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LECTURE NOTES: ANALYSIS 3

License Mathematics

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CHAPTER 1

Infinite Series

1.1 An introduction to series

A series is the result of adding a sequence of numbers together. While you may never have thought of it this way, we deal with series all the time when we write expressions like $\frac{1}{3} = 0.333.....$, since this means that

$$\frac{1}{3} = \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + ...$$

In general we are concerned with infinite series such as

$$\sum_{n\geq 1} a_n = a_1 + a_2 + a_3 + ...a_n + ...$$

First though, we need to decide what it means to add an infinite sequence of numbers together. Clearly we can't just add the numbers together until we reach the end (like we do with finite sums), because we won't ever get to the end.

For any three numbers a, b and c, the following holds: $a + (b + c) = (a + b) + c$. This property has the important theoretical consequence that you can add any three numbers by choosing two, adding them, and then adding their sum to the third number. In other words, **finite sums can be rearranged and regrouped arbitrarily without changing the sum because of the associative property.** What's just as bad is that the associative property doesn't work for infinite series, as we see in the following example: $1 - 1 + 1 - 1 + 1 - 1 + ...$

If you group the terms into pairs, this series can be rewritten as

$$1-1+1-1+1-1+.....=(1-1)+(1-1)+(1-1)+.....=0+0+0+0+0+.....=0.$$

But, if you group the terms differently, you can also obtain

$$1-1+1-1+1-1+.....=1+(-1+1)+(-1+1)+....=1+0+0+0+0+0+....=1.$$

We do not yet know if either of these calculations are valid. But what we do know is that they cannot both be valid (because that would imply $0=1$). Therefore **it cannot be the case than one can legally regroup or rearrange terms in an infinite series.** In other words, the associative property is invalid for infinite series in general.
Instead, we adopt the following limit-based definition.

Definition 1.1.1. If the sequence $\{s_n\}$ of partial sums defined by

$$s_n = a_1 + a_2 + a_3 + ...a_n = \sum_{k=1}^{k=n} a_k$$

has a unique and finite limit as $n \to \infty$, then we say that $\sum_{n\geq 1} a_n = \lim_{n\to\infty} s_n$, and in this case we say that $\sum_{n\geq 1} a_n$ converges. Otherwise, $\sum_{n\geq 1} a_n$ diverges.

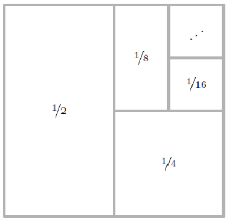
We begin with a particularly simple example.

Example 1.1.1. (Powers of 2). The series $\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges to 1.

We begin by computing a few partial sums:

$$\begin{aligned}
s_1 &= \frac{1}{2} &&= \frac{1}{2} &&= 1-\frac{1}{2} \\
s_2 &= \frac{1}{2}+\frac{1}{4} &&= \frac{3}{4} &&= 1-\frac{1}{4} = 1-\frac{1}{2^2} \\
s_3 &= \frac{1}{2}+\frac{1}{4}+\frac{1}{8} &&= \frac{7}{8} &&= 1-\frac{1}{8} = 1-\frac{1}{2^3} \\
s_4 &= \frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16} &&= \frac{15}{16} &&= 1-\frac{1}{16} = 1-\frac{1}{2^3}......,
\end{aligned}$$

then, $s_n = 1-\frac{1}{2^n}$ and then, $s_{n+1} = s_n + \frac{1}{2^{n+1}} = 1-\frac{1}{2^n}+\frac{1}{2^{n+1}} = 1-\frac{1}{2^{n+1}}$, so the formula is correct for all values of n (this technique of proof is known as mathematical induction). With this formula, we see that $\lim_{n\to\infty} s_n = 1$, so $\sum_{n=1}^{\infty} \frac{1}{2^n} = s = 1$.



Remark 1.1.1. There is an alternative, more geometrical, way to see that this series converges to 1. Divide the unit square in half, giving two squares of area $1/2$. Now divide one of these squares in half, giving two squares of area $1/4$. Now divide one of these in half, giving two squares of area $1/16$. If we continue forever, we will subdivide the unit square (which has area 1) into squares of area $1/2, 1/4, 1/8, \ldots.$, verifying that $\sum_{n=1}^{\infty} \frac{1}{2^n} = s = 1$.

Remark 1.1.2. If s_n does not tend to a unique limit finite or infinite, then series $\sum_{n=1}^{\infty} a_n$ is said to be oscillatory.

Properties 1.1.1. ´

- *If we add or remove finitely many terms in a series, then a convergent series remains convergent and a divergent series remains divergent.*
- *If we multiply each term of the series by a non-zero constant, then a convergent series remains convergent and a divergent series remains divergent.*
- *If* $\sum_{n=1}^{\infty} a_n$ *and* $\sum_{n=1}^{\infty} b_n$ *are convergent, then* $\sum_{n=1}^{\infty} a_n \pm \sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} (a_n \pm b_n)$ *is also convergent.*
- *Unless* a_n *and* b_n *are of the same sign, the divergence of the two series* $\sum_{n=1}^{\infty} a_n$ *and* $\sum_{n=1}^{\infty} b_n$ *does not imply the divergence of the series* $\sum_{n=1}^{\infty} (a_n + b_n)$ *as well shown in the example of* $a_n = (-1)^{n+1}$ *and* $b_n = (-1)^n$.
- *The set of numerical series is a vector space on* $\mathbb{C}$, *that of convergent series is a subspace.*

1.1.1 A necessary condition for convergence

Theorem 1.1.1. *If* $\sum a_n$ *is convergent, then* $\lim_{n\to\infty} a_n = 0$.

Proof. . Since we are assuming that $\sum a_n$ converges, then $a_n = s_n - s_{n-1}$ and $\lim\limits_{n\to\infty} s_n = \lim\limits_{n\to\infty} s_{n-1} = s$. therefore

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} s_n - \lim_{n\to\infty} s_{n-1} = s - s = 0.$$

$\square$

1.1.2 The Test for Divergence

If $\lim\limits_{n\to\infty} a_n \neq 0$ then $\sum a_n$ diverges.

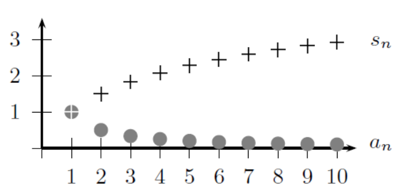
Remark 1.1.3. It is important to remember that the converse to the Test for Divergence is false, i.e., even if the terms of a series tend to 0, the series may still diverge.

Example 1.1.2. (The Harmonic Series). The series $\sum \dfrac{1}{n}$ diverges. Indeed

$$\begin{aligned}\sum_{n\geq 1}\frac{1}{n} &= 1+\frac{1}{2}+\underbrace{\frac{1}{3}+\frac{1}{4}}_{2\times\frac{1}{4}=\frac{1}{2}}+\underbrace{\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}}_{4\times\frac{1}{8}=\frac{1}{2}}\\ &+\underbrace{\frac{1}{9}+\frac{1}{10}+\frac{1}{11}+\frac{1}{12}+\frac{1}{13}+\frac{1}{14}+\frac{1}{15}+\frac{1}{16}}_{8\times\frac{1}{16}=\frac{1}{2}}+\ldots.. \geq 1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\ldots..\end{aligned}$$

and therefore the series diverges.

Example 1.1.3. (Harmonic series) Indeed, the harmonic series is just such a series: $\dfrac{1}{n} \to 0$ as $n \to \infty$, but $\sum \dfrac{1}{n}$ diverges.



Example 1.1.4. If, $a_n = \ln\left(1+\dfrac{1}{n}\right)$. Then $\lim\limits_{n\to\infty} a_n = 0$, but

$$\begin{aligned}
\lim_{n\to\infty} s_n &= \lim_{n\to\infty} \sum_{k=1}^{n} a_k = \lim_{n\to\infty} \sum_{k=1}^{n} (\ln(k+1) - \ln k) \\
&= \lim_{n\to\infty} [(\ln 2 - \ln 1) + (\ln 3 - \ln 2) + (\ln 4 - \ln 2) + ... + (\ln(n+1) - \ln n)] \\
&= \lim_{n\to\infty} \ln(n+1) = \infty.
\end{aligned}$$

Our goal in this chapter is to develop several tests which we can apply to a wide range of series. Our list of tests will grow to include

1. The Test for Divergence (already given)
2. The Integral Test
3. The p-Series Test
4. The Comparison Test
5. The Limit Comparison Test
6. The Ratio Test(d'Alembert test)
7. Cauchy's nth Root Test(or The root test)
8. Raabe's Test
9. The Absolute Convergence Theorem
10. The Alternating Series Test

Remark 1.1.4. It is important to realize that each test has distinct strengths and weaknesses, so if one test is inconclusive, you need to push onward and try more tests until you find one that can handle the series in question.

1.1.3 Geometric series

One of the most important types of infinite series are geometric series. A geometric series is simply the sum of a geometric sequence. Geometric series are some of the only series for which we can not only determine convergence and divergence easily, but also find their sums, if they converge:

Definition 1.1.2. Geometric Series. The geometric series

$$a + ar + ar^2 + ar^3 + ... + ar^n + ... = \sum_{n=0}^{\infty} ar^n,$$

converges to $\frac{a}{1-r}$ if $|r| < 1$, and diverges otherwise.

An easy way to remember this result is

$$\text{geometric series} \sum = \frac{\text{first term}}{1 - \text{ratio between terms}}.$$

Example 1.1.5. Compute

- $12 + 4 + 4/3 + 4/9 + 4/27.....$
- $\sum_{n=6}^{\infty} (-1)^n \frac{2^{n+3}}{3^n}$
- $\sum_{n=1}^{\infty} \frac{2^{n+1} + 9^{n/2}}{5^n}$
- Use geometric series to approximate the decimal expansion of $1/48$.

Proof.

- The first term is 12 and the ratio between terms is $1/3$, so

$$12 + 4 + 4/3 + 4/9 + 4/27.... = \frac{\text{first term}}{1 - \text{ratio between terms}} = \frac{12}{1 - 1/3} = 18.$$

- This series is geometric with common ratio

$$r = \frac{a_{n+1}}{a_n} = \frac{(-1)^{n+1} \frac{2^{n+4}}{3^{n+1}}}{(-1)^n \frac{2^{n+3}}{3^n}} = -2/3,$$

  and so it converges because $|-2/3| < 1$. Its sum is $\sum_{n=6}^{\infty} (-1)^n \frac{2^{n+3}}{3^n} = \frac{2^9/3^6}{1 + 2/3} = 512/1215$.

- We break this series into two

$$\sum_{n=1}^{\infty} \frac{2^{n+1} + 9^{n/2}}{5^n} = \sum_{n=1}^{\infty} \frac{2^{n+1}}{5^n} + \sum_{n=1}^{\infty} \frac{9^{n/2}}{5^n}.$$

The first of these series has common ratio $2/5$, so it converges. To analyze the second series, note that $9^{n/2} = 3^n$, so this series has common ratio $3/5$. Since both series converge, we may proceed with the addition:

$$\sum_{n=1}^{\infty} \frac{2^{n+1} + 9^{n/2}}{5^n} = \frac{2^2/5}{1-2/5} + \frac{3/5}{1-3/5} = 4/3 + 3/2 = 17/6.$$

- We express $1/48$ as $1/50$ times a fraction of the form $1/(1-r)$: $1/48 = 1/(50-2) = (1/50)\dfrac{1}{1-2/50}$. Now we can expand the fraction on the righthand side as a geometric series,

$$1/48 = (1/50)\left(1 + 2/50 + (2/50)^2 + (2/50)^3 + ...\right)$$

Using the first two terms of this series, we obtain the approximation $1/48 \approx 0.02(1 + 0.02) = 0.0204$.

$\square$

Example 1.1.6. (Repeating Decimals). Write the repeating decimal 3.10454545.... as a fraction in lowest terms.
Notice that we can rewrite this decimal as the sum of infinitely many fractions as follows:

$$\begin{aligned}
3.10454545.... &= 3.10 + 0.0045 + 0.000045 + 0.00000045 + ...\\
&= \frac{31}{10} + \frac{45}{10^4} + \frac{45}{10^6} + \frac{45}{10^8} + ...\\
&= \frac{31}{10} + 45 \sum \frac{1}{10^{2n}}\\
&= \frac{31}{10} + \frac{45}{100^2} \frac{1}{1-1/100} = 683/220.
\end{aligned}$$

1.1.4 A necessary and sufficient condition for convergence

Theorem 1.1.2. *Let (a_n) be a sequence in $\mathbb{R}$. The series $\sum a_n$ is convergent if and only if*

$$\forall \varepsilon > 0,\ \exists n_0 \in \mathbb{N} \quad \text{such that } p, q \in \mathbb{N} \text{ and } p > q > n_0, \text{ then } \left| \sum_{n=q+1}^{n=p} a_n \right| \leq \varepsilon.$$

Remark 1.1.5. This theorem is not very simple to handle, nevertheless its importance comes from the fact that it constitutes the only known necessary and sufficient condition which applies generally to any convergent series. It is the last method, when no simpler method is not applicable.

Example 1.1.7. (The Harmonic Series). The series $\sum \frac{1}{n}$ diverges. We have

$$\sum_{n=p+1}^{2p} \frac{1}{n} = \frac{1}{p+1} + \frac{1}{p+2} + \frac{1}{p+3} + ... + \frac{1}{2p} > p\frac{1}{2p},$$

therefore $\sum_{n=p+1}^{2p} \frac{1}{n} > \frac{1}{2}$ and we cannot realize $\left|\sum_{n=p+1}^{2p} \frac{1}{n}\right| < \epsilon$ if $\epsilon < \frac{1}{2}$. And therefore the series diverges.

1.2 Positive Term Series

Definition 1.2.1. If all the terms after some finitely many terms of an infinite series are positive then such a series is called positive term series. e.g.

$$-7 + 8 - 3 - 5 + 9 - 32 + \underbrace{2 + 3 + 5 + 34 + ...}_{\text{positive terms}} \qquad \text{is a positive term series.}$$

Theorem 1.2.1. *Suppose $a_n \geq 0 \quad \forall n$. Then $\sum a_n$ converges if and only if $(s_n)_n$ is bounded above.*

Example 1.2.1. Let $\sum_{n\geq 1} \frac{1}{n^2}$. Since the series is positive and, $\forall n \geq 1$,

$$\begin{aligned} s_n &= \sum_{k=1}^{k=n} \frac{1}{k^2} = 1 + \sum_{k=2}^{k=n} \frac{1}{k^2} \leq 1 + \sum_{k=2}^{k=n} \frac{1}{k(k-1)} = 1 + \sum_{k=1}^{k=n-1} \frac{1}{k(k+1)} \\ &= 1 + \sum_{k=1}^{k=n-1} \left(\frac{1}{k} - \frac{1}{k+1}\right) = 1 + \left(1 - \frac{1}{n}\right) = 2 - \left(1 - \frac{1}{n}\right) \leq 2. \end{aligned}$$

This shows that $(s_n)_n$ is bounded above, so $\sum_{n\geq 1} \frac{1}{n^2}$ is convergent.

