CHAPTER 3

# Fourier Series

## 3.1 Introduction

In the approximate calculation, we often use developments in power series of sufficiently regular functions

$$f(x) \sim f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + .... + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + ....$$

because the powers of $(x-x_0)$ are simple to manipulate and lend themselves well to numerical calculations and operational (derivations, integration,...). However, expanding a function into a power series has a few drawbacks:

- It does not always have the convergence properties often desired;
- It can only be used for infinitely differentiable functions;
- It is only valid near $x_0$.

To overcome these problems, we use expansions in series of simple functions, other than powers of $x$. The choice of these functions is motivated by the fact that many engineering problems involve periodic functions $f$, and it then appears logical to try to develop them in series of $''\sin''$ and $''\cos''$ instead of series of powers of $x$.

$$f(x) \sim \frac{a_0}{2} + (a_1 \cos x + b_1 \sin x) + (a_2 \cos(2x) + b_2 \sin(2x)) + ... + (a_n \cos(nx) + b_n \sin(nx)) + ...$$

## 3.2 Trigonometric series

**Definition 3.2.1.** We call a trigonometric series associated with two sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}^*}$ of complex numbers, the series of functions

$$\frac{a_0}{2}+\sum_{n\in\mathbb{N}^*}(a_n\cos(nx)+b_n\sin(nx)). \tag{3.2.1}$$

The numbers $a_n$ and $b_n$ are called the trigonometric coefficients of this series.

**Remark 3.2.1.**

- For all $N\in\mathbb{N}$, we can write

$$\begin{aligned}\frac{a_0}{2}+\sum_{n=1}^{N}(a_n\cos(nx)+b_n\sin(nx)) &= \frac{a_0}{2}+\sum_{n=1}^{N}\left(a_n\frac{e^{inx}+e^{-inx}}{2}+b_n\frac{e^{inx}-e^{-inx}}{2i}\right)\\ &= \underbrace{\frac{a_0}{2}}_{c_0}e^{i0x}+\sum_{n=1}^{N}\left(\underbrace{\frac{a_n-ib_n}{2}}_{c_n}e^{inx}+\underbrace{\frac{a_n+ib_n}{2}}_{c_{-n}}e^{i(-n)x}\right)\\ &= \sum_{n=-N}^{N}c_ne^{inx},\end{aligned}$$

  by letting $N$ to the infinity we get

$$\frac{a_0}{2}+\sum_{n\in\mathbb{N}^*}(a_n\cos(nx)+b_n\sin(nx))=\sum_{n\in\mathbb{Z}}c_ne^{inx},$$

  where $c_n$ are the exponential coefficients.

- $a_0=2c_0$ and, $\forall n\in\mathbb{N}^*$, $a_n=c_n+c_{-n}$, $b_n=i(c_n-c_{-n})$.

**Proposition 3.2.1.**

1. *The sum $S$ of a convergent trigonometric series is a $2\pi-$periodic function*
2. *If the sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}^*}$ are real and decreasing towards $0$, then the trigonometric series $\frac{a_0}{2}+\sum_{n\in\mathbb{N}^*}(a_n\cos(nx)+b_n\sin(nx))$ converges for all $x\neq 2k\pi$.*
3. *The following three properties are equivalent*

   $(a)$ *The trigonometric series $\frac{a_0}{2}+\sum_{n\in\mathbb{N}^*}(a_n\cos(nx)+b_n\sin(nx))$ converges normally on $\mathbb{R}$.*

$(b)$ *The series* $\sum a_n$ *and* $\sum b_n$ *are absolutely convergent*

$(c)$ *The series* $\sum c_n$ *and* $\sum c_{-n}$ *are absolutely convergent*

**Proof.**

1. We suppose that the series $S(x) = \dfrac{a_0}{2} + \sum\limits_{n\in\mathbb{N}^*} (a_n \cos(nx) + b_n \sin(nx))$ is convergent, and we put $\forall n \in \mathbb{N}\ S_n(x) = \dfrac{a_0}{2} + \sum\limits_{k=1}^{n} (a_k \cos(kx) + b_k \sin(kx))$. Since the functions $x \mapsto \dfrac{a_0}{2}$; $x \mapsto a_k \cos(kx)$ and $x \mapsto b_k \sin(kx)$ are $2\pi-$periodic, the function $S_n$ is $2\pi-$periodic, therefore

$$\forall n \in \mathbb{N},\ \forall x \in \mathbb{R} \quad S_n(x+2\pi) = S_n(x),$$

and this implies that

$$\lim_{n\to\infty} S_n(x+2\pi) = S(x+2\pi) = S(x) = \lim_{n\to\infty} S_n(x),$$

and this means that $S$ is $2\pi-$periodic.

