Power Series

EXERCISE 1.

(1) Determine the interval of convergence : $(a)\ \sum_{n=1}^{\infty} \frac{(-1)^n(x-2)^n}{n}$, $\quad (b)\ \sum_{n=1}^{\infty} \frac{n!x^n}{n^{10}}$.

(2) Consider $\sum_{n=1}^{\infty} \frac{n}{4^n(n+1)} x^{2n}$,

(a): Does the series converge for $x = 2$? Justify your answer.

(b): Based only on your answer from part (a), what can you say about R, the radius of convergence of the series ?

(c): Find the interval of convergence of the series.

(3) Consider the following power series :$\sum_{n=1}^{\infty} \frac{1}{n5^n}(x-4)^{n+1}$.

(a): Find the radius of convergence of the power series. Show all your work.

(b): For which values of x does the series converge absolutely ? For which values of x does it converge conditionally ?

EXERCISE 2.

(1) What is the radius of convergence of the power series $\sum_{n\geq 0} \frac{(-2)^n x^n}{n!}$?

(2) Write f under a usual function : $f(x) = 3\left(1 - \sum_{n\geq 0} \frac{(-2)^n x^n}{n!}\right)$.

(3) Let $y'(x) = 2(3 - y(x))$, with $y(0) = 0$. We suppose that y is expandable as a power series : $y(x) = \sum_{n\geq 0} a_n x^n \quad$ if $|x| < R$. Determine the coefficients a_n, then y.

EXERCISE 3. Use a known series to find a power series in x that has the given function as its sum :

$$a)\ x\sin(x^3), \quad b)\ \frac{\ln(1+x)}{x}, \quad c)\ \frac{x - \arctan x}{x^3}$$

Use a power series to approximate each of the following to within 3 decimal places : $a)$ $\arctan\frac{1}{2}$, $b)$ $\ln(1.01)$, $c)$ $\sin\left(\frac{\pi}{10}\right)$.

Fourier Series

EXERCICE 1.

— Sketch a graph of $f(x)$ in the interval $-2\pi < x < 2\pi$ and find Fourier series representation where $f(x) = \begin{cases} 1, & -\pi < x < 0 \\ 0, & 0 < x < \pi, \text{ and has period } 2\pi \end{cases}$ Pick an appropriate value of x, to show that $\pi/4 = 1 - 1/3 + 1/5 - 1/7 + ...$

— Sketch a graph of $g(x)$ in the interval $-3\pi < x < 3\pi$ and find Fourier series representation where $g(x) = \begin{cases} 0, & -\pi < x < 0 \\ x, & 0 < x < \pi, \text{ and has period } 2\pi \end{cases}$ Pick an appropriate value of x, to show that $\pi/4 = 1 - 1/3 + 1/5 - 1/7 + ...$ and $\pi^2/8 = 1 + 1/3^2 + 1/5^2 + 1/7^2 + ...$

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EXERCISE 2.Consider the function : $h(x) = \begin{cases} x, & 0 \leq x \leq \pi/2 \\ \pi/2, & \pi/2 < x < \pi. \end{cases}$ Continue f on the interval $[-\pi, 0)$ such as to form (a) an even function, (b) an odd function, and (c) a $\pi-$periodic function. For each of these three cases, sketch the graph of f on $(-\pi, \pi)$, and determine the complex-valued Fourier coefficients c_0, c_1 and c_{-1}, as well as the real-valued Fourier coefficients a_0, a_1 and b_1.

EXERCICE 3.Consider the function $g : [0, 2\pi] \to \mathbb{R}$ defined by

$$g(x) = \begin{cases} 2, & \text{if } 0 \leq x < \pi/2, \text{ or } 3\pi/2 < x \leq 2\pi, \\ 1, & \text{if } \pi/2 \leq x \leq 3\pi/2. \end{cases}$$

(a)**:** Denote by $f:\mathbb{R}\to\mathbb{R}$ the $2\pi-$periodic extension of g over $\mathbb{R}$. Sketch f over the interval $x\in[-2\pi,2\pi]$

(b)**:** Show that the Fourier series S of f is

$$S(x)=3/2+2/\pi\left(\sum_{n=0}^{\infty}(-1)^n\frac{\cos((2n+1)x)}{2n+1}\right).$$

(c)**:** For what values of x do we have $S(x)\neq f(x)$ on $[-\pi,\pi]$?

(d)**:** Explain why the Fourier series suggests that

$$\pi/4=1-1/3+1/5-1/7+....=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n-1}$$

and use the Leibnitz Alternating Series Test (AST) to test this series for convergence

EXERCICE 4.Let $f:\mathbb{R}\to\mathbb{R}$ the (period$=2\pi$) function $f(x)=e^x$ for all $x\in\,]-\pi,\pi\,]$.

(1) Show that the Fourier exponential coefficients of the function f are given by

$$c_n(f)=\frac{\sinh(\pi)}{\pi}\frac{(-1)^n}{1-in}\quad\forall n\in\mathbb{N}.$$

(2) Study the convergence (pointwise, uniform) of the fourier series of the function f.

(3) Deduce the values of the sums

$$T_1=\sum_{n=0}^{\infty}\frac{(-1)^n}{n^2+1},\qquad T_2=\sum_{n=0}^{\infty}\frac{1}{n^2+1}.$$

Remember $c_n(f)=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)e^{-inx}dx$.

EXERCICE 4.Let f the function (period$=2\pi-$) defined on $[0,2\pi]$ by : $f(x)=\begin{cases}\pi-x & \text{if } x\in[0,\pi]\\ x-\pi & \text{if } x\in\,]\,\pi,2\pi\,[.\end{cases}$

(1) Sketch f over the interval $[-2\pi,2\pi]$. Is f even ?

(2) Calculate the Fourier coefficients of f.

(3) Show that the Fourier series with odd index has the expression

$$SF(f)(x)=\frac{\pi}{2}+\sum_{p=0}^{+\infty}\frac{4}{\pi(2p+1)^2}\cos((2p+1)x).$$

(We can admit this expression to address the following question...)

(4) Deduce the values ? ?of the following series :

$$(a)\quad\sum_{p=0}^{+\infty}\frac{1}{(2p+1)^2},\qquad (b)\quad\sum_{p=0}^{+\infty}\frac{1}{(2p+1)^4}$$