1.2.1 The integral test

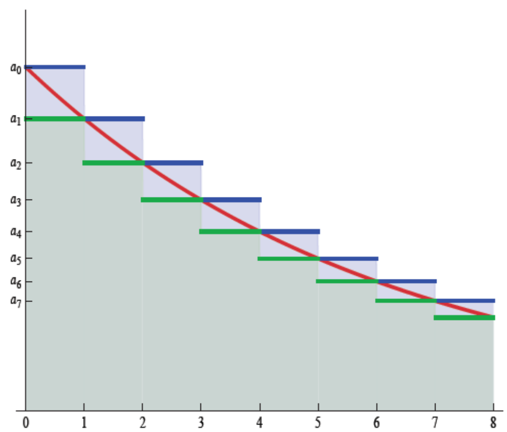
In this subsection we discuss a very simple, but powerful, idea: in order to prove that certain series converge or diverge, we may compare them to integrals.

Definition 1.2.2. In general, if we have a function f defined from $x = a$ to $x = \infty$, we define

$$\int_a^\infty f(x)dx = \lim_{b\to\infty} \int_a^b f(x)dx,$$

and we say that this improper integral converges if the limit converges, and that it diverges if the limit diverges.

Theorem 1.2.2. *Suppose that f is a positive, decreasing, and continuous function, and that $a_n = f(n)$. Then $\sum_{n\geq 1} a_n$ converges if and only if the improper integral $\int_1^\infty f(x)dx$ converges.*



Proof. Suppose we have a function f which is positive and decreasing, such that $f(n) = a_n$ for $n = 0, 1, 2, 3, ...$ Consider the above picture, which shows the graph of f in red. It is clear from the picture that

$$\text{green area } \leq \int_0^\infty f(x)dx \leq \text{green area } + \text{ blue area.} \tag{1.2.1}$$

Now let's figure the area of the green shaded region. This can be subdivided into rectangles of width 1 by drawing vertical line segments from the $x-$axis up to the top of the green area at each integer. If you do this, you will find that (by reading the heights of the rectangles off of the scale on the $y-$axis)

- the area of the first green rectangle is its height times its width, i.e. is $a_1.1 = a_1$,
- the area of the second green rectangle is $a_2.1 = a_2$,

- the area of the third green rectangle is $a_3.1 = a_3$, etc.

Thus the total green area is $a_1 + a_2 + a_3 + ... = \sum_{n=1}^{\infty} a_n$.
Now let's figure the total area of the blue and green regions. As with just the green regions, the combined blue and green regions can be divided into rectangles of width 1. This time, however,

- the area of the left-most rectangle (whose bottom part is green but whose top part is blue) is its height times its width, i.e. is $a_0.1 = a_0$,
- the area of the second rectangle is $a_1.1 = a_1$,
- the area of the third rectangle is $a_2.1 = a_2$, etc.

Thus the total combined blue and green area is $a_0 + a_1 + a_2 + a_3 + ... = \sum_{n=0}^{\infty} a_n$.
Plugging these computations into equation (1.2.1), we see that

$$\sum_{n=1}^{\infty} a_n \leq \int_0^{\infty} f(x)dx \leq \sum_{n=0}^{\infty} a_n.$$

From this inequality we can prove the theorem. First, assume that $\int_0^{\infty} f(x)dx$ converges. This means that the green shaded area, being less than the finite number $\int_0^{\infty} f(x)dx$, is also finite, i.e. $\sum_{n=1}^{\infty} a_n$ converges. Since the starting index of a series is irrelevant to whether or not it converges, $\sum_{n=0}^{\infty} a_n$ converges as well.
Now assume that $\int_0^{\infty} f(x)dx$ diverges, i.e. that the area under the red function is infinite. This means that the combined green and blue shaded area, being greater than the area under the function $f(x)$ (which is $\int_0^{\infty} f(x)dx$), must also be infinite. Therefore $\sum_{n=0}^{\infty} a_n$ diverges. This completes the proof of the Integral Test. □

Example 1.2.2. Does the series $\sum_{n\geq 1} \frac{1}{n^2+1}$ converge or diverge?

We began the section by considering $\sum 1/n$ and $\sum 1/n^2$. What about $\sum 1/n^p$ for other values of p? We can evaluate the integral of $1/x^p$, so the Integral Test can be used to determine which of these series converge. Because series of this form occur so often, we record this fact as its own test.

1.2.2 The p-Series Test

Theorem 1.2.3. *The series* $\sum 1/n^p$ *converges if and only if* $p > 1$.

Proof. When $p = 1$, we already know that the series diverges ($1/n$ is the Harmonic series). For other values of p, we simply integrate the improper integral from the **Integral Test:**

$$\begin{aligned}\int_1^{\infty} \frac{1}{x^p}dx &= \lim_{b\to\infty}\int_1^b \frac{1}{x^p}dx \\ &= \lim_{b\to\infty}\left(\frac{b^{1-p}}{1-p}\right) - \frac{1}{1-p} \\ &= \begin{cases} \dfrac{1}{1-p}, & \text{if } p > 1 \\ \infty, & \text{if } p \leq 1. \end{cases}\end{aligned}$$

□

The Integral Test Remainder Estimates. Suppose that f is a positive, decreasing, and continuous function, and that $a_n = f(n)$. Then the error in the nth partial sum of $\sum a_n$ is bounded by an improper integral:

$$\left| s_n - \sum_{n=1}^{\infty} a_n \right| \leq \int_n^{\infty} f(x)dx.$$

The proof of the Integral Test Remainder Estimate is almost identical to the proof of the Integral Test itself, so we content ourselves with an example.

Example 1.2.3. Bound the error in using the fourth partial sum s_4 to approximate $\sum_{n\geq 1} \frac{1}{n^2}$.

The error in this case is the difference between s_n and the true value of the series:

$$Error = \left| s_4 - \sum_{n=1}^{\infty} a_n \right| = \left| s_n - \sum_{n=1}^{\infty} a_n \right| \leq \int_4^{\infty} \frac{1}{x^2}dx = \lim_{b\to\infty}\int_4^b \frac{1}{x^2}dx = \frac{1}{4}.$$

This is not a very good bound. As we mentioned earlier, EULER approximated the value of this series to within 17 decimal places. How many terms would we need to take to get the upper bound on the error from the Integral Test Remainder Estimates under 10^{-17}?

1.2.3 The Comparison Test

Theorem 1.2.4. *Suppose that* $0 \leq a_n \leq b_n$ *for sufficiently large* n.

- *If* $\sum a_n$ *diverges, then* $\sum b_n$ *also diverges.*
- *If* $\sum b_n$ *converges, then* $\sum a_n$ *also converges.*

Proof. Let s_n denote the nth partial number of a_n and t_n denote the nth partial sum of b_n, so

$$s_n = a_1 + a_2 + ... + a_n, \qquad t_n = b_1 + b_2 + ... + b_n.$$

From our hypotheses (that $0 \leq a_n \leq b_n$ for all n), we know that $s_n \leq t_n$ for all n.

- First suppose that $\sum b_n$ converges, which implies by our definitions that $t_n \to \sum b_n$ as $n \to \infty$. The sequence s_n is nonnegative and monotonically increasing because $s_{n+1} - s_n = a_n \geq 0$ for all n, and

$$0 \leq s_n \leq t_n \leq \sum b_n$$

  so the sequence s_n has a limit by the Monotone Convergence Theorem that the series $\sum a_n$ converges.

- Now suppose that $\sum a_n$ diverges. Because the terms a_n are nonnegative, the only way that $\sum a_n$ can diverge is if $s_n \to \infty$ as $n \to \infty$. Therefore the larger partial sums t_n must also tend to ∞ as $n \to \infty$, so the series $\sum b_n$ diverges as well.

□

Remark 1.2.1. In practice, we will almost always compare with a geometric series or a $p-$series.

Example 1.2.4.
- Show that the series $\sum_{2}^{\infty} \frac{1}{n^2 - 1}$ converges
- Show that the series $\sum_{1}^{\infty} \frac{n}{\sqrt{n^4 + 7}}$ diverges.
- Does the series $\sum_{1}^{\infty} \frac{n \ln n}{\sqrt{(n+3)^5}}$ converge or diverge?

Solution:

- Because $n^2-1 \geq (n-1)^2$, we have that

$$\sum_2^\infty \frac{1}{n^2-1} = \frac{1}{3}+\frac{1}{8}+\frac{1}{15}+.... \leq \frac{1}{1}+\frac{1}{4}+\frac{1}{9}+.... = \sum_1^\infty \frac{1}{n^2},$$

so the series converges by comparison to $\frac{1}{n^2}$.

- We have $\frac{n}{\sqrt{n^4+7}} \geq \frac{n}{\sqrt{n^4+7n^4}} = \frac{1}{\sqrt{8}n}$, so the series we are interested in diverges by comparison to the harmonic series.

- As we know, $\ln n \leq n^{1/4}$ for sufficiently large n and $(n+3)^5 \geq n^5$, we can use the comparison

$$\frac{n\ln n}{\sqrt{(n+3)^5}} \leq \frac{n.n^{1/4}}{n^{5/2}} = \frac{1}{n^{5/4}}.$$

Because $\sum \frac{1}{n^{5/4}}$ is a convergent $p-$series, $\sum_1^\infty \frac{n\ln n}{\sqrt{(n+3)^5}}$ converges by the Comparison Test.

If a series converges by the Comparison Test, then we have the following remainder estimate

The Comparison Test Remainder Estimate. Let $\sum a_n$ and $\sum b_n$ be series with positive terms such that $a_n \leq b_n$ for $n \geq N$. Then for $n \geq N$, the error in the nth partial sum of $\sum a_n$, s_n, is given by

$$\left| s_n - \sum_1^\infty a_n \right| \leq b_{n+1}+b_{n+2}+b_{n+3}+.......$$

Example 1.2.5. How many terms are needed to approximate $\sum_1^\infty \frac{1}{2^n+n}$ to within $\frac{1}{10}$?

1.2.4 The Limit Comparison Test

Theorem 1.2.5. *Let $\sum a_n$ and $\sum b_n$ be two positive term series such that*

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L,$$

then

- *if L is non-zero and finite, then $\sum a_n$ and $\sum b_n$ converge or diverge together,*

- *if* $L = 0$, *then* $\sum a_n$ *is convergent if* $\sum b_n$ *is convergent,*
- *if* $L = \infty$, *then* $\sum a_n$ *is divergent if* $\sum b_n$ *is divergent.*

Proof. □

Example 1.2.6. Examine the convergence of the series:

$$(a)\ \sum_{n\geq 1} \frac{\sqrt{n^2-1}}{n^4+1}, \quad (b)\ \sum_{n\geq 1} \sqrt{n^2+1}-n, \quad (c)\ 1+1/2^2+2^2/3^3+3^3/4^4+...$$

1.2.5 The Ratio Test (D'ALEMBERT Ratio Test)

There are a great many series for which the above tests are not ideally suited, for example, the series

$$\sum_1^\infty \frac{4^n}{n!}.$$

Integrating the terms of this series would be difficult. We could try a comparison, but again, the solution is not particular obvious. Instead, the simplest approach to such a series is the following test due to Jean le Rond d'Alembert (1717–1783).

Theorem 1.2.6. *Suppose that* $\sum a_n$ *is a series with positive terms and let* $L = \lim\limits_{n\to\infty} a_{n+1}/a_n$.

- *If* $L < 1$ *then* $\sum a_n$ *converges.*
- *If* $L > 1$ *then* $\sum a_n$ *diverges.*
- *If* $L = 1$ *or the limit does not exist then the Ratio Test is inconclusive.*

Proof. You should think of the Ratio Test as a generalization of the Geometric Series Test. For example, if $(a_n) = (ar^n$ is a geometric sequence then $\lim\limits_{n\to\infty} a_{n+1}/a_n = r$, and we know these series converge if and only if $|r| < 1$.
If $L > 1$ then the sequence a_n is increasing (for sufficiently large n), and therefore the series diverges by the **Test for Divergence.**
Now suppose that $L < 1$. Choose a number r sandwiched between L and $1 : L < r < 1$. Because $a_{n+1}/a_n \to L$, there is some integer N such that $0 \leq a_{n+1}/a_n \leq r$. For all $n \geq N$. Set $a = a_N$. Then we have

$$a_{N+1} \leq ra_N = ar,$$

and

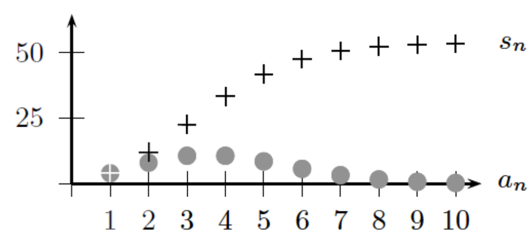
$$a_{N+2} \leq r a_{N+1} < a r^2,$$

and in general, $a_{N+k} < a r^k$. Therefore for sufficiently large n (namely, $n \geq N$), the terms of the series $\sum a_n$ are bounded by the terms of a convergent geometric series (since $0 < r < 1$), and so $\sum a_n$ converges by the **Comparison Test.** $\square$

Example 1.2.7. 1. Does the series $\sum_{1}^{\infty} \frac{4^n}{n!}$ converge or diverge?

2. Does the series $\sum_{1}^{\infty} \frac{n^2}{2^n}$ converge or diverge?

3. Does the series $\sum_{1}^{\infty} \frac{10^n}{n 2^{2n+1}}$ converge or diverge?



Remark 1.2.2.

- It is important to note that the Ratio Test is always inconclusive for series of the form $\sum \frac{\text{polynomial}}{\text{polynomial}}$. As an example, we consider the harmonic series $\sum 1/n$ and $\sum 1/n^2$.
- **When it is a good idea to use the Ratio Test:** The Ratio Test is likely to work well for a series whose terms contain only things that are multiplied and divided, and for series whose terms contain expressions like $2^n, 3^n, c^n, n^n, n!$, etc.

1.2.6 Cauchy's nth Root Test(or The root test)

Theorem 1.2.7. *If $\sum a_n$ is a positive term series such that*

$$\lim_{n\to\infty} (a_n)^{1/n} = L,$$

then

- *If* $L < 1$ *then* $\sum a_n$ *converges.*
- *If* $L > 1$ *then* $\sum a_n$ *diverges.*
- *If* $L = 1$ *or the limit does not exist then the Ratio Test is inconclusive.*

Example 1.2.8. Examine the convergence of the following series:

- $\displaystyle\sum_{n\geq 1} \frac{(n-\ln n)^n}{2^n n^n}$,
- $\displaystyle\sum_{n\geq 1} 3^{(-1)^n - n}$,
- $\displaystyle\sum_{n\geq 1} \left(\frac{n}{n+1}\right)^{n^2}$.