2. We apply Abel's rule to the series $\sum\limits_{n\in\mathbb{N}^*} a_n \cos(nx)$ and $\sum\limits_{n\in\mathbb{N}^*} b_n \sin(nx)$

3. $\forall n \in \mathbb{N}^*$, $|c_n| = \left|\dfrac{a_n - ib_n}{2}\right| \leq \dfrac{|a_n| + |b_n|}{2}$ and $|c_{-n}| = \left|\dfrac{a_n + ib_n}{2}\right| \leq \dfrac{|a_n| + |b_n|}{2}$ then $(b) \Rightarrow (c)$. Likewise $\forall n \in \mathbb{N}^*$, $|a_n| = |c_n + c_{-n}| \leq |c_n| + |c_{-n}|$ and $|b_n| = |i(c_n - c_{-n})| \leq |c_n| + |c_{-n}|$, and this proves that $(c) \Rightarrow (b)$. Consequently we have $(b) \Leftrightarrow (c)$.
   We have also $\forall n \in \mathbb{N}^*$, $\forall x \in \mathbb{R}$, $|a_n \cos nx + b_n \sin nx| \leq |a_n| + |b_n|$ and this shows that $(b) \Rightarrow (a)$. Finally we have $\forall n \in \mathbb{N}^*$, $|a_n| = |a_n \cos n\pi + b_n \sin n\pi| \leq \sup\limits_{x\in\mathbb{R}} |a_n \cos nx + b_n \sin nx|$ and $|b_n| = |a_n \cos n\dfrac{\pi}{2n} + b_n \sin n\dfrac{\pi}{2n}| \leq \sup\limits_{x\in\mathbb{R}} |a_n \cos nx + b_n \sin nx|$ and it follows that $(a) \Rightarrow (b)$, and consequently $(a) \Leftrightarrow (b) \Leftrightarrow (c)$.

$\square$

**Lemma 3.2.1.** *For all* $a \in \mathbb{R}$ *and for all* $m, n$ *in* $\mathbb{N}$, *we have:*

$$\int_a^{a+2\pi} \cos(mx)\sin(nx)dx = 0,$$

$$\int_a^{a+2\pi} \cos(mx)\cos(nx)dx = \int_a^{a+2\pi} \sin(mx)\sin(nx)dx = \begin{cases} 0 \text{ if } n \neq m, \\ \pi \text{ if } n = m. \end{cases}$$

### 3.2.1 Determining Fourier coefficients $a_n$ and $b_n$

**Theorem 3.2.1.** *We assume that the trigonometric series converges and has a continuous function $f(x)$ as its sum on the interval $[a, a+2\pi]$, that is,*

$$f(x) = \frac{a_0}{2} + \sum_{n\in\mathbb{N}^*} (a_n \cos(nx) + b_n \sin(nx)). \tag{3.2.2}$$

*In this case, the trigonometric coefficients of this series are given by*

$$a_0 = \frac{1}{\pi}\int_a^{a+2\pi} f(x)dx, \quad \text{and} \quad \forall n \in \mathbb{N}^*,\ a_n = \frac{1}{\pi}\int_a^{a+2\pi} f(x)\cos(nx)dx,$$

$$\forall n \in \mathbb{N}^*,\ b_n = \frac{1}{\pi}\int_a^{a+2\pi} f(x)\sin(nx)dx.$$

**Proof.** Our aim is to find formulas for the coefficients $a_n$ and $b_n$ in terms of $f$. Recall that for a power series $f(x) = \sum_n c_n(x-a)^n$ we found a formula for the coefficients in terms of derivatives: $c_n = \dfrac{f^{(n)}(a)}{n!}$. Here we use integrals.

If we integrate both sides of Equation (3.2.2) and assume that it's permissible to integrate the series term-by-term, we get

$$\int_a^{a+2\pi} f(x)dx = \underbrace{\int_a^{a+2\pi} \frac{a_0}{2}dx}_{=\pi a_0} + \sum_{n\in\mathbb{N}^*}\left(a_n \underbrace{\int_a^{a+2\pi} \cos(nx)dx}_{=0} + b_n \underbrace{\int_a^{a+2\pi} \sin(nx)dx}_{=0}\right),$$

and solving for $a_0$ gives $a_0 = \dfrac{1}{\pi}\displaystyle\int_a^{a+2\pi} f(x)dx$.

