq 1} (-1)^{n+1} a_n$ *converges.*

An explanation of why the Alternating Series Test works:

We are going for simplicity:

1. that the starting index of the series is $n = 0$, and
2. that the terms $a_0, a_2, a_4, ...$ are all positive and the terms $a_1, a_3, a_5, ...$ are all negative.

Now for each $n \geq 0$, let $S_n = a_0 + a_1 + ... + a_n$ be the n^{th} partial sum. Consider the following picture which plots S_n vertically and n horizontally (an explanation of the picture is below)



The partial sums are plotted with the red and blue points (the S_n where n is even are the red points and the S_n where n is odd are the blue points). Notice that to get from one partial sum to the next, i.e. to get from S_n to S_{n+1}, you have to add a_{n+1}. This is indicated by the green arrows. Now we will use the hypotheses of the Alternating Series Test:

- By hypothesis (1), the an alternate in sign. In our case, all the even an are negative and all the odd an are positive. Therefore, whenever n is odd, S_n is below the previous partial sum, and whenever n is even, S_n is above the previous partial sum. This makes each blue dot lower than each preceding red dot, and each red dot above each preceding blue dot.

- By hypothesis (3), $|a_n| \geq |a_{n+1}|$. Since $|a_n|$ is the length of the n^{th} green arrow, we are assured by hypothesis (3) that the green arrows are getting shorter as n increases. Thus the red and blue dots are getting closer and closer together.

- By hypothesis (2), $\lim\limits_{n\to\infty} |a_n| = 0$. This means that since $|a_n|$ is the length of the green arrows, the length of these green arrows is going to zero as n increases. Thus the red and blue dots are both approaching the same height, so they have the same limit. This limit L is the limit of the partial sums, so by definition the infinite series converges to L.

This concludes the explanation of why the Alternating Series Test works.

Example 1.3.1. Does the series $\sum\limits_{n\geq 1}(-1)^{n+1}\dfrac{2n+3}{3n+4}$ converge or diverge?

This series does alternate in sign, and $\dfrac{2n+3}{3n+4}$ is decreasing, but $\dfrac{2n+3}{3n+4} \to 2/3 \neq 0$, so the series diverges by the Test for Divergence.

Remark 1.3.1. Note that in the solution of Example 1.3.1, we did not appeal to the Alternating Series Test, but instead used the Test for Divergence. The Alternating Series Test never shows that series diverge.

1.3.2 Absolute and Conditional Convergence

Definition 1.3.2. A series $\sum\limits_{n\geq 1} a_n$ is said to be **absolutely convergent** if the series $\sum\limits_{n\geq 1} |a_n|$ is convergent.

Definition 1.3.3. If $\sum_{n\geq 1} a_n$ is converges but $\sum_{n\geq 1} |a_n|$ diverges, then the series $\sum_{n\geq 1} a_n$ is called **conditionally convergent.**

Example 1.3.2. • Test the convergence of the series: $5 - 10/3 + 20/9 - 40/27 + ...$

Test the convergence and absolute convergence of the series:

- $\dfrac{1}{1\times 3} - \dfrac{1}{2\times 4} + \dfrac{1}{3\times 5} - \dfrac{1}{4\times 6} + ...$
- $\sum_{n\geq 1} \dfrac{(-1)^{n-1}}{\sqrt{n}}.$

Remark 1.3.2. • Every absolutely convergent series is a convergent series but the converse is not true.

- If $\sum_{n\geq 1} a_n$ is a series of positive terms, then $\sum_{n\geq 1} |a_n| = \sum_{n\geq 1} a_n$. Therefor the concepts of **convergence** and **absolutely convergence** are the same. Thus, any convergent series of positive terms is also absolutely convergent.

The proof of the **Alternating Series Test** implies the following very simple bound on remainders of these series.