Remark 1.2.3. Let $\sum a_n$ be a positive term series

1. If $\lim\limits_{n\to\infty} a_{n+1}/a_n$ exists, then $\lim\limits_{n\to\infty} a_n^{1/n}$ exists also, and we have
$$\lim_{n\to\infty} a_{n+1}/a_n = \lim_{n\to\infty} a_n^{1/n},$$
2. The reciprocal of (1) is false, in general.

1.2.7 Raabe's Test

Theorem 1.2.8. *If* $\sum a_n$ *is a positive term series such that*
$$\lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}} - 1\right) = L,$$
then

- $\sum a_n$ *is convergent if* $L > 1$,
- $\sum a_n$ *is divergent if* $L < 1$,
- *Test fails if* $L = 1$.

Remark 1.2.4. The Raabe's test is used when D'Alembert's ratio test is failed and the ratio a_n/a_{n+1} does not contains the number e.

Example 1.2.9. Examine the convergence of the series:

$$1+\frac{3}{7}+\frac{3.6}{7.10}+\frac{3.6.9}{7.10.13}+\frac{3.6.9.12}{7.10.13.16}+.....$$

In the case where the limit L of the Raabe's Test is equal to 1, a refinement is still possible:

Theorem 1.2.9. *If $\sum a_n$ is a positive term series such that*

$$\lim_{n\to\infty}\left[n\left(\frac{a_n}{a_{n+1}}-1\right)-1\right]\ln n = L,$$

then

- *$\sum a_n$ is convergent if $L>1$,*
- *$\sum a_n$ is divergent if $L<1$,*
- *Test fails if $L=1$.*

Proposition 1.2.1. ***(Bertrand' series)***[1]
Let α and β two real numbers. The series $\sum_{n\geq 2}\frac{1}{n^\alpha(\ln n)^\beta}$ converges if and only if $(\alpha>1)$, or $(\alpha=1$ and $\beta>1)$.

Proof.

1. If $\alpha>1$, there exists a real constant such that $1<\gamma<\alpha$. Then

$$n^\gamma\frac{1}{n^\alpha(\ln n)^\beta}=\frac{1}{n^{\alpha-\gamma}(\ln n)^\beta}\to_{n\to\infty}0,$$

since $\alpha-\gamma>0$. then, from Riemann's rule the series converges.

2. If $\alpha<1$. With the same manner, we have

$$n\frac{1}{n^\alpha(\ln n)^\beta}=\frac{1}{n^{\alpha-1}(\ln n)^\beta}\to_{n\to\infty}\infty,$$

so the series diverges.

3. assume that $\alpha=1$.

[1] Joseph Bertrand (1822-1900), French mathematician

a If $\beta \leq 0$, for all $n \geq 3$ we get

$$\frac{1}{n(\ln n)^{\beta}} \geq \frac{1}{n},$$

thus, from the comparaison test the series $\sum_{n\geq 2} \frac{1}{n(\ln n)^{\beta}}$ diverges.

b If $\beta > 0$. The function $x \mapsto f(x) = \frac{1}{x(\ln x)^{\beta}}$ is positive and decreasing on $[2, +\infty[$, thus by using integral test (see theorem 1.2.2).

(i) If $\beta \neq 1$, we have

$$\int_2^x \frac{dt}{t(\ln t)^{\beta}} = \left[\frac{(\ln t)^{1-\beta}}{1-\beta}\right]_2^x = \frac{1}{1-\beta}\left((\ln x)^{1-\beta} - (\ln 2)^{1-\beta}\right),$$

we deduce that

$$\lim_{x\to\infty} \int_2^x \frac{dt}{t(\ln t)^{\beta}} = \begin{cases} \dfrac{(\ln 2)^{1-\beta}}{1-\beta}, & \text{if } \beta > 1 \\ +\infty, & \text{if } 0 < \beta < 1. \end{cases}$$

Then, the series converges if $\beta > 1$, and diverges if $0 < \beta < 1$.

(ii) Now, if $\beta = 1$.

$$\int_2^x \frac{dt}{t(\ln t)} = \ln(\ln(x)) - \ln(\ln(2)) \longrightarrow_{x\to\infty} = \infty.$$

Thus, the series $\sum_{n\geq 2} \frac{1}{n(\ln n)}$ diverges.

□

1.3 Alternating Series

We have focused almost exclusively on series with positive terms up to this point. In this short section we begin to delve into series with both positive and negative terms, presenting a test which works for many series whose terms alternate in sign.

Definition 1.3.1. A series with terms alternately positive and negative is called an alternating series. For example, $1 - 1/2 + 1/3 - 1/4 + 1/5 - 1/6 + 1/7 -$ .... The general form of alternating series is given by

$$a_1 - a_2 + a_3 - a_4 + a_5 - a_6 + a_7 - a_8 + ... = \sum_{n\geq 1}(-1)^{n+1}a_n, \quad (a_n > 0)$$

or

$$-a_1 + a_2 - a_3 + a_4 - a_5 + a_6 - a_7 + a_8 - .... = \sum_{n\geq 1}(-1)^{n}a_n, \quad (a_n > 0)$$

1.3.1 The Alternating Series Test

Theorem 1.3.1. *Suppose that the sequence (a_n) satisfies the three conditions:*

1. *$a_n \geq 0$ for sufficiently large n,*
2. *$a_{n+1} \leq a_n$ for sufficiently large n (i.e., a_n is monotonically decreasing), and*
3. *$a_n \to 0$ as $n \to \infty$.*

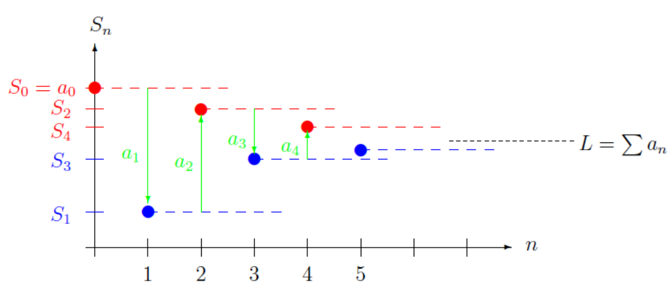
Then the alternating series $\sum_{n\geq 1}(-1)^{n+1}a_n$ converges.

An explanation of why the Alternating Series Test works:

We are going for simplicity:

1. that the starting index of the series is $n = 0$, and
2. that the terms $a_0, a_2, a_4, ...$ are all positive and the terms $a_1, a_3, a_5, ...$ are all negative.

Now for each $n \geq 0$, let $S_n = a_0 + a_1 + ... + a_n$ be the n^{th} partial sum. Consider the following picture which plots S_n vertically and n horizontally (an explanation of the picture is below)



The partial sums are plotted with the red and blue points (the S_n where n is even are the red points and the S_n where n is odd are the blue points). Notice that to get from one partial sum to the next, i.e. to get from S_n to S_{n+1}, you have to add a_{n+1}. This is indicated by the green arrows. Now we will use the hypotheses of the Alternating Series Test:

- By hypothesis (1), the an alternate in sign. In our case, all the even an are negative and all the odd an are positive. Therefore, whenever n is odd, S_n is below the previous partial sum, and whenever n is even, S_n is above the previous partial sum. This makes each blue dot lower than each preceding red dot, and each red dot above each preceding blue dot.

- By hypothesis (3), $|a_n| \geq |a_{n+1}|$. Since $|a_n|$ is the length of the n^{th} green arrow, we are assured by hypothesis (3) that the green arrows are getting shorter as n increases. Thus the red and blue dots are getting closer and closer together.

- By hypothesis (2), $\lim\limits_{n\to\infty} |a_n| = 0$. This means that since $|a_n|$ is the length of the green arrows, the length of these green arrows is going to zero as n increases. Thus the red and blue dots are both approaching the same height, so they have the same limit. This limit L is the limit of the partial sums, so by definition the infinite series converges to L.

This concludes the explanation of why the Alternating Series Test works.

Example 1.3.1. Does the series $\sum\limits_{n\geq 1} (-1)^{n+1} \dfrac{2n+3}{3n+4}$ converge or diverge?

This series does alternate in sign, and $\dfrac{2n+3}{3n+4}$ is decreasing, but $\dfrac{2n+3}{3n+4} \to 2/3 \neq 0$, so the series diverges by the Test for Divergence.

Remark 1.3.1. Note that in the solution of Example 1.3.1, we did not appeal to the Alternating Series Test, but instead used the Test for Divergence. The Alternating Series Test never shows that series diverge.

1.3.2 Absolute and Conditional Convergence

Definition 1.3.2. A series $\sum\limits_{n\geq 1} a_n$ is said to be **absolutely convergent** if the series $\sum\limits_{n\geq 1} |a_n|$ is convergent.

Definition 1.3.3. If $\sum_{n\geq 1} a_n$ is converges but $\sum_{n\geq 1} |a_n|$ diverges, then the series $\sum_{n\geq 1} a_n$ is called **conditionally convergent.**

Example 1.3.2.
- Test the convergence of the series: $5 - 10/3 + 20/9 - 40/27 + ...$

Test the convergence and absolute convergence of the series:

- $\dfrac{1}{1\times 3} - \dfrac{1}{2\times 4} + \dfrac{1}{3\times 5} - \dfrac{1}{4\times 6} + ...$
- $\sum_{n\geq 1} \dfrac{(-1)^{n-1}}{\sqrt{n}}.$

Remark 1.3.2.
- Every absolutely convergent series is a convergent series but the converse is not true.
- If $\sum_{n\geq 1} a_n$ is a series of positive terms, then $\sum_{n\geq 1} |a_n| = \sum_{n\geq 1} a_n$. Therefor the concepts of **convergence** and **absolutely convergence** are the same. Thus, any convergent series of positive terms is also absolutely convergent.

The proof of the **Alternating Series Test** implies the following very simple bound on remainders of these series.

Theorem 1.3.2. ***The Alternating Series Remainder Estimates.***
Suppose that the sequence (b_n) satisfies the three conditions of the Alternating Series Test:

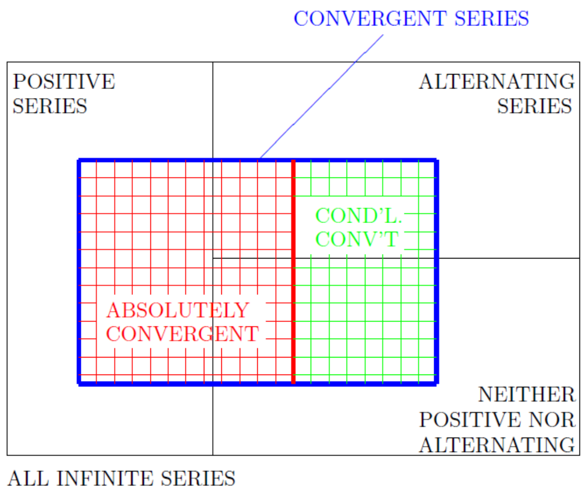
1. $b_n \geq 0$ *for sufficiently large* n*,*
2. $b_{n+1} \leq b_n$ *for sufficiently large* n *(i.e.,* b_n *is monotonically decreasing), and*
3. $b_n \to 0$ *as* $n \to \infty$*.*

Then if $n \geq N$*, the error in the* n*th partial sum of* $\sum_{n\geq 1} (-1)^{n+1} b_n$ *is bounded by* b_{n+1}

$$\left| s_n - \sum_{n\geq 1} (-1)^{n+1} b_n \right| \leq b_{n+1}.$$

Example 1.3.3. How many terms of the alternating series must we add to approximate the true sum with error less than $1/10000$?

Notice that there are now three disjoint classes of infinite series: those which converge **absolutely,** those which converge **conditionally**, and those which **diverge**. Putting all this together, we have the following diagram illustrating the various possibilities for infinite series. This diagram is extremely important to understand:



1.3.3 The Cauchy Product of Infinite Series

The Cauchy product of two infinite series $\sum_n a_n$ and $\sum_n b_n$ is defined to be the series

$$\sum_n c_n, \quad \text{where } c_n = \sum_{j=0}^{n} a_j b_{n-j} = a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + ... + a_{n-1} b_1 + a_n b_0.$$

The convergence of $\sum_n a_n$ and $\sum_n b_n$ is **not in itself sufficient to ensure the convergence of the Cauchy product of these series**. Convergence is however assured provided that the series $\sum_n a_n$ and $\sum_n b_n$ are absolutely convergent.

Theorem 1.3.3. *The Cauchy product $\sum_n c_n$ of two absolutely convergent infinite series $\sum_n a_n$ and $\sum_n b_n$ is absolutely convergent, and*

$$\sum_n c_n = \left(\sum_n a_n\right)\left(\sum_n b_n\right).$$

Proof. For each non-negative integer m, let

$$S_m = \{(j,k) \in \mathbb{Z} \times \mathbb{Z} : \quad 0 \leq j \leq m, \quad 0 \leq k \leq m\},$$

$$T_m = \{(j,k) \in \mathbb{Z} \times \mathbb{Z} : \quad 0 \le j,\ 0 \le m, \quad 0 \le j+k \le m\},$$

Now

$$\sum_{n=0}^{m} c_n = \sum_{(j,k)\in T_m} a_j b_k \text{ and } \left(\sum_{n=0}^{m} a_n\right)\left(\sum_{n=0}^{m} b_n\right) = \sum_{(j,k)\in S_m} a_j b_k.$$

Also

$$\sum_{n=0}^{m} |c_n| \le \sum_{(j,k)\in T_m} |a_j||b_k| \le \sum_{(j,k)\in S_m} |a_j||b_k| \le \left(\sum_n |a_n|\right)\left(\sum_n |b_n|\right).$$

since $|c_n| \le= \sum_{j=0}^{n} |a_j||b_{n-j}|$ and the infinite series $\sum_n a_n$ and $\sum_n b_n$ are absolutely convergent. It follows that the Cauchy product $\sum_n c_n$ is absolutely convergent, and is thus convergent. Moreover

$$\begin{aligned}
\left|\sum_n c_n^{2m} - \left(\sum_n^m a_n\right)\left(\sum_n^m b_n\right)\right| &= \left|\sum_{(j,k)\in T_{2m}\setminus S_m} a_j b_k\right| \\
&\le \sum_{(j,k)\in T_{2m}\setminus S_m} |a_j b_k| \\
&\le \sum_{(j,k)\in S_{2m}\setminus S_m} |a_j b_k| \\
&= \left(\sum_n^{2m} |a_n|\right)\left(\sum_n^{2m} |b_n|\right) - \left(\sum_n^{m} |a_n|\right)\left(\sum_n^{m} |b_n|\right),
\end{aligned}$$

since $S_m \subset T_{2m} \subset S_{2m}$. But

$$\begin{aligned}
\lim_{m\to\infty}\left(\sum_n^{2m} |a_n|\right)\left(\sum_n^{2m} |b_n|\right) &= \left(\sum_n^{\infty} |a_n|\right)\left(\sum_n^{\infty} |b_n|\right) \\
&= \lim_{m\to\infty}\left(\sum_n^{m} |a_n|\right)\left(\sum_n^{m} |b_n|\right),
\end{aligned}$$

since the infinite series $\sum_n a_n$ and $\sum_n b_n$ are absolutely convergent. It follows that

$$\lim_{m\to\infty}\left(\sum_n^{2m} c_n - \left(\sum_n^{m} a_n\right)\left(\sum_n^{m} b_n\right)\right) = 0,$$

and hence

$$\sum_n^{\infty} c_n = \lim_{m\to\infty}\sum_n^{2m} c_n = \left(\sum_n^{\infty} a_n\right)\left(\sum_n^{\infty} b_n\right),$$

as required. □

1.4 Summary of convergence(divergence) tests

Having completed our discussion of methods to classify infinite series as **absolutely convergent, conditionally convergent, or divergent,** we now summarize the procedure one should use when trying to solve such a classification problem. To classify an infinite series $\sum a_n$ as absolutely convergent, conditionally convergent or divergent, follow these steps:

1. First, check to see whether the series is a **p-series** or **a geometric** series (or is a sum or difference of series of this type). If it is, use the p..series Test and/or Geometric Series Test (together with linearity properties) to classify the series.