To determine $a_n$ for $n \geq 1$ we multiply both sides of Equation (3.2.2) by $\cos(mx)$ (where $m$ is an integer and $m \geq 1$) and integrate term-by-term from $a$ to $a + 2\pi$

$$\begin{aligned}\int_a^{a+2\pi} f(x)\cos(nx)dx &= a_0/2 \underbrace{\int_a^{a+2\pi} \cos(nx)dx}_{=0} + \underbrace{\sum_{m\in\mathbb{N}^*} a_m \int_a^{a+2\pi} \cos(mx)\cos(nx)dx}_{=a_n\pi} \\ &+ \underbrace{\sum_{m\in\mathbb{N}^*} b_m \int_a^{a+2\pi} \sin(mx)\cos(nx)dx}_{=0},\end{aligned}$$

Solving for $a_n$, we get the desired relation. Similarly, if we multiply both sides of Equation (3.2.2) by $\sin(mx)$ and integrate from $a$ to $a + 2\pi$, we get $b_n = \dfrac{1}{\pi}\displaystyle\int_a^{a+2\pi} f(x)\sin(nx)dx$. $\square$

## 3.3 Fourier series

When the French mathematician Joseph Fourier (1768-1830) was trying to solve a problem in heat conduction, he needed to express a function as an infinite series of sine and cosine functions.

Earlier, **Daniel Bernoulli and Leonard Euler** had used such series while investigating problems concerning vibrating strings and astronomy. The series in Equation (3.2.2) is called a trigonometric series or Fourier series and it turns out that expressing a function as a Fourier series is sometimes more advantageous than expanding it as a power series. In particular, astronomical phenomena are usually periodic, as are heartbeats, tides, and vibrating strings, so it makes sense to express them in terms of periodic functions.

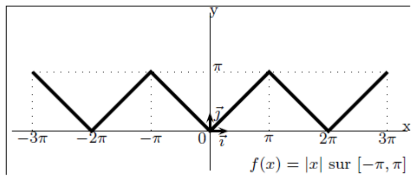
**Definition 3.3.1.** A function $f$ is said to have a period $T$ or to be periodic with period $T$ if for all $x \in \mathbb{R}$, $f(x+T) = f(x)$, where $T$ is a positive constant. The least value of $T > 0$ is called the least period or simply the period of $f$.

**Example 3.3.1.**

- The functions $\sin x$ has periods $2\pi, 4\pi, 6\pi, ...,$ since $\sin(x+2\pi)$, $\sin(x+4\pi)$, $\sin(x+6\pi); ...$ all equal $\sin x$. However, $2\pi$ is the least period or the period of $\sin x$.

- The period of $\sin(nx)$ or $\cos(nx)$, where $n$ is a positive integer, is $2\pi/n$. The period of $\tan x$ is $\pi$.

- A constant has any positive number as period.




**Definition 3.3.2.** $f:\mathbb{R}\to\mathbb{R}$ is even if $f(-x)=f(x)$ for all $x$ and $f$ is odd if $f(-x)=-f(x)$ for all $x$.

**Example 3.3.2.** $x^3,\ x^5-3x^3+2x,\ \sin x,\ \tan(3x)$ are odd functions.
Note that $(even)(even)=even,\ (odd)(odd)=even$ and $(even)(odd)=odd$.

In the previous paragraph, we have introduced the notion of trigonometric series and we have seen, that in the case of pointwise convergence we can calculate its coefficients (trigonometric and exponential)

**Definition 3.3.3.** Let $f$ be defined in the interval $(-L,L)$ and outside of this interval by $f(x+2L)=f(x)$, i.e., $f$ is $2L-$periodic. It is through this avenue that a new function on an infinite set of real numbers is created from the image on $(-L,L)$. The **Fourier series or Fourier expansion** corresponding to $f$ is given by

$$\frac{a_0}{2}+\sum_{n\in\mathbb{N}^*}\left(a_n\cos\left(\frac{n\pi x}{L}\right)+b_n\sin\left(\frac{n\pi x}{L}\right)\right). \tag{3.3.1}$$

where the Fourier coefficients(called also: trigonometric coefficients) $a_n$ and $b_n$ are

$$\begin{cases} a_n = \dfrac{1}{L}\displaystyle\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx \\ b_n = \dfrac{1}{L}\displaystyle\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx \end{cases} \qquad n=0,1,2,.... \tag{3.3.2}$$

