Math 262
Practice Problems Solutions
Power Series and Taylor Series

1. For each of the following power series, find the interval of convergence and the radius of convergence:

   (a) $\sum_{n=1}^{\infty}(-1)^n n^2 x^n$

   Notice that $a_{n+1} = (-1)^{n+1}(n+1)^2 x^{n+1}$. Then $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{(n+1)^2|x|^{n+1}}{n^2|x|^n} = \lim_{n\to\infty}|x|\frac{n^2+2n+1}{n^2}$ $= |x|\lim_{n\to\infty}\frac{2n+2}{2n} = |x|\lim_{n\to\infty}\frac{2}{2} = |x|$, so this series converges absolutely for $-1 < x < 1$.

   Notice when $x = 1$, we have $\sum_{n=1}^{\infty}(-1)^n n^2 1^n = \sum_{n=1}^{\infty}(-1)^n n^2$ which diverges by the nth term test.

   Similarly, when $x = -1$, we have $\sum_{n=1}^{\infty}(-1)^n n^2(-1)^n = \sum_{n=1}^{\infty}(-1)^2 nn^2 = \sum_{n=1}^{\infty} 1$ which diverges by the nth term test.

   Hence, the interval of convergence is: $(-1, 1)$ and the radius convergence is: $R = 1$.

   (b) $\sum_{n=1}^{\infty}\frac{2^n}{n^2}(x-3)^n$

   Notice that $a_{n+1} = \frac{2^{n+1}}{(n+1)^2}(x-3)^{n+1}$. Then $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{2^{n+1}|x-3|^{n+1}}{(n+1)^2}\cdot\frac{n^2}{2^n|x-3|^n}$ $= \lim_{n\to\infty}|x-3|\cdot 2\cdot\frac{n^2+2n+1}{n^2} = 2|x-3|\lim_{n\to\infty}\frac{2n+2}{2n} = 2|x-3|\lim_{n\to\infty}\frac{2}{2} = 2|x-3|$, so this series converges absolutely when $|x-3| < \frac{1}{2}$, or for $\frac{5}{2} < x < \frac{7}{2}$.

   Notice when $x = \frac{5}{2}$, we have $\sum_{n=1}^{\infty}\frac{2^n}{n^2}(-\frac{1}{2})^n = \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$ Thus, since $\sum_{n=1}^{\infty}\frac{1}{n^2}$ is a convergent p-series, the original series converges absolutely.

   Similarly, when $x = \frac{7}{2}$, we have $\sum_{n=1}^{\infty}\frac{2^n}{n^2}(\frac{1}{2})^n = \sum_{n=1}^{\infty}\frac{(1)^n}{n^2} = \sum_{n=1}^{\infty}\frac{1}{n^2}$, which is a convergent p-series.

   Hence, the interval of convergence is: $\left[\frac{5}{2}, \frac{7}{2}\right]$ and the radius convergence is: $R = \frac{1}{2}$.

   (c) $\sum_{n=1}^{\infty}\frac{n^3}{3^n}(x+1)^n$

   Notice that $a_{n+1} = \frac{(n+1)^3}{3^{n+1}}(x+1)^{n+1}$. Then $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{(n+1)^3|x+1|^{n+1}}{3^{n+1}}\cdot\frac{3^n}{n^3|x+1|^n}$ $= \frac{1}{3}|x+1|\lim_{n\to\infty}\frac{(n+1)^3}{n^3}$, which, after a few applications of L'Hôpital's Rule, is $\frac{|x+1|}{3}$, so this series converges absolutely when $|x+1| < 3$ or for $-4 < x < 2$.

   Notice when $x = -4$, we have $\sum_{n=1}^{\infty}\frac{n^3}{3^n}(-3)^n = \sum_{n=1}^{\infty}(-1)^n n^3$, which diverges by the nth term test.

   Similarly, when $x = 2$, we have $\sum_{n=1}^{\infty}\frac{n^3}{3^n}3^n = \sum_{n=1}^{\infty} n^3$ which diverges by the nth term test.

   Hence, the interval of convergence is: $(-4, 2)$ and the radius convergence is: $R = 3$.