2. If the terms of the series contain only **multiplication** and **division** and contain **exponentials or factorial** terms, use the **Ratio Test.** (If the terms of the series are all polynomials in n, avoid the Ratio Test.)

3. Otherwise, classify the series as positive, negative, alternating, or none of these. If the series is negative, factor out -1 from the series and treat what remains as a positive series.

4. If the series is positive:

   (a) If the terms of the series contain **addition/subtraction** in the denominator, or if they contain **sines or cosines,** try the **Comparison Test.**

   (b) If $\lim_{n\to\infty} a_n \neq 0$, then the series diverges by the n^{th}-Term Test.

   (c) If the terms of the series look like a function you can integrate, try the Integral Test (use this only as a last resort).

5. If the series is alternating, compute $\lim_{n\to\infty} |a_n|$.

   (a) If this limit is not zero, then the series diverges by the n^{th}-Term Test.

   (b) If this limit is zero, you can usually verify that $|a_n| \geq |a_{n+1}|$, then the series converges by the **Alternating Series Test.** In this case, you then have to examine the series $\sum_n |a_n|$:

(i) If $\sum_n |a_n|$ converges, then $\sum_n a_n$ **converges absolutely.**

(ii) If $\sum_n |a_n|$ diverges, then $\sum_n a_n$ **converges conditionally.**

6. If the series is neither positive, negative nor alternating:

(a) If you can show that $\lim_{n\to\infty} |a_n| \neq 0$, then the series diverges by the n^{th}-Term Test.

(b) Forget the original series and try to show that the positive series $\sum_n |a_n|$ converges; in this case the original series **converges absolutely** by definition.

CHAPTER 2

Sequences and Series of Functions

In this chapter, we will use the concepts developed in Chapter 1 to define and study functions which are written as infinite series. What we will find is that several functions we know (like the exponential function, sine, cosine, arctangent, etc.) can be written as an infinite series which is relatively easy to work with. Furthermore, the representation of these and other functions by a class of infinite series called "power series" has many applications.

2.1 Sequences Functions

2.1.1 Uniform Convergence of a Sequence of Functions

Definition 2.1.1. (Pointwise Convergence)
For each $n \in \mathbb{N}$, let $f_n : A \subseteq \mathbb{R} \to \mathbb{R}$ be a real valued function on A. The sequence (f_n) of functions converges pointwise on A to a function f if, for all $x \in A$, the sequence of real numbers $(f_n(x))$ converges to the real number $f(x)$. We often write

$$\lim_{n\to\infty} f_n(x) = f(x) \qquad \text{or} \qquad \lim_{n\to\infty} f_n = f.$$

Thus we have

$$\forall x \in A,\ \forall \epsilon > 0,\ \exists n_0 \in \mathbb{N} \text{ such that, } \forall n \in \mathbb{N},\ \text{if } n > n_0,\ \text{then } |f_n(x) - f(x)| < \epsilon.$$

Remark 2.1.1. There are several notations for the sequences of functions

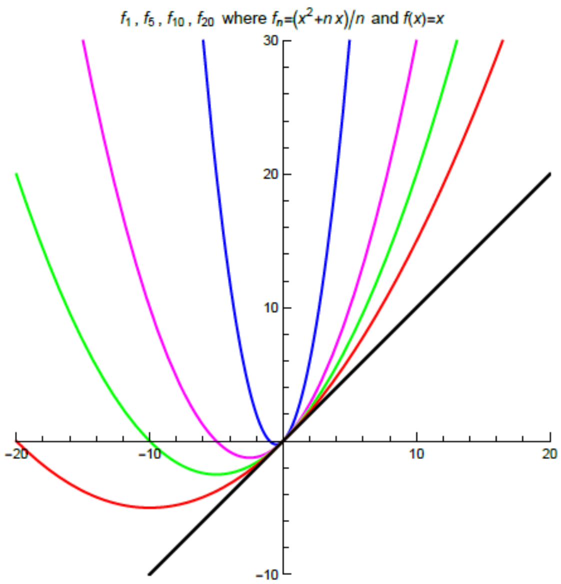
$$(f_n)_{n\in\mathbb{N}}, \qquad (f_n)_{n\geq 0}, \qquad (f_n), \quad \text{or} \quad f_0, f_1, f_2, ..., f_n, ...$$

There is a difference between (f_n) and f_n: (f_n) is the sequence and f_n is the term of rank n, or general term of this sequence.

Example 2.1.1. Let $f_n(x) = \dfrac{x^2+nx}{n} = \dfrac{x^2}{n} + x$, and

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty}\left(\frac{x^2}{n} + x\right) = 0 + x = x.$$

If $f(x) = x$, then $f_n \to f$ as $n \to \infty$. In this case, the functions f_n are everywhere continuous and differentiable, and the limit function is also everywhere continuous and differentiable.



Example 2.1.2. 1. Let $g_n(x) = x^n$ on the set $[0,1]$.

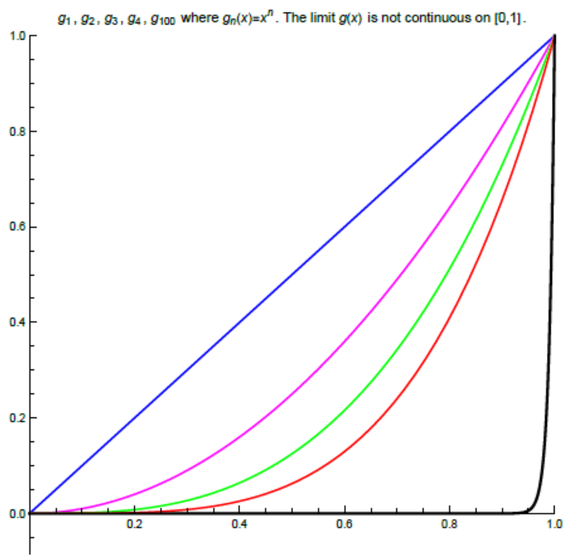
$$\lim_{n\to\infty} g_n(x) = \lim_{n\to\infty} x^n = g(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1, \\ 1 & \text{if } x = 1. \end{cases}$$

In this case, the functions $g_n(x)$ are continuous on $[0,1]$, but the limit function $g(x)$ is not continuous at $x = 1$.

2. $\forall n \in \mathbb{N},\ \forall x \in \mathbb{R}^+, \quad f_n(x) = \dfrac{x^n}{1+x^n}$

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{x^n}{1+x^n} = f(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1, \\ 1/2 & \text{if } x = 1, \\ 1 & \text{if } x > 1. \end{cases}$$

In this case, the functions $f_n(x)$ are continuous on $\mathbb{R}^+$, but the limit function $f(x)$ is not continuous at $x = 1$.



Example 2.1.3. 1. $\forall n \in \mathbb{N}^*,\ \forall x \in \mathbb{R}^+; \quad g_n(x) = \dfrac{\sin(nx)}{n}$: Since we have

$$\forall n \in \mathbb{N}^*,\ \forall x \in \mathbb{R}^+; \quad 0 \leq |g_n(x)| \leq \frac{1}{n}, \quad \text{then, } g_n(x) \to g(x) \equiv 0.$$

We have also, $\forall n \in \mathbb{N}^*,\ \forall x \in \mathbb{R}^+,\ g_n'(x) = \cos(nx)$, and this clearly shows that, in the case where $x \neq 2k\pi (k \in \mathbb{N})$ the functions $(g_n'(x))_{n\geq 1}$ has no limit when n tends to the infinity. For all $x \in \mathbb{R}^+$ and $x \neq 2k\pi (k \in \mathbb{N})$, we have

$$\frac{d}{dx}\left\{\lim_{n\to\infty} g_n\right\}(x) \neq \lim_{n\to\infty} \frac{d}{dx}\left\{(g_n)(x)\right\}.$$

2. Let $(h_n)_{n\geq 0}$ defined on $]0,1[$ such that,

$$\forall n \in \mathbb{N}, \quad \forall x \in]0,1[; \qquad h_n(x) = nx^n.$$

For all $x \in]0,1[$, we have

$$\lim_{n\to\infty} h_n(x) = \lim_{n\to\infty} nx^n = \lim_{n\to\infty} \frac{n}{e^{-n\ln x}} = \lim_{n\to\infty} \frac{-1}{e^{-n\ln x}\ln x} = 0,$$

therefore the sequence $(h_n)_n$ converges on $]0,1[$ to the function $h = 0$. But

$$\int_0^1 [\lim_{n\to\infty} h_n(x)]dx = \int_0^1 0dx = 0 \neq 1 = \lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \int_0^1 h_n(x)dx.$$

Important Question

Suppose $f(x) = \lim_{n\to\infty} f_n(x)$ for all $x \in A$. What **additional hypothesis** would ensure the following?

(i) If each f_n is continuous on A, then f is continuous on A.

(ii) if each f_n is differentiable on A, then f is differentiable on A.

The best general answer to these questions has to do with the concept of **uniform convergence.**

Definition 2.1.2. (Uniform Convergence)
Let (f_n) be a sequence of functions defined on $A \subset \mathbb{R}$. We say that (f_n) converges uniformly on A to the limit function f defined on A if for every $\epsilon > 0$ there exists an $n_0 \in \mathbb{N}$ such that

$$|f_n(x) - f(x)| < \epsilon \qquad \text{for all} \quad x \in A,$$

whenever $n \geq n_0$. Which is equivalent to

$$\forall \epsilon > 0,\ \exists n_0 \in \mathbb{N}\ (n_0 = n_0(\epsilon))/(n > n_0) \Longrightarrow (\forall x \in A, \quad |f_n(x) - f(x)| < \epsilon).$$

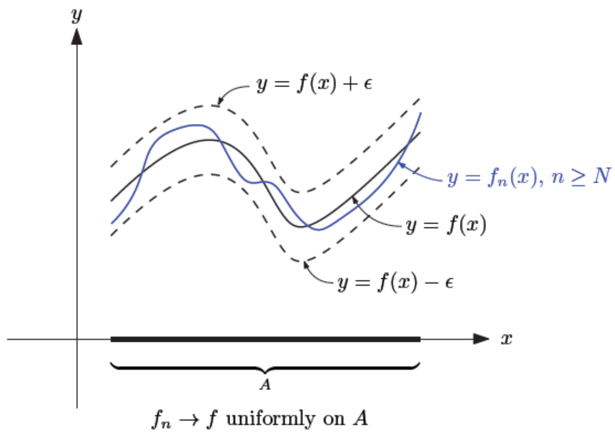
or

$$\lim_{n\to\infty} \|f_n - f\|_\infty = \lim_{n\to\infty} \sup_{x\in A} |f_n(x) - f(x)| = 0.$$

This means that for every $n \geq n_0$, the difference between $f_n(x)$ and $f(x)$ is less than ϵ for every $x \in A$.

Remark 2.1.2. In the definition, the value of n_0 is **independent of** x.

Here is a figure that graphically depicts the definition:



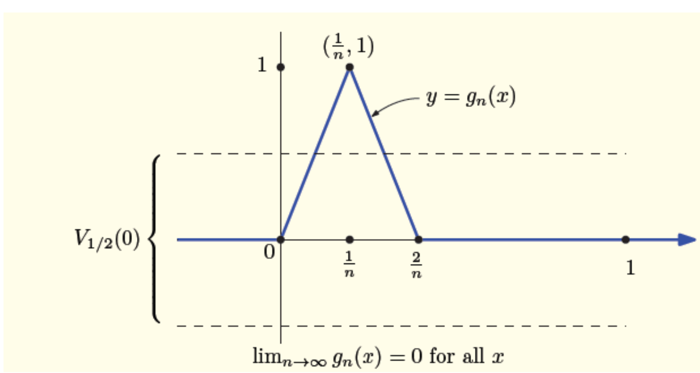
Example 2.1.4. Example of Non-Uniform Convergence

$$g_n(x) = \begin{cases} nx & \text{if } 0 \leq x \leq \dfrac{1}{n} \\ 2 - nx & \text{if } \dfrac{1}{n} \leq x \leq \dfrac{2}{n} \\ 0 & \text{otherwise.} \end{cases} \qquad \lim_{n\to\infty} g_n(x) = g(x) = 0.$$

If $g(x) := 0$, then $(g_n) \to g$ pointwise. Let $\epsilon = \dfrac{1}{2}$ and $x_n = \dfrac{1}{n}$. Then

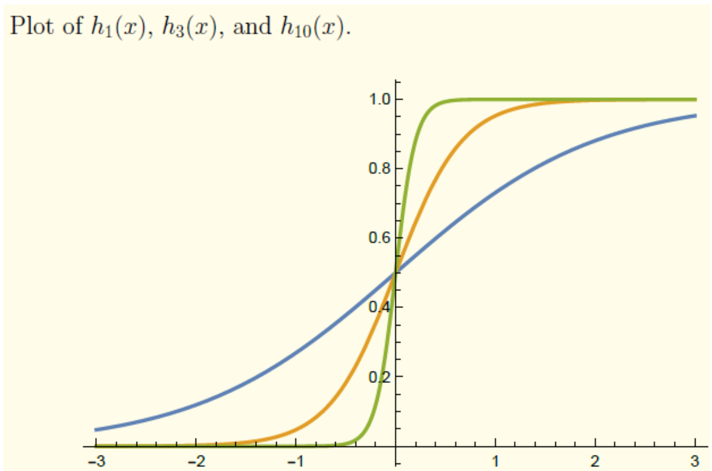
$$|g_n(x_n) - g(x_n)| = |1 - 0| = 1 > \epsilon = \frac{1}{2}.$$

So, it is not true that for all $\epsilon > 0$, there exist an $n_0 \in \mathbb{N}$ large enough such that $n \geq n_0$ implies $|g_n(x) - g(x)| < \epsilon$ for all x. So, (g_n) does not converge to g uniformly.