Similarly, if the **Fourier series or Fourier expansion** corresponding to $f$ is given by

$$c_0+\sum_{n\in\mathbb{Z}^*}c_n e^{i\frac{n\pi x}{L}}. \tag{3.3.3}$$

where the Fourier coefficients(called also: exponential coefficients) $c_n$ are

$$c_n = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-i\frac{n\pi x}{L}}dx \qquad n = .... -3,-2,-1,0,1,2,3,.... \tag{3.3.4}$$

**Example 3.3.3.** Obtain the Fourier series of the following function defined in $(0,2\pi)$, by

$$f(x) = \begin{cases} x & \text{if } 0 < x < \pi, \\ \pi & \text{if } \pi \le x < 2\pi. \end{cases} \qquad \text{(and has period } 2\pi).$$

**Solution.**

- **Step one.**

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_0^{2\pi} f(x)dx \\ &= \frac{1}{\pi}\int_0^{\pi} xdx + \frac{1}{\pi}\int_{\pi}^{2\pi} \pi dx \\ &= \frac{1}{\pi}\left[x^2/2\right]_0^{\pi} + \frac{1}{\pi}\left[\pi x\right]_{\pi}^{2\pi} \\ &= \frac{3\pi}{2}. \end{aligned}$$

- **Step two.**

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\cos(nx)dx \\ &= \frac{1}{\pi}\int_0^{\pi} x\cos(nx)dx + \frac{1}{\pi}\int_{\pi}^{2\pi} \pi\cos(nx)dx \\ &= \frac{1}{\pi}\left[\frac{1}{n}\left(\pi\sin(n\pi) - 0\sin(0.n)\right) - \left(-\frac{\cos(nx)}{n^2}\right)_0^{\pi}\right] + \frac{1}{n}(\sin(2n\pi) - \sin(n\pi)) \\ &= \frac{1}{n^2\pi}(\cos(n\pi) - 1) = \begin{cases} \dfrac{-2}{n^2\pi}, & n \text{ odd} \\ 0 & n \text{ even} \end{cases} \end{aligned}$$

- **Step three.**

$$
\begin{aligned}
b_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\sin(nx)dx \\
&= \frac{1}{\pi}\int_0^{\pi} x\sin(nx)dx + \frac{1}{\pi}\int_{\pi}^{2\pi} \pi\sin(nx)dx \\
&= \frac{1}{\pi}\left[\left(\frac{-\pi\cos(n\pi)}{n}+0\right)+\left(\frac{\sin(nx)}{n^2}\right)_0^{\pi}\right] - \frac{1}{n}(\cos(2n\pi)-\cos(n\pi)) \\
&= -\frac{1}{n}(-1)^n + 0 - \frac{1}{n}(1-(-1)^n) = -\frac{1}{n}.
\end{aligned}
$$

we now have

$$f(x) = \frac{3\pi}{4} + \sum_{n=1}^{\infty}\left(-\frac{2}{\pi(2n+1)^2}\cos((2n+1)x) - \frac{1}{n}\sin(nx)\right)$$

### 3.3.1 The Dirichlet Test:

A Theorem on the Convergence of a Fourier Series

**Theorem 3.3.1.** *1. Let $f$ be a function that is defined and finite on $(-L, L)$, except possibly at a finite number of points inside this interval.*

*2. Let $f$ be periodic of period $2L$ outside $(-L, L)$.*

*3. Assume that $f, f'$ are piecewise continuous in $(-L, L)$ (this means that $f$ and its derivative are each continuous except possibly at a finite number of points).*

*Then the Fourier series of $f$ converges to:*

$(a)$ *$f(x)$, if $f$ is continuous at $x$.*

$(b)$ *$\frac{f(x+0)+f(x-0)}{2}$ if $f$ is not continuous at $x$ (i.e., **It converges to the average value of $f$ at $x$**).*

Here $f(x+0)$ and $f(x-0)$ are the right and left hand limits of $f(x)$ at $x$ and represent $\lim_{\epsilon\to 0^+} f(x+\epsilon)$ and $\lim_{\epsilon\to 0^+} f(x-\epsilon)$, respectively.
The conditions (1), (2), and (3) imposed on $f$ are **sufficient but not necessary,** and are generally satisfied in practice. **There are at present no known necessary and sufficient conditions for convergence of Fourier series.** It is of interest that continuity of $f$ does not alone ensure convergence of a Fourier series.