(d) $\sum_{n=1}^{\infty}(-1)^n \frac{10^n}{n!}(x-10)^n$

Notice that $a_{n+1} = (-1)^{n+1}\frac{10^{n+1}}{(n+1)!}(x-10)^{n+1}$. Then $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{10^{n+1}|x-10|^{n+1}}{(n+1)!}\cdot\frac{n!}{10^n|x-10|^n}$
$= |x-10|\lim_{n\to\infty}\frac{10}{n+1} = 0$

Hence the interval of convergence is $(-\infty,\infty)$ and $R=\infty$.

(e) $\sum_{n=1}^{\infty}(-1)^n \frac{1}{n10^n}(x-2)^n$

Notice that $a_{n+1} = (-1)^{n+1}\frac{1}{(n+1)10^{n+1}}(x-2)^{n+1}$. Then $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{|x-2|^{n+1}}{(n+1)10^{n+1}}\cdot\frac{n10^n}{|x-2|^n}$
$= \frac{1}{10}|x-2|\lim_{n\to\infty}\frac{n}{n+1} = \frac{1}{10}|x-2|$, so this series converges absolutely when $|x-2|<10$ or for $-8<x<12$.

Notice when $x=-8$, we have $\sum_{n=1}^{\infty}(-1)^n\frac{1}{n10^n}(-10)^n = \sum_{n=1}^{\infty}(-1)^n\frac{1}{n}(-1)^n = \sum_{n=1}^{\infty}\frac{1}{n}$, which diverges since it is the harmonic series.

Similarly, when $x=10$, we have $\sum_{n=1}^{\infty}(-1)^n\frac{1}{n10^n}10^n = \sum_{n=1}^{\infty}(-1)^n\frac{1}{n}$ which converges by the Alternating Series Test.

Hence, the interval of convergence is: $(-8,10]$ and the radius convergence is: $R=10$.

2. Use a known series to find a power series in x that has the given function as its sum:

(a) $x\sin(x^3)$

Recall the Maclaurin series for $\sin u = \sum_{n=0}^{\infty}(-1)^n\frac{u^{2n+1}}{(2n+1)!}$

Therefore, $\sin(x^3) = \sum_{n=0}^{\infty}(-1)^n\frac{(x^3)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty}(-1)^n\frac{(x)^{6n+3}}{(2n+1)!}$.

Hence $x\sin(x^3) = \sum_{n=0}^{\infty}(-1)^n\frac{(x)^{6n+4}}{(2n+1)!}$.

(b) $\frac{\ln(1+x)}{x}$

Recall the Maclaurin series for $\ln(1+x) = \sum_{n=0}^{\infty}(-1)^n\frac{x^{n+1}}{n+1}$

Therefore, $\frac{\ln(1+x)}{x} = \sum_{n=0}^{\infty}(-1)^n\frac{x^n}{n+1}$

(c) $\frac{x-\arctan x}{x^3}$

Recall the Maclaurin series for $\arctan(x) = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\cdots$

Therefore, $x-\arctan(x) = x - \left(x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\cdots\right) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^{2n+1}}{2n+1}$

Hence $\frac{x-\arctan x}{x^3} = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^{2n-2}}{2n+1}$

3. Use a power series to approximate each of the following to within 3 decimal places:

   (a) $\arctan\dfrac{1}{2}$

   Notice that the Maclaurin series $\arctan(x) = \displaystyle\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1}$ is an alternating series satisfying the hypotheses of the alternating series test when $x = \frac{1}{2}$. Then to find our approximation, we need to find n such that $\dfrac{(.5)^{2n+1}}{2n+1} < .0005$.

   $a_0 = \dfrac{1}{2}$, $a_1 = -\dfrac{1}{24} \approx 0.04667$, $a_3 = \dfrac{1}{160} = 0.00625$, $a_4 = -\dfrac{1}{896} \approx -0.001116$, and $a_5 \approx 0.00217$

   Hence $\arctan\dfrac{1}{2} \approx \dfrac{1}{2} - \dfrac{1}{24} + \dfrac{1}{160} - \dfrac{1}{896} \approx 0.463$

   (b) $\ln(1.01)$

   Notice that the Maclaurin series $\ln(1+x) = \displaystyle\sum_{n=0}^{\infty}(-1)^n \frac{x^{n+1}}{n+1}$ is an alternating series satisfying the hypotheses of the alternating series test when $x = 0.01$. Then to find our approximation, we need to find n such that $\dfrac{(0.1)^{n+1}}{n+1} < .0005$.

