EXERCICE 1.Let $(f_n)_n$ be the sequence of functions defined on $\mathbb{R}_+$ by

$$\forall n \in \mathbb{N}^*,\ \forall x \in \mathbb{R}_+, \quad f_n(x) = (x^2+1)e^{-x}\frac{nx}{nx+1}.$$

(1) Calculate $f_n(0)$, then show that $(f_n)_n$ converges on $\mathbb{R}_+$ to a function f that we will determine.

(2) Is the convergence of $(f_n)_n$ to f uniform on $\mathbb{R}_+$?

(3) For all $x \neq 0$, calculate $f(x) - f_n(x)$. Deduce that, $\forall a > 0$, and $\forall x > a$, we have

$$0 \leq f(x) - f_n(x) \leq \frac{f(x)}{(na+1)}.$$

(4) Let $g_n(x) = \dfrac{f(x)}{(na+1)}$, calculate $g_n'(x)$, then show the uniform convergence of f_n towards f on $\mathbb{R}_+^*$

(5) Calculate $\lim\limits_{n\to\infty} \displaystyle\int_a^1 f_n(x)dx$ where $a \in \,]\,0,1\,[$.

EXERCICE 2.Let $(f_n)_{n\in\mathbb{N}^*}$ be the sequence of functions defined on $[0,1]$ by

$$f_n(x) = \begin{cases} n^2x(1-nx) & \text{if } x \in [0, \frac{1}{n}] \\ 0 & \text{if } x \in \,]\,\frac{1}{n}, 1\,]\,. \end{cases}$$

(1) Calculate the limit of the sequence (f_n).

(2) Calculate $\int_1^0 f_n(t)dt$. Is there uniform convergence on $[0,1]$?

(3) Study the uniform convergence on $[a,1]$ for $a \in]0,1]$.

EXERCICE 3. We consider the sequence of functions $f_n : [0,1] \to \mathbb{R}$ defined by

$$f_n(x) = \frac{ne^x}{n+x}.$$

(c.1): Determine the simple limit of f_n. Is there uniform convergence ?

(c.2): Determine $\lim\limits_{n\to\infty} \displaystyle\int_0^1 f_n(x)dx$.

EXERCICE 4.For all integer $n \geq 1$ and all $x \in \mathbb{R}^+$, we put

$$f_n(x) = x^2e^{-nx} \qquad \text{and} \qquad g_n(x) = \int_0^x f_n(t)dt = \int_0^x t^2e^{-nt}dt.$$

(1) Calculate $f_n'(x)$ and $\sup_{x\in\mathbb{R}} |f_n(x)|$, then show that the series of functions $\sum_{n\geq1} f_n(x)$ is normally convergent on $\mathbb{R}^+$.

(2) Let $f(x) = \sum_{n\geq1} f_n(x)$. Show that f is continuous on $\mathbb{R}^+$ and show that its sum $f(x) = \dfrac{x^2}{e^x-1}$.(Indication : use the sum of a geometric series).

(3) Using two integrations by parts, show that

$$g_n(x) = -\left(\frac{x^2}{n} + \frac{2x}{n^2}\right)e^{-nx} + \frac{2}{n^3}\left(1 - e^{-nx}\right), \quad \text{for } x > 0.$$

(4) Show that $|g_n(x)| \leq \dfrac{6}{n^3}$ (We can use the fact that $\dfrac{p!}{n^p} \geq x^pe^{-nx}$ for all integer $p \geq 0$ and all $x \in \mathbb{R}^+$).

(5) Deduce that the series of functions $\sum_{n\geq1} g_n(x)$ is normally convergent on $\mathbb{R}^+$.

(6) Let $g(x) = \sum_{n\geq1} g_n(x)$. Calculate $\lim_{x\to+\infty} g_n(x)$. Express $\lim_{x\to+\infty} g(x)$ as a sum of series.

(7) Show that $g(x) = \int_0^x f(t)dt$. Deduce the following identity : $\displaystyle\int_0^{+\infty} \frac{t^2}{e^t-1}dt = 2\sum_{n\geq1}\frac{1}{n^3}$.