Example 2.1.5. Another Example of Non-Uniform Convergence For $x \in \mathbb{R}$ and $n \in \mathbb{N}$, let

$$h_n(x) = \frac{e^{nx}}{1+e^{nx}} \quad \Longrightarrow \quad \lim_{n\to\infty} h_n(x) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{1}{2} & \text{if } x = 0 \\ 1 & \text{if } x > 0. \end{cases}$$



Example 2.1.6. Example of Uniform Convergence

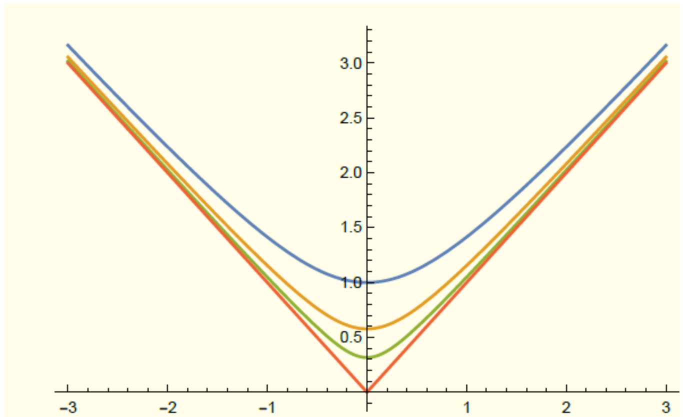
$$f_n(x) = \sqrt{x^2 + \frac{1}{n^2}} \qquad \lim_{n\to\infty} f_n(x) = \sqrt{x^2+0} = \sqrt{x^2} = |x|.$$

So, $f_n(x) \to f(x) = |x|$ pointwise. Let $\epsilon > 0$ be given. Choose $n_0 \in \mathbb{N}$ large enough such that $\frac{1}{n_0} < \epsilon$. Then for any $x \in \mathbb{R}$ and $n \geq n_0$ we have

$$\begin{aligned} |f_n(x) - f(x)| &= \left|\sqrt{x^2+\frac{1}{n^2}} - |x|\right| = \left|\sqrt{x^2+\frac{1}{n^2}} - |x|\right| \left(\frac{\sqrt{x^2+\frac{1}{n^2}}+|x|}{\sqrt{x^2+\frac{1}{n^2}}+|x|}\right) \\ &= \frac{\frac{1}{n^2}}{\sqrt{x^2+\frac{1}{n^2}}+|x|} \leq \frac{\frac{1}{n^2}}{\sqrt{0+\frac{1}{n^2}}+0} \\ &= \frac{1}{n} < \epsilon. \end{aligned}$$

This shows that $(f_n) \to f$ uniformly on $\mathbb{R}$. Note that each $f_n(x)$ is both continuous and differentiable on $\mathbb{R}$, but $f(x) = |x|$ is continuous on $\mathbb{R}$ and not differentiable at $x = 0$.

Plot of $f_1(x), f_3(x), f_{10}(x)$, and $|x|$.



Cauchy Criterion for Uniform Convergence

Theorem 2.1.1. *A sequence of functions (f_n) defined on a set $A \subseteq \mathbb{R}$ converges uniformly on A if and only if for every $\epsilon > 0$ there exists an $n_0 \in \mathbb{N}$ such that $|f_n(x) - f_m(x)| < \epsilon$ whenever $m, n \geq n_0$ and $x \in A$.*

Proof. $(\Longrightarrow)$ Assume the sequence (f_n) converges uniformly on A to a limit function f. Let $\epsilon > 0$ **be given.** Then there exists an $n_0 \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \frac{\epsilon}{2}$, whenever $n \geq n_0$ and $x \in A$. Then if $n, m \geq n_0$ and $x \in A$, we have

$$\begin{aligned} |f_n(x) - f_m(x)| &= |f_n(x) - f(x) + f(x) - f_m(x)| \\ &\leq |f_n(x) - f(x)| + |f(x) - f_m(x)| = \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

$(\Longleftarrow)$ Conversely, assume that for every $\epsilon > 0$ there exists an $n_0 \in \mathbb{N}$ such that $|f_n(x) - f_m(x)| < \epsilon$. whenever $m, n \geq n_0$ and $x \in A$. This hypothesis implies that, for each $x \in A$, $(f_n(x))$ is a Cauchy sequence. By Cauchy's Criterion, this sequence converges to a point, which we will call $f(x)$. So, the uniformly Cauchy sequence converges pointwise to the function $f(x)$. We must show that the convergence is also uniform. For the value of ϵ given above, we use the corresponding n_0. Then for $n, m \geq n_0$ and all $x \in A$,

$$|f_n(x) - f_m(x)| < \epsilon.$$

Taking the limit as $m \to \infty$ gives

$$|f_n(x) - f(x)| \leq \epsilon \qquad \text{for all} \quad x \in A,$$

which shows that (f_n) converges uniformly to f on A. This completes the proof. $\square$

Proposition 2.1.1. *Let $(f_n) \to f$ pointwise on $A \subseteq \mathbb{R}$. If there exists a real positive sequence (α_n) such that*

- $\lim\limits_{n\to\infty} \alpha_n = 0,$
- $|f_n(x) - f(x)| \leq \alpha_n \qquad \text{for all } x \in A.$

Then $(f_n) \to f$ uniformly on A.

Example 2.1.7. Consider the sequence

$$f_n(x) = \frac{e^{-nx}}{n} \qquad \text{for all } x \in \mathbb{R}^+ \text{ and } n \geq 1.$$

$(f_n) \to f(x) = 0$ pointwise on $\mathbb{R}^+$. for all $n \geq 1$, we have:

$$|f_n(x) - f(x)| = \frac{e^{-nx}}{n} \leq \frac{1}{n} = \alpha_n \qquad \text{for all } x \in \mathbb{R}^+$$

Since $\dfrac{1}{n} \to_{n\to\infty} 0$, we conclude that $(f_n) \to 0$ uniformly on $\mathbb{R}^+$.

2.1.2 Uniform Convergence and Continuity

Theorem 2.1.2. ***Continuous Limit Theorem***
Let (f_n) be a sequence of functions defined on $A \subseteq \mathbb{R}$ that converges uniformly on A to a function f. If each f_n is continuous at $c \in A$, then f is continuous at c.

Proof. Let $\epsilon > 0$ be given. Fix $c \in A$. Since $f_n \to f$ uniformly, there exists an $n_0 \in \mathbb{N}$ such that

$$|f_{n_0}(x) - f(x)| < \frac{\epsilon}{3} \qquad \text{for all} \quad x \in A.$$

Since f_{n_0} is continuous at c, there exists $\delta > 0$ such that $|f_{n_0}(x) - f_{n_0}(c)| < \dfrac{\epsilon}{3}$ whenever

$|x-c|<\delta$. If $|x-c|<\delta$, then

$$\begin{aligned}
|f(x)-f(c)| &= |f(x)-f_{n_0}(x)+f_{n_0}(x)-f_{n_0}(c)+f_{n_0}(c)-f(c)| \\
&\leq |f(x)-f_{n_0}(x)|+|f_{n_0}(x)-f_{n_0}(c)|+|f_{n_0}(c)-f(c)| \\
&< \frac{\epsilon}{3}+\frac{\epsilon}{3}+\frac{\epsilon}{3}=\epsilon.
\end{aligned}$$

The first and third $\frac{\epsilon}{3}$ are due to uniform convergence and the choice of n_0. The second $\frac{\epsilon}{3}$ is due to the choice of δ. This shows that f is continuous at c, as desired. □

Remark 2.1.3. The converse of theorem 2.1.2 is generally false: A sequence of continuous functions can converge to a continuous function, without the convergence being uniform.

Example 2.1.8. Let for all $n \in \mathbb{N}$ $f_n(x)=\dfrac{1}{nx+1}$, $x \in I = \,]0,1[$ It's clear that $(f_n) \to f(x)=0$ pointwise on I that all f_n are continuous on I and f is continuous also, but (f_n) does not converge uniformly to $f(x)=0$ on I. Since

$$\sup_{x\in I}|f_n(x)-f(x)| = 1 \not\longrightarrow 0 \text{ as } n\to\infty.$$

Theorem 2.1.3. ***(Dini's theorem)***

Let (f_n) be a sequence of real functions defined on the bounded and closed interval $[a,b]$, and assume that

- *Each (f_n) is continuous on $[a,b]$ for large n,*
- $f_n \xrightarrow[n\to\infty]{P.C} f$ *on $[a,b]$,*[1]
- *f is continuous on $[a,b]$,*
- *(f_n) is increasing(or decreasing) on $[a,b]$.*

$$\forall n \geq N_1\ \forall x \in [a,b],\ f_{n+1}(x) \geq f_n(x), \qquad \text{or,}\ \ f_{n+1}(x) \leq f_n(x).$$

Then (f_n) converges uniformly on $[a,b]$.

Remark 2.1.4. The condition "$I=[a,b]$ is closed and bounded" is really important in Dini's theorem. It is thanks to her that we were able to write, in the proof of this theorem, that each function continues f_n is bounded on I and that it reaches its upper limit at a point x_n of I.

[1]P.C means Pointwise Convergence.

2.1.3 Uniform Convergence and Integration

Theorem 2.1.4. *Let $f_n : A \longrightarrow \mathbb{R}$ be a real-valued function on A. Suppose that*

- *f_n converges uniformly on $[a,b]$ to a function f and*
- *each f_n is continuous on $[a,b]$.*

Then

$$\lim_{n\to\infty}\int_a^b f_n(x)dx = \int_a^b \lim_{n\to\infty} f_n(x)dx = \int_a^b f(x)dx.$$

Example 2.1.9. Find the value of $\lim\limits_{n\to\infty}\int_0^1 \frac{nx+1}{nx^2+x+n}dx$.

Let $f_n(x) = \frac{nx+1}{nx^2+x+n} = \frac{x+1/n}{x^2+x/n+1} \to \frac{x}{x^2+1} = f(x)$, as $n\to\infty$. Moreover,

$$\begin{aligned} |f_n(x)-f(x)| &= \left|\frac{(nx+1)(x^2+1)-x(nx^2+x+n)}{(x^2+1)(nx^2+x+n)}\right| \\ &= \left|\frac{1}{(x^2+1)(nx^2+x+n)}\right| \\ &= \frac{1}{n}\left|\frac{1}{(x^2+1)(x^2+x/n+1)}\right| \leq \frac{1}{n}, \end{aligned}$$

since $1+x^2 \geq 1$ and $x^2+x/n+1 \geq 1$. Clearly this means that

$$\sup_{0\leq x\leq 1} |f_n(x)-f(x)| \leq \frac{1}{n},$$

as $n\to\infty$. The convergence is therefore uniform and

$$\begin{aligned} \lim_{n\to\infty}\int_0^1 \frac{nx+1}{nx^2+x+n}dx &= \int_0^1 \lim_{n\to\infty}\frac{nx+1}{nx^2+x+n}dx \\ &= \int_0^1 \frac{x}{x^2+1} \\ &= \left[\frac{1}{2}\ln(1+x^2)\right]_0^1 = \frac{\ln 2}{2}. \end{aligned}$$

2.1.4 Uniform Convergence and Differentiation

Theorem 2.1.5. ***Differentiable Limit Theorem***
Let $(f_n)\to f$ pointwise on the closed interval $[a,b]$ and assume each f_n is differentiable. If $(f_n')\to g$ uniformly on $[a,b]$, then f is differentiable and $f'=g$.

Proof. Fix $c \in [a,b]$ and let $\epsilon > 0$. We will show that there exists $\delta > 0$ such that

$$\left|\frac{f(x)-f(c)}{x-c} - g(c)\right| < \epsilon,$$

whenever $0 < |x-c| < \delta$ and $x \in [a,b]$.
For $x \neq c$, consider the following:

$$\begin{aligned} \left|\frac{f(x)-f(c)}{x-c} - g(c)\right| &\leq \underbrace{\left|\frac{f(x)-f(c)}{x-c} - \frac{f_n(x)-f_n(c)}{x-c}\right|}_{iii} \\ &+ \underbrace{\left|\frac{f_n(x)-f_n(c)}{x-c} - f_n'(c)\right|}_{ii} + \underbrace{\left|f_n'(c) - g(c)\right|}_{i} \end{aligned} \tag{2.1.1}$$

Since $\lim_{n\to\infty} f_n'(c) = g(c)$, there exists $n_1 \in \mathbb{N}$ such that

$$\left|f_n'(c) - g(c)\right| < \frac{\epsilon}{3} \qquad \text{for all } n \geq n_1. \tag{2.1.2}$$

From Cauchy's Criterion for uniform convergence, since the sequence (f_n') converges uniformly to g, there exists an $n_2 \in \mathbb{N}$ such that

$$\left|f_n'(x) - f_m'(x)\right| < \frac{\epsilon}{3} \quad \text{whenever } m,n \geq n_2 \text{ and } x \in [a,b].$$

Set $N = \max\{n_1, n_2\}$, the function f_N is differentiable at c. So there exists $\delta > 0$ such that

$$\left|\frac{f_N(x)-f_N(c)}{x-c} - f_N'(c)\right| < \frac{\epsilon}{3} \qquad \text{whenever } 0 < |x-c| < \delta \text{ and } x \in [a,b]. \tag{2.1.3}$$

We'll show this δ will suffice.
Suppose $0 < |x-c| < \delta$ and $m \geq N$. By the Mean Value Theorem applied to $f_m - f_N$ on the interval $[c,x]$ (if $x < c$ the argument is the same) there exists $\alpha \in (c,x)$ such that

$$f_m'(\alpha) - f_N'(\alpha) = \frac{[f_m(x) - f_N(x)] - [f_m(c) - f_N(c)]}{x-c}.$$

By our choice of N, $\left|f'_m(\alpha) - f'_N(\alpha)\right| < \dfrac{\epsilon}{3}$ and so

$$\left|\frac{[f_m(x) - f_N(x)] - [f_m(c) - f_N(c)]}{x - c}\right| < \frac{\epsilon}{3}.$$

Since $f_m \to f$ as $m \to \infty$, by the **Algebraic Order Limit Theorem**

$$\left|\frac{f(x) - f(c)}{x - c} - \frac{f_N(x) - f_N(c)}{x - c}\right| \leq \frac{\epsilon}{3}. \tag{2.1.4}$$

Combining inequalities (2.1.1), (2.1.2), (2.1.3), and (2.1.4), we obtain for $0 < |x - c| < \delta$ and $x \in [a, b]$

$$\left|\frac{f(x) - f(c)}{x - c} - g(c)\right| \leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3}.$$

This proves that $f = \lim\limits_{n\to\infty} f_n$ is differentiable and that $f' = g = \lim\limits_{n\to\infty} f'_n$. □

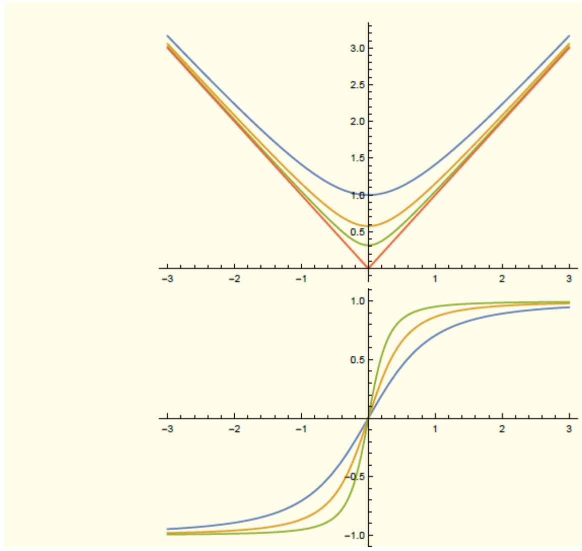
Example 2.1.10. Earlier, we studied the example

$$h_n(x) = \sqrt{x^2 + \frac{1}{n^2}}$$

We showed that $(h_n(x)) \to h(x) = |x|$ uniformly on $\mathbb{R}$. However, since the function $h(x) = |x|$ is not differentiable at $x = 0$, by the previous theorem, we know that $h'_n(x)$ does not converge uniformly to a limit function on $\mathbb{R}$. Note that

$$\lim_{n\to\infty} h'_n(x) = \lim_{n\to\infty} \frac{x}{\sqrt{x^2 + \frac{1}{n^2}}} = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0. \end{cases}$$

Plots of $h_1(x), h_3(x), h_{10}(x)$, and $|x|$. Plots of $h'_1(x), h'_3(x)$, and $h'_{10}(x)$.