   $a_0 = 0.01$, $a_1 = -0.00005$

   Hence $\ln(1.01) \approx 0.010$

   (c) $\sin\left(\frac{\pi}{10}\right)$

   Notice that the Maclaurin series $\sin x = \displaystyle\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!}$ is an alternating series satisfying the hypotheses of the alternating series test when $x = \frac{\pi}{10}$. Then to find our approximation, we need to find n such that $\dfrac{(\frac{\pi}{10})^{2n+1}}{(2n+1)!} < .0005$.

   $a_0 = \frac{\pi}{10} \approx 0.314159$, $a_1 \approx -0.0051677$, $a_2 \approx 0.0000255$

   Hence $\sin\left(\dfrac{\pi}{10}\right) \approx 0.314159 - 0.0051677 \approx 0.309$

4. For each of the following functions, find the Taylor Series about the indicated center and also determine the interval of convergence for the series.

   (a) $f(x) = e^{x-1}$, $c = 1$

   Notice that $f'(x) = e^{x-1}$ and $f''(x) = e^{x-1}$. In fact, $f^{(n)}(x) = e^{x-1}$ for every n.

   Then $f^{(n)}(1) = e^0 = 1$ for every n, and hence $a_n = \frac{1}{n!}$ for every n.

   Thus $e^{x-1} = \displaystyle\sum_{n=0}^{\infty} \frac{(x-1)^n}{n!}$.

   To find the interval of convergence, notice that $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{|x-1|^{n+1}}{(n+1)!}\cdot\frac{n!}{|x-1|^n} = |x-1|\lim_{n\to\infty}\frac{1}{n+1} = 0$

   Thus this series converges on $(-\infty, \infty)$ and $R = \infty$.

   (b) $f(x) = \cos x$, $c = \dfrac{\pi}{2}$

   $f'(x) = -\sin x$, $f''(x) = \cos x$, $f'''(x) = \sin x$, $f^4(x) = -\cos x$, and the same pattern continues from there.

   Therefore, $f\left(\dfrac{\pi}{2}\right) = \cos\dfrac{\pi}{2} = 0$ $f'\left(\dfrac{\pi}{2}\right) = -\sin\dfrac{\pi}{2} = -1$, $f''\left(\dfrac{\pi}{2}\right) = -\cos\dfrac{\pi}{2} = 0$, $f'''\left(\dfrac{\pi}{2}\right) = \sin\dfrac{\pi}{2} = 1$, $f^4\left(\dfrac{\pi}{2}\right) = \cos\dfrac{\pi}{2} = 0$, and the pattern continues from there.

   Therefore, $a_0 = 0$, $a_1 = -1$, $a_2 = 0$, $a_3 = \frac{1}{3!} = \frac{1}{6}\cdots$

   Hence the series is: $\cos x = \displaystyle\sum_{n=0}^{\infty}(-1)^{n+1}\frac{1}{(2n+1)!}(x - \frac{\pi}{2})^{2n+1}$

   To find the interval of convergence, notice that $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{|x-\frac{\pi}{2}|^{n+1}}{(2n+3)!}\cdot\frac{(2n+1)!}{|x-\frac{\pi}{2}|^n}$

   $= |x - \dfrac{\pi}{2}|\displaystyle\lim_{n\to\infty}\frac{1}{(2n+3)(2n+2)} = 0$

   Thus this series converges on $(-\infty, \infty)$ and $R = \infty$.

(c) $f(x) = \dfrac{1}{x}$, $c = -1$

$f'(x) = -x^{-2}$, $f''(x) = 2x^{-3}$, $f'''(x) = -6x^{-4}$, so $f^n(x) = (-1)^n x^{-(n+1)}$

Then $f(-1) = -1$, $f'(-1) = -1$, $f''(-1) = -2$, $f'''(-1) = -6$, and $f^n(-1) = -n!$.

Therefore, $a_0 = -1$, $a_1 = -1$, $a_2 = -1$, $a_3 = -1$, and, in fact, $a_n = -1$ for all n.

Hence $\dfrac{1}{x} = \displaystyle\sum_{n=0}^{\infty}(-1)(x-1)^n$

To find the interval of convergence, notice that $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{(-1)|x-1|^{n+1}}{(-1)|x-1|^n} = |x-1|$, so this series converges absolutely for $0 \le x \le 2$

When $x = 0$, we have $\displaystyle\sum_{n=0}^{\infty}(-1)(-1)^n$, which diverges by the nth term test.