### 3.3.2 Half range Fourier sine or cosine series

A half range Fourier sine or cosine series is a series in which only sine terms or only cosine terms are present, respectively. When a half range series corresponding to a given function is desired, the function is generally defined in the interval $(0, L)$ [which is half of the interval $(-L, L)$, thus accounting for the name half range] and then the function is specified as odd or even, so that it is clearly defined in the other half of the interval, namely, $(-L, 0)$. In such case, we have

$$\begin{cases} f \text{ even}: & b_n = 0,\ a_n = \dfrac{2}{L}\displaystyle\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx & \text{for half range cosine series} \\ f \text{ odd}: & a_n = 0,\ b_n = \dfrac{2}{L}\displaystyle\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx & \text{for half range sine series} \end{cases} \tag{3.3.5}$$

If $f(x)$ is any function defined on an interval of the form $[0, L]$ we define its even extension to $[-L, L]$ by setting $f(x) = f(-x)$ for $x$ in $[-L, 0]$ (or by reflecting its graph about the $y-$axis). Similarly, we define its odd extension to $[-L, L]$ by setting $f(x) = -f(-x)$ for $x$ in $[-L, 0]$, or by reflecting its graph about the origin. The reason for these definitions is that is that we can create an even function (over $(-L, L)$) out of a function that is given only on half-the-range, i.e., $(0, L)$. Similarly, we can create an odd function (over $(-L, L)$) out of a function that is given only on half-the-range, i.e., $(0, L)$. We do this because we may want to expand a function in terms of a pure cosine series only (in which case we use the even extension since we don't want any sine terms) or in terms a pure sine series (in which case we use the odd extension since we don't want any cosine terms).

**Example 3.3.4.** Find the even and odd extension of the function $f$ defined by $f(x) = x(\pi - x)$ for $0 \leq x \leq \pi$.

**Solution:**

- We recall the definition of an even function: For f to be even on $[-\pi, \pi]$ we must have $f(x) = f(-x)$. To get the form of $f$ on the part $[-\pi, 0]$ we replace $x$ by $''-x''$ in the definition of $f(x)$ : This gives $f(-x) = -x(\pi + x)$ for $x$ in $[-\pi, 0]$.

- This case is similar to the first part. We know that $f$ is odd only when $f(x) = -f(-x)$. So to get the form of $f$ on the left interval $[-\pi, 0]$, we calculate the value of $-f(-x)$ using the given expression on $[0, \pi]$. Just as before we replace $x$ by $''-x''$ in the definition of $f(x)$ : This gives $-f(-x) = x(\pi + x)$ for $x$ in $[-\pi, 0]$ (note the removal of the minus sign).

### 3.3.3 Pareval's identity

If $a_n$ and $b_n$ are the Fourier coefficients corresponding to $f$ and if $f$ satisfies the **Dirichlet conditions**(see, theorem 3.3.1). Then

$$\frac{a_0^2}{2} + \sum_{n\geq 1}(a_n^2 + b_n^2) = \frac{1}{L}\int_{-L}^{L} |f(x)|^2 dx. \tag{3.3.6}$$

In general there always holds Bessel's Inequality, that is,

$$\frac{a_0^2}{2} + \sum_{n\geq 1}(a_n^2 + b_n^2) \leq \frac{1}{L}\int_{-L}^{L} |f(x)|^2 dx. \tag{3.3.7}$$

This result is valid for any function that is piecewise continuous on $(-L, L)$ (whether or not its Fourier series actually equals $f(x)$!)

**Example 3.3.5.** Find the Fourier series of the function defined in pieces (sometimes called a piecewise constant function) by

$$f(x) = \begin{cases} 8, & 0 < x < 2 \\ -8, & 2 < x < 4 \end{cases}$$

where $f$ is periodic with period 4. What does the series converge to at $x = 2$? and at $x = 3$? Using Parseval's Equality, show that $\sum_{n \text{ odd}} \frac{1}{n^2}\frac{\pi^2}{8}$.

**Solution.** Since the function has period 4, the graph of $f$ on the interval $[-4, 0]$ must be the same as (or a translate of) the one on $[0, 4]$. In other words, we must have,

$$f(x) = \begin{cases} 8, & -4 < x < -2 \\ -8, & -2 < x < 0. \end{cases}$$

Therefore $f$ is an odd function (why?) on $[-4, 4]$ and so its Fourier series is a pure sine series. Since the period is $P = 4$ here, we get that $P = 2L$ implies that $L = 2$. The Fourier series looks like,

$$f(x) \underbrace{\sim}_{\text{not necessarily equal}} \sum_{n\geq 1} b_n \sin\left(\frac{n\pi x}{2}\right),$$

where

$$\begin{aligned} b_n &= \frac{1}{2}\int_{-2}^{2} f(x)\sin\left(\frac{n\pi x}{2}\right)dx \\ &= \underbrace{\frac{1}{2}\int_{-2}^{0}(-8)\sin\left(\frac{n\pi x}{2}\right)dx}_{\frac{8}{n\pi}(1-\cos(n\pi))} + \underbrace{\frac{1}{2}\int_{0}^{2}(+8)\sin\left(\frac{n\pi x}{2}\right)dx}_{\frac{8}{n\pi}(1-\cos(n\pi))} \\ &= \frac{16}{n\pi}(1-\cos(n\pi)). \end{aligned}$$

