We would like to compute the powers (resp. the exponential or any matrix function) of an $n \times n$ matrix A.

In both cases we must first compute the characteristic polynomial and determine the eigenvalues and eigenvectors.

- We also use **Minimal polynomial**.

Case 1

The matrix A is **diagonalizable**:

$A_m(\lambda_i) = G_m(\lambda_i)$ for $1 \leq i \leq k$.

We need to apply **Cayley-Hamilton Theorem**:

v is called an eigenvector for λ if $(A - \lambda I)\, v = 0$ with $v \neq 0$.

Theorem (Necessary and sufficient condition for diagonalization)

Let $A \in \mathcal{M}_n(\mathbb{K})$ be a square matrix. A is diagonalizable, if and only if, there exists a basis $\mathcal{B}$ of $\mathbb{K}^n$ formed by n eigenvectors of A.

- If $A = P \cdot D \cdot P^{-1}$, then $f(A) = P \cdot f(D) \cdot P^{-1}$.

Here, P is invertible and D is diagonal.

Example

Assume that $A = PDP^{-1}$, where P is invertible and D is diagonal. We have

$$\begin{cases} \text{For } f(x) = x^k \Rightarrow f(A) = A^k = P \cdot D^k \cdot P^{-1} \quad \text{for } k \geq 0 \\ \text{For } f(x) = \sqrt{x} \Rightarrow f(A) = \sqrt{A} = P \cdot \sqrt{D} \cdot P^{-1} \\ \text{For } f(x) = \cos x \Rightarrow f(A) = \cos A = P \cdot \cos D \cdot P^{-1} \\ \text{For } f(x) = e^x \Rightarrow f(A) = e^A = P \cdot e^D \cdot P^{-1} \\ \text{For } f(x) = \ln x \Rightarrow f(A) = \ln A = P \cdot \ln D \cdot P^{-1} \\ \dots \\ \text{and so on.} \end{cases}$$

We apply the powers of the matrix A in solving **systems of recurrence sequences** and its exponential in solving linear **systems of differential equations** of the first order.

- $X_{n+1} = AX_n \Rightarrow X_n = A^n X_0.$
- $X_{n+1} = AX_n + C \Rightarrow X_n = A^n X_0 + (A^{n-1} + A^{n-2} + \dots + A + I)\, C.$
- $X' = AX \Rightarrow X(t) = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2 + \dots + c_n e^{\lambda_n t} v_n$

Case 2

The matrix A is **not diagonalizable.**
We need to apply **Cayley-Hamilton Theorem**: $p_A(A) = 0$

In this case we must compute the **generalized eigenvectors** of A.

v is called a generalized eigenvector for λ if $(A - \lambda I)^s\, v = 0$ for some $s \geq 2$

We will apply one of the following three famous Theorems:

1. **Shaur decomposition Theorem**: States that every n by n matrix is similar to an upper triangular matrix T.
2. **Dunford Decomposition Theorem:** States that every n by n matrix A can be written as A = D + N, where D is **diagonalizable**, N is **nilpotent** with D.N = N.D. Here, for computing the powers of A, we must use the **Binomial formula** for matrices:

$$A^n = (D+N)^n = C_n^0 D^n + C_n^1 D^{n-1} N + \dots + C_n^{k-1} D^{n-(k-1)} N^{k-1},$$
$$\text{where } N^k = 0.$$

3. **Jordan Decomposition Theorem:** States that every n by n matrix A is similar to a Jordan matrix by **blocks,** i.e.

$$A = P \cdot J \cdot P^{-1}$$

where J is given by:

$$J = \begin{pmatrix} J_{\lambda_1} & & & \\ & J_{\lambda_2} & & \\ & & \ddots & \\ & & & J_{\lambda_k} \end{pmatrix}$$

with

$$J_{\lambda_i} = \begin{pmatrix} \lambda_i & 1 & & \\ & \lambda_i & 1 & \\ & & \ddots & \ddots \\ & & & \lambda_i \end{pmatrix} \text{ or } \begin{pmatrix} \lambda_i & & & \\ 1 & \lambda_i & & \\ & 1 & \ddots & \\ & & \ddots & \lambda_i \end{pmatrix}$$

We have the formulas of the powers and exp of J

J^k and e^J

- If $A = P \cdot J \cdot P^{-1}$, then $e^{At} = P \cdot e^{Jt} \cdot P^{-1}$.

Conclusion. We can compute the powers and exponential of any n by n matrix.