Similarly, when $x = 2$ we have $\displaystyle\sum_{n=0}^{\infty}(-1)(1)^n$, which also diverges by the nth term test.

Thus this series converges on $(0, 2)$ and $R = 1$.

5. For each of the following functions, find the Taylor Polynomial for the function at the indicated center c. Also find the Remainder term.

(a) $f(x) = \sqrt{x}$, $c = 1$, $n = 3$.

First, $f'(x) = \dfrac{1}{2}x^{-\frac{1}{2}}$, $f''(x) = -\dfrac{1}{4}x^{-\frac{3}{2}}$, $f'''(x) = \dfrac{3}{8}x^{-\frac{5}{2}}$, and $f^{(4)}(x) = -\dfrac{15}{16}x^{-\frac{7}{2}}$.

Then $f(1) = 1$, $f'(1) = \dfrac{1}{2}$, $f''(1) = -\dfrac{1}{4}$, $f'''(1) = \dfrac{3}{8}$.

Hence $a_0 = 1$, $a_1 = \frac{1}{2}$, $a_2 = -\frac{1}{8}$, and $a_3 = \frac{1}{16}$

Thus $P_3(x) = 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2 + \frac{1}{16}(x-1)^3$

and $R_3(x) = \frac{f^{(4)}(z)}{4!}(x-1)^4 = \frac{5z^{-\frac{7}{2}}}{128}(x-1)^4$

(b) $f(x) = \ln x$, $c = 1$, $n = 4$.

First, $f'(x) = x^{-1}$, $f''(x) = -x^{-2}$, $f'''(x) = 2x^{-3}$, $f^{(4)}(x) = -6x^{-4}$, and $f^{(5)}(x) = 24x^{-5}$.

Then $f(1) = 0$, $f'(1) = 1$, $f''(1) = -1$, $f'''(1) = 2$, and $f^{(4)}(1) = -6$.

Hence $a_0 = 0$, $a_1 = 1$, $a_2 = -\frac{1}{2}$, $a_3 = \frac{1}{3}$, and $a_4 = -\frac{1}{4}$

Thus $P_4(x) = 0 + (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4$

and $R_4(x) = \frac{f^{(5)}(z)}{5!}(x-1)^5 = \frac{24z^{-5}}{120}(x-1)^5 = \frac{z^{-5}}{5}(x-1)^5$

(c) $f(x) = \sqrt{1+x^2}$, $c = 0$, $n = 4$.

First, $f'(x) = x(1+x^2)^{-\frac{1}{2}}$, $f''(x) = (1+x^2)^{-\frac{1}{2}} - x^2(1+x^2)^{-\frac{3}{2}}$, $f'''(x) = -3x(1+x^2)^{-\frac{3}{2}} + 3x^3(1+x^2)^{-\frac{5}{2}}$, $f^{(4)}(x) = -3(1+x^2)^{-\frac{3}{2}} + 18x^2(1+x^2)^{-\frac{5}{2}} - 15x^4(1+x^2)^{-\frac{7}{2}}$, and $f^{(5)}(x) = 45x(1+x^2)^{-\frac{5}{2}} - 150x^3(1+x^2)^{-\frac{7}{2}} + 105x^5(1+x^2)^{-\frac{9}{2}}$

Then $f(0) = 1$, $f'(0) = 0$, $f''(0) = 1$, $f'''(0) = 0$, and $f^{(4)}(0) = -3$.

Hence $a_0 = 1$, $a_1 = 0$, $a_2 = \frac{1}{2}$, $a_3 = 0$, and $a_4 = -\frac{1}{8}$

Thus $P_4(x) = 1 + \frac{1}{2}x^2 - \frac{1}{8}x^4$

and $R_4(x) = \frac{f^{(5)}(z)}{5!}x^5 = \frac{45z(1+z^2)^{-\frac{5}{2}} - 150z^3(1+z^2)^{-\frac{7}{2}} + 105z^5(1+z^2)^{-\frac{9}{2}}}{120}x^5$

6. Estimate each of the following using a Taylor Polynomial of degree 4. Also find the error for your approximation. Finally, find the number of terms needed to guarantee an accuracy or at least 5 decimal places.

(a) $e^{0.1}$

Recall that $e^x = \displaystyle\sum_{n=0}^{\infty}\frac{x^n}{n!}$.

