EXERCISE 1.

A**:** The sum of the first n terms of a series with general term a_n is given by

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + ... + a_n = S_n = n^2(n+1)(n+2), \quad n \geq 1.$$

(1) Calculate a_1, a_2, and $S_n - S_{n-1}$. Deduce a_n.

(2) Calculate $S_{2n} - S_n$, then show clearly that $\displaystyle\sum_{k=n+1}^{2n} a_k = 3n^2(n+1)(5n+2)$.

B**:** Let $n \geq 1$. and $S_n = 1 + 11 + 111 + 1111 + 11111 + ... \underbrace{11111....1}_{n\,times}$.

(1) Find the sum of $A = 10^1 + 10^2 + 10^3 + ... + 10^n$.

(2) Find in exact simplified form an exact expression for the sum of S_n.

hint : use the fact that $1 = \frac{1}{9} \times 9,\ 11 = \frac{1}{9} \times 99, 111 = \frac{1}{9} \times 999$, and so on.

EXERCISE 2.Let

$$f(x) = \begin{cases} x, & 0 \leq x \leq \pi \\ -x, & -\pi \leq x \leq 0. \end{cases}$$

(a)**:** Sketch f over the interval $[-3\pi, 3\pi]$. Is f even ?

(b)**:** Find the Fourier series of f.

(c)**:** State Dirichlet theorem and Parseval's identity for the Fourier series of f.

(d)**:** Deduce the values of the following series : $\displaystyle\sum_{p=0}^{p=+\infty} \frac{1}{(2p+1)^2}$ and $\displaystyle\sum_{p=0}^{p=+\infty} \frac{1}{(2p+1)^4}$.

EXERCISE 3.Determine if the following series converge or diverge

$$a) \sum_{n=1}^{\infty} \frac{4^n}{(n+2)^n}, \quad b) \sum_{n=1}^{\infty} n! \left(\frac{3}{n}\right)^n, \quad c) \sum_{n=1}^{\infty} \frac{(-3)^n}{(2n)!}, \quad d) \sum_{n=1}^{\infty} \frac{4(-n)^n}{(n+5)^n}.$$

EXERCISE 4.

(A)**:** Let $f_n : [a, +\infty\ ] \to \mathbb{R} \ \ x \mapsto f_n(x) = \ln\left(x + \dfrac{1}{n}\right), \quad \forall n \in \mathbb{N}^*, a > 0.$ and

$g_n(x) = \ln\left(\dfrac{nx+1}{xn}\right)$.

a_1**:** Calculate $f(x) = \lim\limits_{n\to\infty} f_n(x)$, and show that $g_n(x)$ is decreasing function $\forall x \geq a$.

(a_2)**:** Show that the sequence of functions (f_n) converges uniformly to its limit $f(x)$.

(B)**:** Consider the power series given by $\sum\limits_{n\geq1} (-1)^n(2x-3)^n$.

(1) Write the power series as the form $\sum\limits_{n\geq1} a_n(x-a)$(i ;e : find a_n and a).

(2) Find the radius of convergence, then the interval of convergence.

(3) Find the sum of the series (**Hint :** use the sum of a geometric series).

(4) For $x \in [1, 2]$, express $f(x) = \frac{-2x+3}{2x-2}$ as a power series.

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Licence : (2[nd] **year) 13/01/2025** **Key :Final exam : Analysis 3** **11 :30 - 13 :30**

EXERCISE 1.**(Solution)**

A**:** The sum of the first n terms of a series with general term a_n is given by

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + ... + a_n = S_n = n^2(n+1)(n+2), \quad n \geq 1.$$

(1) We have $a_1 = S_1 = 1^2 \times (1+1)(1+2) = 6$.

(2) $S_n - S_{n-1} = n^2(n+1)(n+2) - (n-1)^2 n(n+1) = n(n+1)[n(n+2) - (n-1)^2] = n(n+1)[n^2 + 2n - n^2 + 2n - 1] = n(n+1)[4n-1]$. and $S_{2n} - S_n = (2n)^2(2n+1)(2n+2) - n^2(n+1)(n+2) = 4n^2(2n+1)2(n+1) - n^2(n+1)(n+2) = n^2(n+1)[8(2n+1) - (n+2)] = n^2(n+1)(15n+6) = 3n^2(n+1)(5n+2)$.

— We know that $S_{2n} - S_n = \sum\limits_{k=1}^{2n} a_k - \sum\limits_{k=1}^{n} a_k = a_1 + a_2 + ... + a_n + a_{n+1} + a_{n+2} + ... + a_{2n} - (a_1 + a_2 + ... + a_n) = a_{n+1} + a_{n+2} + ... + a_{2n} = \sum\limits_{k=n+1}^{2n} a_k$. From the above question we have $\sum\limits_{k=n+1}^{2n} a_k = 3n^2(n+1)(5n+2)$.

B**:** (1) $A = 10^1 + 10^2 + 10^3 + ... + 10^n = \frac{a(r^n-1)}{r-1} = \frac{10(10^n-1)}{10-1}$.