Example 2.1.11. Let $g_n(x) = \dfrac{x}{2} + \dfrac{x^2}{2n}$, then

$$g(x) = \lim_{n\to\infty} g_n(x) = \lim_{n\to\infty}\left(\frac{x}{2} + \frac{x^2}{2n}\right) = \frac{x}{2} \qquad \text{(pointwise)},$$

and $g'(x) = \dfrac{1}{2}$. On the other hand,

$$g_n'(x) = \frac{1}{2} + \frac{x}{n} \to h(x) = \frac{1}{2} = g'(x) \quad (\text{pointwise for all} x \in \mathbb{R}).$$

We'll now examine how the previous theorem applies to this example.
Consider the interval $[-M, M]$ where $M > 0$. Let $h(x) = \dfrac{1}{2}$ for all $x \in \mathbb{R}$. Let $\epsilon > 0$ be given. Let $N \in \mathbb{N}$ be large enough such that $\dfrac{M}{N} < \epsilon$. Then if $x \in [-M, M]$ and $n \geq N$, we have

$$\left|g_n'(x) - h(x)\right| = \left|\left(\frac{1}{2} + \frac{x}{n}\right) - \frac{1}{2}\right| = \left|\frac{x}{n}\right| \leq \frac{M}{N} < \epsilon.$$

This shows that g_n' converges uniformly to h on $[-M, M]$. Because we verified that (g_n') converges uniformly on $[-M, M]$, the theorem tells us that

$$\lim_{n\to\infty} g_n'(x) = h(x) = g'(x) \quad \text{for } x \in [-M, M].$$

Since M is arbitrary we can conclude that

$$\lim_{n\to\infty} g_n'(x) = h(x) = g'(x) \quad \text{for } x \in \mathbb{R}.$$

Theorem 2.1.6. ***Theorem Related to Differentiable Limit Theorem***
Let (f_n) be a sequence of differentiable functions defined on the closed interval $[a,b]$, and assume (f_n') converges uniformly on $[a,b]$. If there exists a point $x_0 \in [a,b]$ where the sequence $(f_n(x_0))$ converges, then (f_n) converges uniformly on $[a,b]$.

Proof. Let $x \in [a,b]$ where $x \neq x_0$. Both x and x_0 will be fixed real numbers throughout the proof. Without loss of generality, we may assume $x > x_0$. (If $x < x_0$, the argument is the same.) By the Mean Value Theorem applied the function $f_n - f_m$ on the interval $[x_0, x]$, there exists some $\alpha \in (x_0, x)$ (α depends on m and n) such that

$$f_n'(\alpha) - f_m'(\alpha) = \frac{[f_n(x) - f_m(x)] - [f_n(x_0) - f_m(x_0)]}{x - x_0},$$

which implies

$$[f_n(x) - f_m(x)] - [f_n(x_0) - f_m(x_0)] = \left(f_n'(\alpha) - f_m'(\alpha)\right)(x - x_0). \tag{2.1.5}$$

Let $\epsilon > 0$ be given. Since (f_n') converges uniformly, by Cauchy's Criterion for uniformly convergent sequences of functions, there exists some $n_1 \in \mathbb{N}$ such that

$$|f_n(c) - f_m(c)| < \frac{\epsilon}{2(b-a)} \qquad \text{for all } n, m \geq n_1 \text{ and } c \in [a,b].$$

By hypothesis, the sequence $(f_n(x_0)$ converges. So there exists an $n_2 \in \mathbb{N}$ such that

$$|f_n(x_0) - f_m(x_0)| < \frac{\epsilon}{2} \qquad \text{for all } m, n \geq n_2.$$

Let $N = \max\{n_1, n_2\}$. Then if $m, n \geq N$, we have

$$\begin{aligned}
|f_n(x) - f_m(x)| &\leq |(f_n(x) - f_m(x)) - (f_n(x_0) - f_m(x_0))| + |f_n(x_0) - f_m(x_0)| \\
&=^{(2.1.5)} \left(f_n'(\alpha) - f_m'(\alpha)\right)(x - x_0) + |f_n(x_0) - f_m(x_0)| \\
&< \frac{\epsilon}{2}(x - x_0) + \frac{\epsilon}{2} \\
&\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

Since the choice of N was independent of x, this proves that (f_n) converges uniformly on $[a,b]$. □

Remark 2.1.5. Combining the previous two theorems gives a stronger version of the Differentiable Limit Theorem.

Theorem 2.1.7. ***Better Version of Differentiable Limit Theorem***
Let (f_n) be a sequence of differentiable functions defined on the closed interval $[a,b]$, and assume that the sequence (f_n') converges uniformly to a function g on $[a,b]$. If there exists a point $x_0 \in [a,b]$ for which $(f_n(x_0))$ converges, then (f_n) converges uniformly. Moreover, the limit function $f = \lim f_n$ is differentiable and satisfies $f' = g$.

2.2 Series of Functions

Definition 2.2.1. 1. Let f and f_n for $n \in \mathbb{N}$ be functions defined on a set $A \subseteq \mathbb{R}$.

(a) The infinite series $\sum\limits_{n\geq 1} f_n(x) = f_1(x) + f_2(x) + f_3(x) + ...$ **converges pointwise** on A to $f(x)$ if the sequence of partial sums $s_k(x) = f_1(x) + f_2(x) + ... + f_k(x)$ converges pointwise to $f(x)$ on A.

(b) The infinite series converges uniformly on A to $f(x)$ if the sequence of partial sums converges uniformly on A to $f(x)$.

2. Let
$$D = \left\{ x \in A : \quad \text{such that } \sum_{n\geq 1} f_n(x) \text{ converges} \right\}.$$
D is the domain of pointwise convergence of the series $\sum\limits_{n\geq 1} f_n(x)$.

3. We say that the series $\sum\limits_{n\geq 1} f_n(x)$ is absolutely convergeent on A if the series $\sum\limits_{n\geq 1} |f_n(x)|$ is pointwise convergent for all $x \in A$.

Remark 2.2.1. Since an infinite series of functions is defined in terms of the limit of a sequence of partial sums, everything we already know about sequences applies to series. For the sum $\sum\limits_{n\geq 1} f_n(x)$, we merely restate all of the previous theorems for the sequence of k^{th} partial sums
$$s_k(x) = f_1(x) + ... + f_k(x).$$

Example 2.2.1. • We consider the series of functions $\sum\limits_{n\geq 1} \dfrac{e^{nx}}{n}$. The functions $f_n(x) =$

$\dfrac{e^{nx}}{n}$ are positive. We have

$$\lim_{n\to\infty}\frac{f_{n+1}(x)}{f_n(x)} = \lim_{n\to\infty}\frac{n}{n+1}e^x = e^x.$$

Therefore, according to d'Alembert's test(LCT), the series is pointwise convergent if $e^x < 1$, in other words the domain of pointwise convergence is $\mathbb{R}_-^*$.

- Let $\sum_{n\geq 1}\dfrac{\sin(nx)}{n^2+|x|}$ be a function series defined on $\mathbb{R}$. We have for all $n\in\mathbb{N}^*$ and $x\in\mathbb{R}$

$$\left|\frac{\sin(nx)}{n^2+|x|}\right| \leq \frac{1}{n^2}.$$

The series $\sum_{n\geq 1}\dfrac{1}{n^2}$ is convergent (Riemann series), therefore according to the comparison test the given series is absolutely convergent on $\mathbb{R}$.

Definition 2.2.2. We say that the series $\sum_{n\geq 1} f_n(x)$ is normally convergent on A if and only if the series $\sum_{n\geq 1}\|f_n\|_\infty$ converges such that $\|f_n\|_\infty = \sup_{x\in A}|f_n(x)|$.

Example 2.2.2.

- The normal convergence of the series of the general term $f_n(x) = \dfrac{\cos(nx)}{n^2\ln(n)}$ on $\mathbb{R}$ for $n\geq 2$. the function cos is bounded for all $x\in\mathbb{R}$, thus $\|f_n\|_\infty = \dfrac{1}{n^2\ln(n)}$. As the series $\sum_{n\geq 2}\dfrac{1}{n^2\ln(n)}$ is convergent (Bertrand series), we conclude that the series of functions $\sum_{n\geq 2}\dfrac{\cos(nx)}{n^2\ln(n)}$ converges normally on $\mathbb{R}$.

- We consider the function series defined by $\sum_n nx^2e^{-x\sqrt{n}}$ for all $x\in\mathbb{R}^+$.
To study the normal convergence we must calculate $\|f_n\|_\infty = \sup_{x\in\mathbb{R}^+}|f_n(x)|$. The function f_n is differentiable on $\mathbb{R}^+$ and for all $x\in\mathbb{R}^+$

$$f_n'(x) = nx(2-x\sqrt{n})e^{-x\sqrt{n}} = 0 \Rightarrow x = \frac{2}{\sqrt{n}},$$

then $f_n \nearrow$ for $x\in[0,\dfrac{2}{\sqrt{n}}]$ and $f_n \searrow$ for $x\in[\dfrac{2}{\sqrt{n}},\infty]$, which implies that

$$\|f_n\|_\infty = \sup_{x\in\mathbb{R}^+}|f_n(x)| = f_n\left(\frac{2}{\sqrt{n}}\right) = 4e^{-2}.$$

The series $\sum_n 4e^{-2}$ is divergent, then the function series $\sum_n nx^2e^{-x\sqrt{n}}$ does not converge normally on $\mathbb{R}^+$.

2.2.1 Weierstrass M-test

Theorem 2.2.1. *Let $A \subset \mathbb{R}$. Suppose that there exists positive constants M_n, $n = 1, 2, ...$, such that $|f_n(x)| \leq M_n$ for $x \in A$. If $\sum_{n\geq 1} M_n < \infty$, then $\sum_{n\geq 1} f_n(x)$ converges uniformly on A.*

Proof. Since $|f_n(x)| \leq M_n$ and $\sum_{n\geq 1} M_n$ is convergent, it is clear that[2] $f(x) = \sum_{n\geq 1} f_n(x)$ exists for every $x \in A$. Now

$$\begin{aligned}
\|f(x) - s_n\|_\infty &= \left\|\sum_{k=1}^{\infty} f_k(x) - \sum_{k=1}^{n} f_k(x)\right\|_\infty = \left\|\sum_{k=n+1}^{\infty} f_k(x)\right\|_\infty \\
&\leq \sum_{k=n+1}^{\infty} \|f_k(x)\|_\infty \\
&\leq \sum_{k=n+1}^{\infty} M_k \longrightarrow 0,
\end{aligned}$$

as $k \to \infty$. By definition, this implies that the series is uniformly convergent. □

Example 2.2.3. Solution. Let $0 < a < 1$ and $ab > 1$. Show that $f(x) = \sum_{k=1}^{\infty} a^k \sin(b^k \pi x)$ is uniformly convergent.

We see that $\left|a^k \sin(b^k \pi x)\right| \leq a^k, \qquad k = 1, 2, 3, ...,$ since $|\sin(b^k \pi x)| \leq 1$. As $\sum_{k=1}^{\infty} a^k$ is a geometric series with quotient a and $|a| < 1$, we know that this series is convergent. Thus, by Weierstrass' $M-$test, it follows that the original series is uniformly convergent

2.2.2 Abel Uniform Criterion

Theorem 2.2.2. *Let (f_n) and (g_n) two sequences of functions defined from A to $\mathbb{R}$ such that:*

1. *There exists $M > 0$ such that for all $n \in \mathbb{N}$,* $\quad \sup_{x\in A} \left|\sum_{k=0}^{n} f_k(x)\right| \leq M,$

[2] The absolute convergence implies the pointwise convergence

2. *for all* $x \in A$, (g_n) *is decreasing.*

3. $(g_n) \to 0$ *uniformly on* A.

Then, the series $\sum_n f_n(x)g_n(x)$ *converges uniformly on* A.

2.2.3 Term-by-term Continuity Theorem

Theorem 2.2.3. *Let* f_n *be continuous functions defined on a set* $A \subseteq \mathbb{R}$, *and assume* $\sum_{n\geq 1} f_n$ *converges uniformly on* A *to a function* f. *Then* f *is continuous on* A.

Proof. Apply the Continuous Limit Theorem (2.1.2) to the partial sums $s_k = f_1 + f_2 + ... + f_k$. □

Example 2.2.4. Show that $f(x) = \sum_{k=1}^{\infty} \frac{2x}{x^2+k^4}$ is continuous.