Then $P_4(x) = 1 + x + \dfrac{x^2}{2} + \dfrac{x^3}{6} + \dfrac{x^4}{24}$, and $R_4 = \dfrac{e^z}{5!}x^5$

When $x = 0.1$, $P_4(x) \approx 1 + 0.1 + 0.005 + 0.0001667 + .000004167 = 1.105170867$

In general, $R_n(x) = \dfrac{f^{(n+1)(z)}}{(n+1)!}x^{n+1} = \dfrac{e^z}{(n+1)!}(0.1)^{n+1}$, where $0 \leq z \leq 0.1$.

Since e^x is increasing, we need to find n so that $\dfrac{e^{0.1}}{(n+1)!}(0.1)^{n+1} < 0.000005$

When we use $P_4(x)$, our error is at most $\dfrac{e^{0.1}}{5!}(0.1)^5 \approx 0.000000092$ (in fact, one would only need $P_3(x)$ to get within 5 decimal places).

(b) $\ln 0.9$

Recall that $\ln(1+x) = \displaystyle\sum_{n=0}^{\infty}(-1)^n \frac{x^{n+1}}{n+1}$.

We will take $x = -0.1$ so that $\ln(1+x) = \ln(.9)$

Then $P_4(x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4}$. Also, $f^{(5)}(x) = 24(1+x)^{-5}$.

Therefore, $R_4 = \dfrac{24(1+z)^{-5}}{5!}x^5$. In general, $R_n(x) = (-1)^n \dfrac{(1+z)^{-(n+1)}}{n+1}x^{n+1}$.

When $x = -0.1$, $P_4(x) \approx -0.1 - 0.005 - 0.000333333 - .000025 = -0.105358333$

Since $R_n(x) = \dfrac{f^{(n+1)(z)}}{(n+1)!}x^{n+1} = (-1)^n \dfrac{(1+z)^{-(n+1)}}{n+1}x^{n+1}$, where $-0.1 \leq z \leq 0$.

Since $\ln(1+x)$ is negative and increasing when $-.1 < x < 0$, we need to find n so that $(-1)^n \dfrac{(1-.1)^{-(n+1)}}{n+1}x^{n+1} < 0.000005$

When we use $P_4(x)$, our error is at most $\dfrac{(1-.1)^{-(5)}}{5}(0.1)^5 \approx 0.000084675$.

If we use $P_5(x)$, our error is at most $\dfrac{(1-.1)^{-(6)}}{6}(0.1)^6 \approx 0.000000314$, so this is a sufficient number of terms to approximate to at least 5 decimal places.

(c) $\sqrt{1.2}$

We will use $f(x) = \sqrt{x}$ centered at $c = 1$ and we will take $x = 1.2$.

Then $f'(x) = \dfrac{1}{2}x^{-\frac{1}{2}}$, $f''(x) = -\dfrac{1}{4}x^{-\frac{3}{2}}$, $f'''(x) = \dfrac{3}{8}x^{-\frac{5}{2}}$, $f^{(4)}(x) = -\dfrac{15}{16}x^{-\frac{7}{2}}$, and $f^{(5)}(x) = -\dfrac{105}{32}x^{-\frac{9}{2}}$.

Then $f(1) = 1$, $f'(1) = \dfrac{1}{2}$, $f''(1) = -\dfrac{1}{4}$, $f'''(1) = \dfrac{3}{8}$, and $f^{(4)(1) = -\frac{15}{16}}$.

Hence $a_0 = 1$, $a_1 = \frac{1}{2}$, $a_2 = -\frac{1}{8}$, $a_3 = \frac{1}{16}$, and $a_4 = -\frac{5}{128}$

Thus $P_4(x) = 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2 + \frac{1}{16}(x-1)^3 - \frac{5}{128}(x-1)^4$

and $R_4(x) = \frac{f^{(5)}(z)}{5!}(x-1)^5 = \frac{7z^{-\frac{9}{2}}}{256}(x-1)^5$

Thus $\sqrt{1.2} \approx P_4(1.2) = 1 + \dfrac{1}{2}(0.2) - \dfrac{1}{8}(0.2)^2 + \dfrac{1}{16}(0.2)^3 - \dfrac{5}{128}(0.2)^4 \approx 1.0954375$

The error of this approximation is at most: $\dfrac{7(1.2)^{-\frac{9}{2}}}{256}(0.2)^5 \approx .000003852$

Hence this estimate is already sufficient to approximate to 5 decimal places (one can easily verify that $P_3(x)$ is only accurate to 4 decimal places).