Therefore,

$$f(x) \sim \sum_{n\geq 1}\frac{16}{n\pi}(1-\cos(n\pi))\sin\left(\frac{n\pi x}{2}\right).$$

Next, we need to use the **Dirichlet Test**: Now, this function $f$ is **NOT** continuous at $x = 2$ (since its left limit is 8 while its right limit is $-8$). It follows that when $x = 2$ the Fourier series converges to $\dfrac{f(x+0)+f(x-0)}{2} = \dfrac{-8+8}{2} = 0$. This result is easy to verify directly since at $x = 2$ the sine term in the Fourier series is $\sin(n\pi) = 0$ since $n$ is always an integer! However, at $x = 3$ the function **IS** continuous and its value there is $f(3) = -8$. Thus, we find

$$\sum_{n\geq 1}\frac{16}{n\pi}(1-\cos(n\pi))\sin\left(\frac{3n\pi}{2}\right) = -8.$$

We know the $b_n$'s and $f(x)$. So we can conclude that:

$$\begin{aligned} \sum_{n\geq 1} b_n^2 = \sum_{n\geq 1}\frac{16^2}{n^2\pi^2}(1-\cos(n\pi))^2 &= \frac{1}{L}\int_{-L}^{L}(f(x))^2dx \\ &= \frac{1}{2}\int_{-2}^{2}64dx = 128, \end{aligned}$$

i.e.,

$$\sum_{n\geq 1}\frac{(1-\cos(n\pi))^2}{n^2\pi^2} = \frac{128}{256} = \frac{1}{2}.$$

Since $(1-\cos(n\pi)) = 0$ whenever $n$ is even and $(1-\cos(n\pi)) = 2$ whenever $n$ is odd, we obtain the desired equality.

**Example 3.3.6.** Find the Fourier "cosine" series of the function defined by $f(x) = x(\pi - x)$, for $x$ in $(0,\pi)$.

**Solution:** Since we want a **cosine** series for $f(x)$ the extension of $f$ to $(-\pi, 0)$ must be **even** (no sine terms allowed in the Fourier series expansion). Now refer to (Example (3.3.4)). We know that the even extension of $f(x)$ looks like

$$f(x)=\begin{cases} x(\pi-x), & 0<x<\pi, \\ -x(\pi+x), & -\pi<x<0, \end{cases}$$

This extended function is even and periodic with period $\pi$, so $L=\pi/2$. The Fourier cosine coefficients are now given by

$$a_0=\frac{2}{L}\int_0^L f(x)dx=\frac{4}{\pi}\int_0^{\pi/2} x(\pi-x)dx=\pi^2/3.$$

and

$$a_n=\frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx=\frac{4}{\pi}\int_0^{\pi/2} x(\pi-x)\cos(2nx)\,dx=-\frac{1}{n^2}.$$

So, the Fourier cosine series of this function $f(x)$ is given by

$$f(x)=\pi^2/6-\sum_{n\geq 1}\frac{1}{n^2}\cos(2nx),$$

with convergence properties according to the **Dirichlet Test.** In particular, since $f$ is continuous at $x=0$ and $f(0)=0$, it follows that $\pi^2/6=\sum_{n\geq 1}\frac{1}{n^2}$.

**Example 3.3.7.** Use Example 3.3.6 and Parseval's Equality to show that $\sum_{n\geq 1}\frac{1}{n^4}=\frac{\pi^4}{90}$.