Theorem 1.3.2. ***The Alternating Series Remainder Estimates.***
Suppose that the sequence (b_n) satisfies the three conditions of the Alternating Series Test:

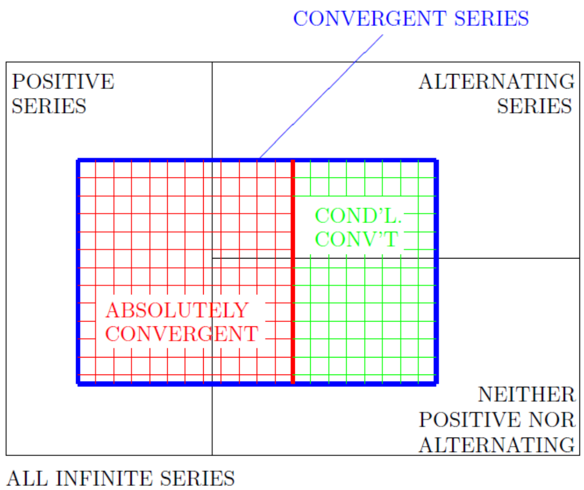
1. $b_n \geq 0$ *for sufficiently large* n,
2. $b_{n+1} \leq b_n$ *for sufficiently large* n *(i.e.,* b_n *is monotonically decreasing), and*
3. $b_n \to 0$ *as* $n \to \infty$.

Then if $n \geq N$, *the error in the* n*th partial sum of* $\sum_{n\geq 1} (-1)^{n+1} b_n$ *is bounded by* b_{n+1}

$$\left| s_n - \sum_{n\geq 1} (-1)^{n+1} b_n \right| \leq b_{n+1}.$$

Example 1.3.3. How many terms of the alternating series must we add to approximate the true sum with error less than $1/10000$?

Notice that there are now three disjoint classes of infinite series: those which converge **absolutely,** those which converge **conditionally**, and those which **diverge**. Putting all this together, we have the following diagram illustrating the various possibilities for infinite series. This diagram is extremely important to understand:



1.3.3 The Cauchy Product of Infinite Series

The Cauchy product of two infinite series $\sum_n a_n$ and $\sum_n b_n$ is defined to be the series

$$\sum_n c_n, \quad \text{where } c_n = \sum_{j=0}^{n} a_j b_{n-j} = a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + ... + a_{n-1} b_1 + a_n b_0.$$

The convergence of $\sum_n a_n$ and $\sum_n b_n$ is **not in itself sufficient to ensure the convergence of the Cauchy product of these series**. Convergence is however assured provided that the series $\sum_n a_n$ and $\sum_n b_n$ are absolutely convergent.

Theorem 1.3.3. *The Cauchy product $\sum_n c_n$ of two absolutely convergent infinite series $\sum_n a_n$ and $\sum_n b_n$ is absolutely convergent, and*

$$\sum_n c_n = \left(\sum_n a_n\right)\left(\sum_n b_n\right).$$

Proof. For each non-negative integer m, let

$$S_m = \{(j,k) \in \mathbb{Z} \times \mathbb{Z} : \quad 0 \le j \le m, \quad 0 \le k \le m\},$$

$$T_m = \{(j,k) \in \mathbb{Z} \times \mathbb{Z} : \quad 0 \le j,\ 0 \le m, \quad 0 \le j+k \le m\},$$

Now

$$\sum_{n=0}^{m} c_n = \sum_{(j,k)\in T_m} a_j b_k \text{ and } \left(\sum_{n=0}^{m} a_n\right)\left(\sum_{n=0}^{m} b_n\right) = \sum_{(j,k)\in S_m} a_j b_k.$$