Solution. Let $f_k(x) = \frac{2x}{x^2+k^4}$. Then

$$|f_k(x)| \leq \frac{2|x|}{x^2+k^4}, \quad k = 1, 2, 3, ...$$

We need to find constants M_k such that $|f_k(x)| \leq M_k$, so we maximize $f(x) = \frac{2x}{x^2+k^4}$ on $[\,0, \infty\,[$. Note that $f(0) = 0$ and $f(x) \to 0$ as $x \to \infty$. Moreover,

$$f'(x) = \frac{2k^4 - 2x^2}{(x^2+k^4)^2} \Rightarrow f'(x) = 0 \left[\Leftrightarrow x = \pm k^2\right],$$

so

$$|f(x)| \leq \sup_x f(x) = f(k^2) = \frac{2k^2}{k^4+k^4} = \frac{1}{k^2} =: M_k.$$

Since $\sum_{k=1}^{\infty} \frac{1}{k^2} < \infty$, the $M-$test proves that $\sum_{k=1}^{\infty} f_k(x)$ is uniformly convergent. Since f_k are continuous, the uniform convergence proves that also f is continuous.

2.2.4 Term-by-term Integration Theorem

Theorem 2.2.4. *Suppose that* $f(x) = \sum_n u_n(x)$ *is uniformly convergent for* $x \in [a, b]$. *If* $f_0, f_1, f_2, ...$ *are continuous functions on* $[a, b]$, *then we can exchange the order of sum-*

mation and integration:

$$\int_a^b f(x)dx = \int_a^b \sum_n f_n(x)dx = \sum_n \int_a^b f_n(x)dx = \sum_n \int_a^b f_n(x)dx.$$

Proof. Apply the theorem 2.1.4 of the uniform convergence and integration to the partial sums $s_n(x) = \sum_{k=0}^{n} f_k(x)$. □

Example 2.2.5. Let $f(x) = \sum_{k=1}^{\infty} \frac{2x}{x^2+k^4}$ prove that $\int_0^1 f(x)dx = \sum_{k=1}^{\infty} \ln\left(1+1/k^4\right)$.

Solution. From the above example (2.2.4) f is uniformly convergent. Moreover, the uniform convergence implies that we can integrate the series termwise, so

$$\int_0^1 f(x)dx = \sum_{k=1}^{\infty} \int_0^1 \frac{2x}{x^2+k^4}dx = \sum_{k=1}^{\infty} \left[\ln(x^2+k^4)\right]_{x=0}^{x=1} = \sum_{k=1}^{\infty} \ln\left(1+1/k^4\right).$$

2.2.5 Term-by-term Differentiability Theorem

Theorem 2.2.5. *Suppose the following three statements:*

1. *Let f_n be differentiable functions defined on an interval $A = [a,b]$.*
2. *Assume $\sum_n f_n'(x)$ converges uniformly to a limit $g(x)$ on A.*
3. *$\sum_n f_n(x)$ converges pointwise on A.*

Then, the series $\sum_n f_n(x)$ converges uniformly to a differentiable function $f(x)$ satisfying $f'(x) = g(x)$ on A. In other words,

$$f(x) = \sum_n f_n(x) \qquad \text{and} \qquad f'(x) = \sum_n f_n'(x).$$

Proof. Apply the stronger version of the Differentiable Limit Theorem 2.1.7 to the partial sums $s_k = f_1 + f_2 + ... + f_k$. □

Example 2.2.6. Show that $f(x) = \sum_{k=1}^{\infty} \frac{1}{x^2+k^2}$ is continuously differentiable (that is, show that $f \in C^1$).

Solution. Let $f_k(x) = \dfrac{1}{x^2+k^2}$. Clearly $|f_k(x)| \leq \dfrac{1}{k^2}$, $k = 1,2,3,...$, so the series defining $f(x)$ is convergent for all $x \in \mathbb{R}$ (actually uniformly convergent by the $M-$test). To show that $f(x)$ is differentiable, we prove the uniform convergence of the series

$$\sum_{k=1}^{\infty} f_k'(x) = \sum_{k=1}^{\infty} \frac{-2x}{(x^2+k^2)^2}.$$

Clearly $f_k'(x) \to 0$ as $x \to \pm\infty$ and

$$f_k''(x) = \frac{6x^2-2k^2}{(x^2+k^2)^3} \Rightarrow \left[f_k''(x) = 0 \Leftrightarrow x^2 = k^2/3\right],$$

so

$$|f'k(x)| \leq |\sup_x f_k'(x)| = |f_k'(\pm 3^{-1/2}k)| = \frac{3\sqrt{3}}{8k^3}.$$

Since $\sum_{k=1}^{\infty} \dfrac{1}{k^3} < \infty$, the $M-$test proves that $g(x) = \sum_{k=1}^{\infty} f_k'(x)$ is uniformly convergent. Moreover, f_k' are continuous for $k = 1,2,3,...$, so g is a continuous function. This is sufficient for claiming that f is differentiable with $f'(x) = g(x)$ for all x. Thus f inf C^1.

2.2.6 Cauchy Criterion for Uniform Convergence of Series

Theorem 2.2.6. *A series $\sum_n f_n$ converges uniformly on $A \subseteq \mathbb{R}$ if and only if for every $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that*

$$|s_n - s_m| = |f_{m+1}(x) + f_{m+2}(x) + ... + f_n(x)| < \epsilon.$$

whenever $n > m \geq N$ and $x \in A$.

2.3 Taylor and Power Series

2.3.1 Power Series

Definition 2.3.1. A power series (in x) centered at $x = a$ is a function of the form

$$f(x) = \sum_n a_n(x-a)^n = a_0 + a_1(x-a) + a_2(x-a)^2 + a_3(x-a)^3 + a_4(x-a)^4 + ...$$

where the $a_0, a_1, \ldots.$ are real numbers called the coefficients of the power serie.

Example 2.3.1.

$$\sum_{n\geq 0} \frac{x^n}{n!} \text{ centered at } 0, \qquad \sum_{n\geq 1} \frac{(x-1)^n}{n} \text{ centered at } 1$$
$$\sum_{n\geq 0} nx^n \text{ centered at } 0, \quad \sum_{n\geq 1} \frac{(x+2)^n}{n^2} \text{ centered at } -2.$$

Important Question: For which $x \in \mathbb{R}$ does the series $\sum_n a_n(x-a)^n$ converge?

Theorem 2.3.1. *If a power series $\sum_n a_n x^n$ converges at some nonzero point $x_0 \in \mathbb{R}$, then it converges absolutely for any x satisfying $|x| < |x_0|$.*

Proof. If $\sum_n a_n x_0^n$ converges, then $\lim_{n\to\infty} a_n x_0^n = 0$. So there exists some $M > 0$ such that $|a_n x_0^n| < M$ for all $n \geq 0$. If x satisfies $|x| < |x_0|$, then

$$|a_n x^n| = |a_n x_0^n| \left|\frac{x}{x_0}\right|^n \leq M \left|\frac{x}{x_0}\right|^n.$$

By the Comparison Test the series $\sum_n a_n x^n$ converges absolutely as follows:

$$\left|\sum_n a_n x^n\right| \leq \sum_n |a_n x^n| \leq \sum_n M \left|\frac{x}{x_0}\right|^n = \frac{M}{1-\left|\frac{x}{x_0}\right|} < \infty.$$

Thus, $\sum_n a_n x^n$ converges absolutely for x satisfying $|x| < |x_0|$. □

Example 2.3.2. For each power series, state where the power series is centered, identify its second coefficient, its first term, its sixth term, and its ninth coefficient:

$$(a)\ f(x) = \sum_{n=0}^{\infty} \frac{(x-2)^n}{n^2+1}, \ (b)\ f(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{3n+1}.$$

Definition 2.3.2. (Radius of Convergence) Let

$$R = \sup\left\{ |x_0| : \quad \sum_n a_n x_0^n \text{ converges} \right\}.$$

Then R is called the radius of convergence of the series $\sum_n a_n x^n$.

Remark 2.3.1. From the previous theorem and the definition of the radius of convergence, it is clear that if $0 < R < \infty$, then the series converges for $|x| < R$ and diverges for $|x| > R$.

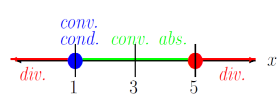
Example 2.3.3. For what x does the given series converge?

$$(a)\ \sum_{n=1}^{\infty} \frac{(x-3)^n}{n2^n},\quad (b)\ \sum_{n=0}^{\infty} n!(x+4)^n.$$

Solution: (a) We can apply the **Ratio Test** to the terms of this series. First, we have

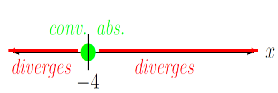
$$\rho = \lim_{n\to\infty} \frac{|f_{n+1}(x)|}{|f_n(x)|} = \lim_{n\to\infty} \frac{\left|\frac{(x-3)^{n+1}}{(n+1)2^{n+1}}\right|}{\left|\frac{(x-3)^n}{n2^n}\right|} = \lim_{n\to\infty} (1/2)|x-3|\frac{n}{n+1} = (1/2)|x-3|.$$

By the Ratio Test, this series converges absolutely when $\rho < 1$. Since $\rho = (1/2)|x-3|$, this corresponds to the inequality $(1/2)|x-3| < 1 \Leftrightarrow 1 < x < 5$. So the power series converges absolutely when $1 < x < 5$. Similarly, the series diverges when $x < 1$ or $x > 5$.
We analyze the cases $x = 1$ and $x = 5$ individually: If $x = 1 \Rightarrow f(1) = \sum_n \frac{(-1)^n}{n}$. This series converges conditionally (it is alternating harmonic). If $x = 5 \Rightarrow f(5) = \sum_n \frac{1}{n}$. This series diverges since it is harmonic. **In conclusion**, we have determined that the series converges absolutely when $x \in (1, 5)$, the series converges conditionally when $x = 1$, and the series diverges for all other x. We summarize this with the following picture,



(b) we have

$$\rho = \lim_{n\to\infty} \frac{\left|\frac{(x-3)^{n+1}}{(n+1)2^{n+1}}\right|}{\left|\frac{(x-3)^n}{n2^n}\right|} = \lim_{n\to\infty} (n+1)|x+4| = \begin{cases} 0 & \text{if } x = -4 \\ 1 & \text{else} \end{cases}$$



Corollary 2.3.1. ***(Cauchy-Hadamard (C-H) Theorem).***
If the series $\sum_n a_n x^n$ has radius of convergence R, then the set of all x for which the series converges is one of the following intervals:

- *If $R = 0$, the series converges only for $x = 0$.*
- *If $0 < R < \infty$, the series converges for all x in one of the following four intervals:*

$$(-R, R), \ [-R, R), \ (-R, R], \ [-R, R].$$

- *If $R = \infty$, then the series converges for all $x \in \mathbb{R}$.*

Proof. This is an immediate consequence of the previous theorem and the definition of the radius of convergence. If the corollary is not obvious, go back and review the previous theorem and definition. □

Theorem 2.3.2. ***(Abel's Formula).***
Let $\sum_{n\geq 0} a_n (x-a)^n$ be a power series centered at a. Then, the radius of convergence of this power series is given by

$$R = \lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|},$$

(assuming this limit exists).

Proof. **(Proof of the Cauchy-Hadamard Theorem and Abel's Formula):** Given power series $\sum_{n=0}^{\infty} a_n (x-a)^n$, let $R = \lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|}$. We'll begin by proving statement (1) of the C-H Theorem, so we assume for now that $0 < R < 1$. Now try to determine the convergence of the power series using the Ratio Test; first compute:

Theorem 2.3.3. *If a power series $\sum_{n\geq 0} a_n x^n$ converges absolutely at a point x_0, then it converges uniformly on the closed interval $[-|x_0|, |x_0|]$.*

Proof. For each $n \geq 0$, let $M_n = |a_n x_0^n|$. By hypothesis the series $\sum_{n\geq 0} a_n x_0^n$ converges absolutely and so $\sum_{n\geq 0} |a_n x_0^n| = \sum_{n\geq 0} M_n$ converges. Then for any $x \in [-|x_0|, |x_0|]$, we have

$$\left| \sum_{n\geq 0} a_n x^n \right| \leq \sum_{n\geq 0} |a_n x^n| \leq \sum_{n\geq 0} |a_n x_0^n| = \sum_{n\geq 0} M_n < \infty.$$

By the Weierstrass M-test, the series converges uniformly on the closed interval $[-|x_0|, |x_0|]$.

$\square$

Theorem 2.3.4. ***Abel's Theorem about Uniform Convergence***
Let $g(x) = \sum_{n\geq 0} a_n x^n$ *be a power series that converges at the point* $x = R > 0$*. Then the series converges uniformly on the interval* $[0, R]$*. A similar result holds if the series converges at* $x = -R$*.*

We have seen that the sum $f(x) = \sum_{n\geq 0} a_n(x-a)^n$ of a power series is continuous in the interior $(a-R, a+R)$ of its interval of convergence. But what happens if the series converges at an endpoint $a \pm R$?

Before we turn to the proof, we need a lemma that can be thought of as a discrete version of integration by parts.

Lemma 2.3.1. ***(Abel's Summation Formula)*** *Let* $(a_n)_n$ *and* $(b_n)_n$ *be two sequences of real numbers, and let* $s_n = \sum_{k=0}^{n} a_k$*. Then*

$$\sum_{n=0}^{N} a_n b_n = s_N b_N + \sum_{n=0}^{N-1} s_n(b_n - b_{n+1}).$$

If the series $\sum_n a_n$ *converges, and* $b_n \to 0$ *as* $n \to \infty$*, then*

$$\sum_{n=0}^{\infty} a_n b_n = \sum_{n=0}^{\infty} s_n(b_n - b_{n+1}).$$

Proof. Note that $a_n = s_n - s_{n-1}$ for $n \geq 1$, and that this formula even holds for $n = 0$ if we define $s_{-1} = 0$. Hence

$$\sum_{n=0}^{N} a_n b_n = \sum_{n=0}^{N} (s_n - s_{n-1}) b_n = \sum_{n=0}^{N} s_n b_n - \sum_{n=0}^{N} s_{n-1} b_n,$$

Changing the index of summation and using that $s_{-1} = 0$, we see that $\sum_{n=0}^{N} s_{n-1} b_n =$

$\sum_{n=0}^{N-1} s_n b_{n+1}$ Putting this into the formula above, we get

$$\sum_{n=0}^{N} a_n b_n = \sum_{n=0}^{N} s_n b_n - \sum_{n=0}^{N-1} s_n b_{n+1} = s_N b_N + \sum_{n=0}^{N-1} s_n (b - n - b_{n+1})$$

and the first part of the lemma is proved. The second follows by letting $N \to \infty$. $\square$We are now ready to prove:

Theorem 2.3.5. *The sum of a power series $f(x) = \sum_n a_n (x-a)^n$ is continuous in its entire interval of convergence. This means in particular that if R is the radius of convergence, and the power series converges at the right endpoint $a+R$, then $\lim_{x \uparrow a+R} f(x) = f(a+R)$, and if the power series converges at the left endpoint $a-R$, then $\lim_{x \downarrow a-R} f(x) = f(a-R)$.*

Example 2.3.4. Summing a geometric series, we clearly have

$$\frac{1}{1+x^2} = \frac{1}{1-\underbrace{(-x^2)}_{=u}} = \frac{1}{1-u} = \sum_{n=0}^{\infty} u^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}, \quad \text{for } |u| = |-x^2| < 1 \Leftrightarrow |x| < 1.$$

Integrating, we get

$$\int_0^x \frac{1}{1+t^2} dt = \arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \quad \text{for } |x| < 1$$

we see that the series converges even for $x = 1$. By Abel's Theorem

$$\pi/4 = \arctan 1 = \lim_{x \to 1} \arctan x = \lim_{x \to 1} \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$$

Hence we have proved

$$\pi/4 = 1 - 1/3 + 1/5 - 1/7 + \dots.$$

This is often called Leibniz' or Gregory's formula for π.