**solution:** We know $a_0, a_n=-1/n^2,\ b_n=0$. Note that $a_0^2=\pi^4/18$. So, by Parseval's Equality we get

$$\pi^4/18+\sum_{n\geq 1}\frac{1}{n^4}=\frac{1}{L}\int_{-L}^{L}|f(x)|^2dx=2/\pi\int_{-\pi/2}^{\pi/2}|f(x)|^2dx,$$

However,

$$2/\pi\int_{-\pi/2}^{\pi/2}|f(x)|^2dx=2/\pi\int_{-\pi/2}^{0}\left[-x(\pi-x)\right]^2dx+2/\pi\int_0^{\pi/2}\left[x(\pi-x)\right]^2dx=\pi^4/15.$$

Combining these results we obtain

$$\sum_{n\geq 1}\frac{1}{n^4}=\frac{\pi^4}{15}-\frac{\pi^4}{18}=\frac{\pi^4}{90}.$$

### 3.3.4 Integrating and Differentiating Fourier Series

We can integrate a Fourier series term by term provided the conditions of the Dirichlet Test hold. Indeed, if the function $f$ is piecewise continuous on $[-L, L]$ and the points $a, x$ are in $[-L, L]$ and $f(x)$ has the expansion given by (3.3.1), then

$$\int_a^x f(t)dt = a_0/2\int_a^x dt + \sum_{n\in\mathbb{N}^*}\left(a_n\int_a^x \cos\left(\frac{n\pi t}{L}\right)dt + b_n\int_a^x \sin\left(\frac{n\pi t}{L}\right)dt\right),$$

or we can integrate the series term by term after which the new series will converge to the integral of the original series given on the left.

**Example 3.3.8.** Expand $f(x) = \cos x$, for $0 < x < \pi$, in a pure Fourier "sine" series on $(0, \pi)$.

Show that the Fourier cosine series of the sine function on $(0, \pi)$ is given by

$$\sin x = \frac{4}{\pi}\sum_{n\geq 1}\frac{1}{(4n^2-1)} - \sum_{n\geq 1}\frac{\cos(2nx)}{(4n^2-1)}.$$

**Solution:**

1. Note that $''\cos x''$ is an **even** function while only odd functions can have pure sine series expansions.

2. So we must extend $\cos x$ to be an odd function on $(-\pi, \pi)$ by taking its odd extension to $(-\pi, 0)$.
   Write $f(x) = \cos x$ for $x$ in $[0, \pi]$. Then $-f(-x) = -\cos(-x) = -\cos x$ since $\cos(-x) = \cos x$ by trigonometry. It follows that the odd extension of $\cos x$ is given by the modified function.

   $$f(x) = \begin{cases} \cos x, & 0 < x < \pi, \\ -\cos x, & -\pi < x < 0, \end{cases}$$

   Note that this odd extension is not continuous at $x = 0$.

3. The resulting extended $f(x)$ is now an **odd periodic** function of period $\pi$, (**not** $2\pi$ **as one may think!**) i.e., $f(x+\pi) = f(x)$. Since $P = 2L$ it follows that $L = \pi/2$.

Furthermore, $f(x)$ is defined on $(-L, L) = (-\pi/2, \pi/2)$ and formally, its Fourier series representation looks like

$$f(x) \sim \sum_{n\geq 1} b_n \sin\left(\frac{n\pi x}{L}\right) = \sum_{n\geq 1} b_n \sin(2nx),$$

(there can be no $a_n$'s since $f(x)$ is odd on $(-\pi/2, \pi/2)$). The Fourier sine coefficients are given by

$$\begin{aligned}
b_n &= \frac{2}{\pi}\int_{-\pi/2}^{\pi/2} f(x)\sin(2nx)\,dx \\
&= \frac{2}{\pi}\int_{-\pi/2}^{0} -\cos(x)\sin(2nx)\,dx + \frac{2}{\pi}\int_{0}^{\pi/2}\cos(x)\sin(2nx)\,dx \\
&= \frac{4n}{\pi(4n^2-1)} + \frac{4n}{\pi(4n^2-1)} \\
&= \frac{8n}{\pi(4n^2-1)}
\end{aligned}$$

The Fourier series of this extended cosine function is therefore of the form

$$\cos x = \frac{8}{\pi}\sum_{n\geq 1}\frac{n}{(4n^2-1)}\sin(2nx).$$

for any $x$ in $(-\pi/2, \pi/2)$ and outside this interval by periodicity (or periodically repeating the graph). In particular we see that at $x = \pi/4$ we get the result

$$\frac{\pi\sqrt{2}}{16} = \sum_{n\geq 1}\frac{n}{(4n^2-1)}\sin(n\pi/2).$$

Note: At $x = \pi$ the Fourier series converges to $\dfrac{f(x+0)+f(x-0)}{2} = \dfrac{1+(-1)}{2} = 0$ (O.K. by the Dirichlet Test) so, in order to get convergence at this point, we need to define $f(\pi) = 0$.