Also

$$\sum_{n=0}^{m} |c_n| \le \sum_{(j,k)\in T_m} |a_j||b_k| \le \sum_{(j,k)\in S_m} |a_j||b_k| \le \left(\sum_n |a_n|\right)\left(\sum_n |b_n|\right).$$

since $|c_n| \le= \sum_{j=0}^{n} |a_j||b_{n-j}|$ and the infinite series $\sum_n a_n$ and $\sum_n b_n$ are absolutely convergent. It follows that the Cauchy product $\sum_n c_n$ is absolutely convergent, and is thus convergent. Moreover

$$\begin{aligned}
\left|\sum_n c_n^{2m} - \left(\sum_n^{m} a_n\right)\left(\sum_n^{m} b_n\right)\right| &= \left|\sum_{(j,k)\in T_{2m}\setminus S_m} a_j b_k\right| \\
&\le \sum_{(j,k)\in T_{2m}\setminus S_m} |a_j b_k| \\
&\le \sum_{(j,k)\in S_{2m}\setminus S_m} |a_j b_k| \\
&= \left(\sum_n^{2m} |a_n|\right)\left(\sum_n^{2m} |b_n|\right) - \left(\sum_n^{m} |a_n|\right)\left(\sum_n^{m} |b_n|\right),
\end{aligned}$$

since $S_m \subset T_{2m} \subset S_{2m}$. But

$$\begin{aligned}
\lim_{m\to\infty}\left(\sum_n^{2m} |a_n|\right)\left(\sum_n^{2m} |b_n|\right) &= \left(\sum_n^{\infty} |a_n|\right)\left(\sum_n^{\infty} |b_n|\right) \\
&= \lim_{m\to\infty}\left(\sum_n^{m} |a_n|\right)\left(\sum_n^{m} |b_n|\right),
\end{aligned}$$

since the infinite series $\sum_n a_n$ and $\sum_n b_n$ are absolutely convergent. It follows that

$$\lim_{m\to\infty}\left(\sum_n^{2m} c_n - \left(\sum_n^{m} a_n\right)\left(\sum_n^{m} b_n\right)\right) = 0,$$

and hence

$$\sum_n^{\infty} c_n = \lim_{m\to\infty}\sum_n^{2m} c_n = \left(\sum_n^{\infty} a_n\right)\left(\sum_n^{\infty} b_n\right),$$

as required. $\square$

1.4 Summary of convergence(divergence) tests

Having completed our discussion of methods to classify infinite series as **absolutely convergent, conditionally convergent, or divergent,** we now summarize the procedure one should use when trying to solve such a classification problem. To classify an infinite series $\sum a_n$ as absolutely convergent, conditionally convergent or divergent, follow these steps:

1. First, check to see whether the series is a **p-series** or **a geometric** series (or is a sum or difference of series of this type). If it is, use the p..series Test and/or Geometric Series Test (together with linearity properties) to classify the series.

2. If the terms of the series contain only **multiplication** and **division** and contain **exponentials or factorial** terms, use the **Ratio Test.** (If the terms of the series are all polynomials in n, avoid the Ratio Test.)

3. Otherwise, classify the series as positive, negative, alternating, or none of these. If the series is negative, factor out -1 from the series and treat what remains as a positive series.

4. If the series is positive:

   (a) If the terms of the series contain **addition/subtraction** in the denominator, or if they contain **sines or cosines,** try the **Comparison Test.**

   (b) If $\lim\limits_{n\to\infty} a_n \neq 0$, then the series diverges by the n^{th}-Term Test.

   (c) If the terms of the series look like a function you can integrate, try the Integral Test (use this only as a last resort).

5. If the series is alternating, compute $\lim\limits_{n\to\infty} |a_n|$.

   (a) If this limit is not zero, then the series diverges by the n^{th}-Term Test.

   (b) If this limit is zero, you can usually verify that $|a_n| \geq |a_{n+1}|$, then the series converges by the **Alternating Series Test.** In this case, you then have to examine the series $\sum\limits_{n} |a_n|$:

(i) If $\sum_n |a_n|$ converges, then $\sum_n a_n$ **converges absolutely.**

(ii) If $\sum_n |a_n|$ diverges, then $\sum_n a_n$ **converges conditionally.**

6. If the series is neither positive, negative nor alternating:

(a) If you can show that $\lim_{n\to\infty} |a_n| \neq 0$, then the series diverges by the n^{th}-Term Test.

(b) Forget the original series and try to show that the positive series $\sum_n |a_n|$ converges; in this case the original series **converges absolutely** by definition.