Uniqueness of power series

Recall that we asked in the last section if a function could be represented by two different power series centered at a. To address this question, suppose that function f can be represented by some power series on an open interval containing a, i.e. that

$$f(x)=\sum_{n=0}^{\infty}a_n(x-a)^n=a_0+a_1(x-a)+a_2(x-2)^2+a_3(x-a)^3+a_4(x-a)^4+....$$

on $(a-R,a+R)$ where $R>0$ is the radius of convergence of the series. Then, $f'(x)$ can be expressed as a power series centered at a with the same radius of convergence, so f is infinitely differentiable on $(a-R,a+R)$ (i.e. it is a function which can be repeatedly differentiated without anything becoming undefined). Furthermore

$$\begin{aligned}
f'(x) &= a_1+2a_2(x-a)+3a_3(x-a)^2+4a_4(x-a)^3+.....\\
f''(x) &= 2a_2+3.2a_3(x-a)+4.3a_4(x-a)^2+....\\
f^{(3)}(x) &= f'''(x)=3.2a_3+4.3.2a_4(x-a)+....\\
\vdots &= \vdots\\
f^{(n)}(x) &= n!a_n+(n+1)!a_{n+1}(x-a)+....
\end{aligned}$$

Now, plug in a for x in each of the following formulas above. We obtain

$$\begin{aligned}
f'(a) &= a_1+2a_2(a-a)+3a_3(a-a)^2+4a_4(a-a)^3+.....=a_1\\
f''(a) &= 2a_2+3.2a_3(a-a)+4.3a_4(a-a)^2+....=2a_2\\
f^{(3)}(a) &= f'''(a)=3.2a_3+4.3.2a_4(a-a)+....=3.2a_3\\
\vdots &= \vdots\\
f^{(n)}(a) &= n!a_n+(n+1)!a_{n+1}(a-a)+....=n!a_n
\end{aligned}$$

The key formula that has been derived is in the last line above:

$$f^{(n)}(a)=n!a_n \Leftrightarrow a_n=\frac{f^{(n)}(a)}{n!}.$$

We have proven the following theorem:

Theorem 2.3.6. ***(Formula for coefficients of a power series).*** *Suppose* $f(x)=\sum_{n=0}^{\infty}a_n(x-a)^n$ *where this series converges on an open interval containing* a *(equivalently, the series has positive radius of convergence). Then, for every* n *the coefficients* a_n *of the power* *series must satisfy*

$$a_n = \frac{f^{(n)}(a)}{n!}.$$

Theorem 2.3.7. ***(Uniqueness of coefficients).*** *Suppose* $\sum_{n=0}^{\infty} a_n(x-a)^n = \sum_{n=0}^{\infty} b_n(x-a)^n$ *on an open interval containing* $x = a$*. Then* $a_n = b_n$ *for all* n*.*

Proof. By Theorem 2.3.6, both a_n and b_n must be equal to $\frac{f^{(n)}(a)}{n!}$ for all n, thus they are equal to one another. □

Example 2.3.5. Suppose $f(x) = \sum_{n=0}^{\infty} \frac{3}{(n+1)^2} x^n$. Find $f^{(9)}(0)$.

By Theorem 2.3.6 with $n = 9$, we know that $a_9 = \frac{f^{(9)}(0)}{9!}$. Now a_9 can be found by the formula for f that is given; it is the coefficient on the x^9 term which is $\frac{3}{(9+1)^2} = 3/100$. Thus we have $f^{(9)}(0) = \frac{3.9!}{100}$.

The next question we ask is the converse: if you start with a function f which is infinitely differentiable on $(a-R, a+R)$, is it the case that f is representable by a power series? **This leads to the discussion in the next section.**

2.3.2 Taylor Series

Definition 2.3.3. Given a function f which is infinitely differentiable on some open interval containing a, the Taylor series of f centered at a is the power series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + ...$$

If $a = 0$, then the series in this definition, namely

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + ...$$

is called the Taylor series of f or the Maclaurin series of f.

Remark 2.3.2. It is easy to confuse the terms "power series" and "Taylor series". A power series is any expression of the form $\sum_{n=0}^{\infty} a_n(x-a)^n$. A Taylor series is a particular power series associated to some function f which is specified in advance.

Main questions related to Taylor series:

1. For what x does the Taylor series of a function f centered at a converge?

2. What function does the Taylor series of f converge to?

At this point, we know enough to answer the first question. The Taylor series of a function f centered at a is an example of a power series centered at a. Therefore, by the Cauchy-Hadamard Theorem, the Taylor series converges (absolutely) to $f(a)$ when $x = a$. This is because

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \Big|_{x=a} = f(a) + 0 + 0 + 0 + ... = f(a).$$

We also know that there is some interval $(a-R, a+R)$ centered at a on which the Taylor series of f converges to something. Ideally, the Taylor series of f should converge to f itself (since it is the only possible power series representation of f). But we don't know at this point whether or not this happens, or under what circumstances this happens.

Example 2.3.6. Prototype Example 1: $f(x) = e^x$, and $a = 0$.

Here, we see that $f^{(n)}(x) = e^x$ for all n. Therefore $f^{(n)}(a) = f^{(n)}(0) = 1$ for all n and therefore the Taylor series of f is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = \sum_{n=0}^{\infty} \frac{1}{n!}x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

To find where this series converges, we use Abel's Formula:

$$R = \lim_{n\to\infty} |a_n|/|a_{n+1}| = \lim_{n\to\infty} (n+1) = \infty$$

Since $R = \infty$, this series converges for all x by the Cauchy-Hadamard Theorem.

Example 2.3.7. Prototype Example 2: $g(x) = \sin x$, and $a = 0$.

So the Taylor series of $\sin x$ is

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!}x^n &= g(0) + g'(0)x + \frac{g''(0)}{2}x^2 + \frac{g'''(0)}{3!}x^3 + ... \\
&= 0 + x + 0x^2 - \frac{1}{3!}x^3 + 0x^4 + \frac{1}{5!}x^5 + 0x^6 - \frac{1}{7!}x^7 + ... \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1}
\end{aligned}$$

By an argument similar to the previous example (Abel's Formula gives $R = \infty$), this series converges absolutely for all x.

Remark 2.3.3. To study the convergence of Taylor series for arbitrary functions, we return to the basics of infinite series. Recall from Chapter 1 that a series converges if the limit of its partial sums exists and is finite. Therefore, to understand the convergence of Taylor series, it makes sense to talk about the partial sums of a Taylor series. These partial sums are called **Taylor polynomials**

Definition 2.3.4. Let $n \geq 0$. Given a function f which can be differentiated n times on an open interval containing a, we can define the Taylor polynomial of order n centered at a, also called the nth Taylor polynomial centered at a to be

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + ... + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Properties of Taylor polynomials

Given any function f, where P_n denotes the nth Taylor polynomial centered at a, the following hold:

1. $P_n(x)$ is a polynomial of degree $\leq n$,

2. If $f^{(n)}(a) \neq 0$, then $P_n(x)$ is a polynomial whose degree is exactly n,

3. $P_0(x)$ is the constant function $f(a)$,

4. $P_1(x) = f(a) + f'(a)(x-a)$ is the tangent line to f when $x = a$,

5. $P_n(x)$ is the nth partial sum of the Taylor series of f centered at a, therefore

$$\lim_{n\to\infty} P_n(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n,$$

   if the limit exists.

Example 2.3.8. $f(x) = e^x, \ a = 0$, recall that the Taylor series of f is

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + x^2/2 + x^3/(3!) + x^4/(4!) + ...$$

Given this, we see that

$$P_0(x) = 1, \quad P_1(x) = 1 + x, \quad P_2(x) = 1 + x + x^2/2, \quad P_3(x) = 1 + x + x^2/2 + x^3/(3!),$$

$$....P_n(x) = 1 + x + x^2/2 + x^3/(3!) + ... + x^n/(n!).$$

Next, we turn to the problem of determining whether the Taylor series of a function f converges to f, or to something else. To do this, we introduce a new function, called the nth remainder, which measures the difference between the original function f and its nth Taylor polynomial

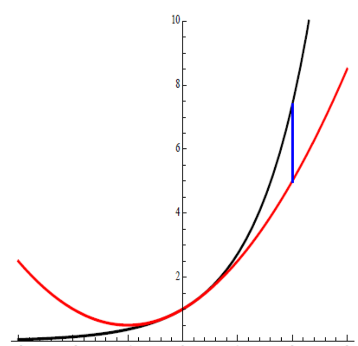
Definition 2.3.5. Let f be infinitely differentiable on $(a - R, a + R)$ and let P_n be the nth Taylor polynomial of f, centered at $x = a$. Define the nth remainder (of f centered at a) to be the function

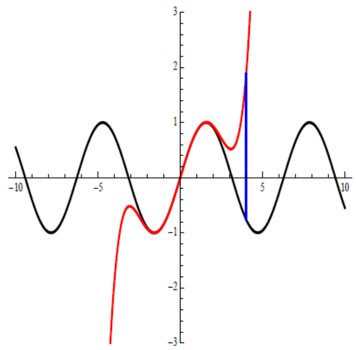
$$R_n(x) = f(x) - P_n(x).$$

Example 2.3.9.

1. $f(x) = e^x$, $a = 0$, recall that $P_2(x) = 1 + x + x^2/2$. In the picture below, f is graphed in black, P_2 is graphed in red, and $R_2(2)$ is the length of the blue line segment

2. $g(x) = \sin x$, $a = 0$, recall that $P_5(x) = x - x^3/(3!) + x^5/(5!)$. In the picture below, f is graphed in black, P_5 is graphed in red, and $R_5(4)$ is the length of the blue line segment





Theorem 2.3.8. ***(Remainder Theorem).*** *Let f be infinitely differentiable on $(a-R, a+R)$ and let P_n and R_n be the nth Taylor polynomial and nth remainder of f, centered at $x = a$. Then if $\lim\limits_{n\to\infty} R_n(x) = 0$, we have*

$$f(x) = \sum_{n=0}^{n} \frac{f^{(n)}(a)}{n!}(x-a)^n,$$

i.e. f is equal to its Taylor series on $(a-R, a+R)$.

Proof. Recall that $P_n(x)$ is the nth partial sum of the Taylor series of f. Therefore, since any infinite series is defined to be the limit of its partial sums, we have

$$\begin{aligned}
\sum_{n=0}^{n} \frac{f^{(n)}(a)}{n!}(x-a)^n &= \lim_{n\to\infty} P_n(x) = \lim_{n\to\infty}(f(x) - R_n(x)) \quad \text{(by definition of } R_n\text{)} \\
&= f(x) - \lim_{n\to\infty} R_n(x) \\
&= f(x). \qquad \text{(by hypothesis)}
\end{aligned}$$

□The Remainder Theorem sufficiently (for our purposes) answers (at least theoretically) the second main question related to Taylor series, because it gives a condition under which the Taylor series of f converges to f itself. In particular, the Remainder Theorem tells us that to show an infinitely differentiable function is equal to its Taylor series, we need only to show that $\lim\limits_{n\to\infty} R_n(x) = 0$. However, the definition of $R_n(x)$ alone is insufficient to evaluate this limit. We need an alternate representation of the nth remainders which will allow us to show that $\lim\limits_{n\to\infty} R_n(x) = 0$. To get this alternate representation, we first recall a theorem from Calculus I

Theorem 2.3.9. ***(Mean Value Theorem (MVT)).*** *Let f be differentiable on the interval*

$[a, x]$. *Then, there exists a* $z \in (a, x)$ *such that*

$$\frac{f(x) - f(a)}{x - a} = f'(z)$$

Theorem 2.3.10. ***(Taylor's Theorem).*** *Suppose* f *can be differentiated* $n+1$ *times in an open interval* $(a - R, a + R)$ *(where* $R > 0$*). Then, for all* $x \in (a - R, a + R)$ *and all* $n \geq 0$*, there exists a* z *between* a *and* x *such that*

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x - a)^{n+1}.$$

Proof. **First, a remark:** this proof will use the Mean Value Theorem. **The proof of the Mean Value Theorem is deep; take an advanced calculus course if you want to see that.** Now, let's prove the theorem. Fix $x \in (a - R, a + R)$ and recall that $R_n(x) = f(x) - P_n(x)$. Define a new function g, whose input variable will be called t, by setting $g(t)$ equal to

$$f(x) - \left[f(t) + f'(t)(x - t) + \frac{f''(t)}{2!}(x - t)^2 + ... + \frac{f^{(n)}(t)}{n!}(x - t)^n \right] - R_n(x)\frac{(x - t)^{n+1}}{(x - a)^{n+1}}.$$

Observe that $g(a) = 0$ and $g(x) = 0$. Now apply the Mean Value Theorem to g to find a point z between a and x such that

$$g'(z) = \frac{g(x) - g(a)}{x - a} = \frac{0 - 0}{x - a} = 0,$$

Last, evaluate the derivative of g. We have

$$\begin{aligned} g'(z) &= \frac{d}{dt} g(t)\,|_{t=z} \\ &= \Big(0 - \Big[f'(t) + (f''(t)(x - t) - f'(t)) + \Big(f'''(t)/(2!)(x - t)^2 - f''(t)(x - t) \Big) \; + \\ &\vdots \\ &+ \left(\frac{f^{(n+1)}(t)}{n!}(x - t)^n - \frac{f^{(n)}(t)}{(n-1)!}(x - t)^{n+1} \right) \Big] + \frac{R_n(x)}{(x - a)^{n+1}}(n+1)(x - t)^n \Big) \Big|_{t=z} \end{aligned}$$

notice that the terms inside the brackets cancel out to leave

$$g'(z) = -\frac{f^{(n+1)}(t)}{n!}(x - t)^n + \frac{R_n(x)}{(x - a)^{n+1}}(n+1)(x - t)^n.$$

Since $g'(z) = 0$, we have

$$0 = -\frac{f^{(n+1)}(t)}{n!}(x-t)^n + \frac{R_n(x)}{(x-a)^{n+1}}(n+1)(x-t)^n \Leftrightarrow R_n(x) = \frac{f^{(n+1)}(t)}{(n+1)!}(x-a)^{n+1}.$$

□