We know that

$$\cos t = \frac{8}{\pi}\sum_{n\geq 1}\frac{n}{(4n^2-1)}\sin(2nt).$$

is valid for $t$ in $(-\pi, \pi)$. Since this function satisfies all the conditions of Dirichlet's Test we can choose $a = 0$ and fix a value of $x$ in $(-\pi, \pi)$. We now integrate both sides of this last display over $[0, x]$ and find

$$\sin x = \int_0^x \cos t dt = \frac{8}{\pi} \sum_{n \geq 1} \frac{n}{(4n^2 - 1)} \int_0^x \sin(2nt) dt = \frac{4}{\pi} \sum_{n \geq 1} \frac{1 - \cos 2nx}{(4n^2 - 1)}.$$

**Example 3.3.9. Additional example.**

1. Find the Fourier series of the function $f$ defined by $f(x) = x^2$ on the interval $[-2, 2]$. What does the series converge to when $x = 0$?

2. Use $(1)$ to calculate the Fourier series of the function $f(x) = x^3$ defined on $[-2, 2]$.

### 3.3.5 Differentiating a Fourier series

Differentiating a Fourier series can be a risky business! This is because the differentiated series may not converge at all (let alone to the function it is supposed to represent) as we will see in the next example.

**Example 3.3.10.** Calculate the Fourier sine series of the function $f(x) = x$ for $x$ in $(0, 2)$ and show that its differentiated series does not converge at all except for $x = 0$.
**Solution:** Since we want a Fourier sine series we must extend this $f$ to the interval $(-2, 0)$ using its odd extension. This means that $f(x) = -f(-x) = -(-x) = x$ for $x$ in $(-2, 0)$. But this means that $f(x) = x$ for all $x$ in $(-2, 2)$. Of course, this means that $f(x) = x$ is already an odd function at the outset (but we may not have noticed this). A simple calculation (we omit the details) shows that, since $P = 2$, then, $b_n = -2\frac{(-1)^n}{n\pi}$, it follows that

$$x = \frac{2}{\pi} \sum_{n \geq 1} \frac{(-1)^{n+1}}{n} \sin(n\pi x).$$

Differentiating this series "formally" (this means "without paying any attention to the details") we find the Ą"equality"

$$1 = 2 \sum_{n \geq 1} (-1)^{n+1} \sin(n\pi x).$$

Unfortunately, the series on the right **CANNOT** converge since

$$\lim_{n \to \infty} \left| (-1)^{n+1} \sin(n\pi x) \right|$$

does not exist! So, how does one handle the differentiation of Fourier series? There is a test we can cite that can be used without too much effort.

**Test for Differentiating a Fourier Series** Let $f$ be a continuous function for all $x \in [-L, L]$ and assume that f$(-L) = f(L)$. Extend $f$ to a periodic function of period $2L$ outside $[-L, L]$ by periodicity. Assume that $f$ is piecewise differentiable in $(-L, L)$ having finite left and right derivatives at Ą$\pm L$. Then the **differentiated Fourier series converges to** $f'(x)$ **on** $[-L, L]$**.**

**Example 3.3.11.** Find the Fourier sine series of the function $\dfrac{\pi x(\pi - x)}{8}$ valid on $(0, \pi)$ and find the value of its differentiated series.

**Solution:** Note that here, $L = \pi$. The odd extension of this function is given by

$$f(x) = \begin{cases} \dfrac{1}{8}\pi x(\pi - x), & 0 < x < \pi, \\ \dfrac{1}{8}\pi x(\pi + x), & -\pi < x < 0, \end{cases}$$

The Fourier coefficients are given by

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)dx = \frac{(1 + (-1)^{n+1})}{2n^3},$$

and so the Fourier series is

$$\sin x + \sin 3x/(3^3) + \sin 5x/(5^3) + .... = \sum_{n\geq 1} \frac{(1 + (-1)^{n+1})}{2n^3} \sin nx.$$

Note that f satisfies the conditions of the Test, above. It is continuous everywhere and it fails to have a derivative at points of the form $\pm\pi$, where $n$ is an integer. The differentiated series looks like

$$\cos x + \cos 3x/(3^2) + \cos 5x/(5^2) + .... = \sum_{n\geq 1} \frac{(1 + (-1)^{n+1})}{2n^2} \sin nx.$$

and so we can conclude that

$$f'(x) = \pi^2/8 - \pi x/4 = \sum_{n\geq 1} \frac{(1 + (-1)^{n+1})}{2n^2} \sin nx,$$

holds for $x$ in the range $[-\pi, \pi]$. When $x = 0$ we recover the result of Example 3.3.5 using a different method.