(2)

$$\begin{aligned} S_n &= \left(\frac{1}{9} \times 9\right) + \left(\frac{1}{9} \times 99\right) + \left(\frac{1}{9} \times 999\right) + ... + \left(\frac{1}{9} \times \underbrace{99...9}_{ntimes}\right) \\ &= \frac{1}{9}\left[9 + 99 + 999 + ... + \underbrace{99...9}_{ntimes}\right] \\ &= \frac{1}{9}\left[(10^1 - 1) + (10^2 - 1) + (10^3 - 1) + ... + (10^n - 1)\right] \\ &= \frac{1}{9}\left[(10^1 + 10^2 + 10^3 + ... + 10^n) - (\underbrace{1 + 1 + 1 + ... + 1}_{n\,times})\right]. \end{aligned}$$

By using the previous question we get

$$\begin{aligned} S_n &= \frac{1}{9}\left[\frac{10(10^n - 1)}{10 - 1} - n\right] = \frac{1}{9}\left[\frac{10}{9}(10^n - 1) - n\right] \\ &= \frac{1}{81}\left[10^{n+1} - 9n - 10\right]. \end{aligned}$$

EXERCISE 2. **Solution**

(a)**:** Sketch f over the interval $[-3\pi, 3\pi]$. Yes f is even ?

(b)**:** Since f is even, then $b_n = 0$.

— $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int_{-\pi}^{\pi}|x|dx = \frac{2}{\pi}\int_0^{\pi} xdx = \pi$.

— $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx = \frac{2}{\pi}\int_{-\pi}^{\pi} x\cos(nx)dx$, using integration by parts we get ($u(x) = x$, $v'(x) = \cos(nx)$)

$$\begin{aligned} a_n &= \frac{2}{\pi}\left(\left[\frac{x\sin(nx)}{n}\right]_0^{\pi} - \int_0^{\pi}\frac{\sin(nx)}{n}dx\right) = \frac{2}{\pi}\left(0 + \left[\frac{1}{n^2}\cos(nx)\right]_0^{\pi}\right) \\ &= \frac{2}{\pi n^2}(\cos(n\pi) - 1) = \frac{2}{\pi n^2}((-1)^n - 1) = \begin{cases} 0, & \text{if } n = 2p \\ \frac{-4}{\pi n^2} & \text{if } n = 2p+1 \end{cases} \end{aligned}$$

Since f is continuous, then $f(x) = |x| = \frac{\pi}{2} - \sum\limits_{p=0}^{\infty}\frac{4}{\pi(2p+1)^2}\cos\{(2p+1)x\}$.

(c)**:** The Parseval identity states $\frac{1}{L}\int_{-L}^{L}|f(x)|^2\,dx = \frac{a_0^2}{2} + \sum\limits_{n=1}^{\infty}(a_n^2 + b_n^2)$.

(d)**:**

$$\begin{aligned} \frac{1}{\pi}\int_{-\pi}^{\pi}|x|^2\,dx &= \frac{\pi^2}{2} + \sum_{n=1}^{\infty}a_n^2 \\ \frac{2}{\pi}\int_0^{\pi}x^2dx &= \frac{\pi^2}{2} + \sum_{p=0}^{\infty}\frac{16}{\pi^2(2p+1)^4} \\ \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi} &= \frac{\pi^2}{2} + \frac{16}{\pi^2}\sum_{p=0}^{\infty}\frac{1}{(2p+1)^4} \\ \frac{2}{3}\pi^2 &= \frac{\pi^2}{2} + \frac{16}{\pi^2}\sum_{p=0}^{\infty}\frac{1}{(2p+1)^4} \\ \frac{\pi^2}{6} &= \frac{16}{\pi^2}\sum_{p=0}^{\infty}\frac{1}{(2p+1)^4}. \end{aligned}$$

Thus $\sum\limits_{p=0}^{p=+\infty}\frac{1}{(2p+1)^4} = \frac{\pi^4}{96}$.

$\mathcal{EXERCISE}$ 3. **Solution**

— $a)$ $\sum\limits_{n=1}^{\infty}\frac{4^n}{(n+2)^n}$. Using Cauchy's test we have

$$\lim_{n\to\infty}\sqrt[n]{a_n} = \lim_{n\to\infty}\sqrt[n]{\frac{4^n}{(n+2)^n}} = \lim_{n\to\infty}\frac{4}{(n+2)} = 0 < 1,$$

the series converges.

— $b)$ $\sum\limits_{n=1}^{\infty} n!\left(\frac{3}{n}\right)^n$. By D'alembert test we have

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}(n+1)!\left(\frac{3}{n+1}\right)^{n+1}\times\frac{\left(\frac{n}{3}\right)^{n+1}}{n!} = \lim_{n\to\infty}3\left(\frac{n}{n+1}\right)^n = 3 > 1,$$

the series diverges.

— $c)$ $\sum\limits_{n=1}^{\infty}\frac{(-3)^n}{(2n)!}$. Let $|a_n| = \frac{3^n}{(2n)!}$, then, by D'alembert test we have

$$\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty}\frac{3^{n+1}}{(2n+2)!}\times\frac{(2n)!}{3^n} = \lim_{n\to\infty}\frac{3^n 3}{(2n+2)(2n+1)(2n)!}\times\frac{(2n)!}{3^n} = \lim_{n\to\infty}\frac{3}{(2n+2)(2n+1)} = 0 < 1,$$

the series converges.

— $d)$ $\sum\limits_{n=1}^{\infty} \frac{4(-n)^n}{(n+5)^n}$. Since

$$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} 4^{\frac{1}{n}} \left(\frac{n}{n+5}\right) = 1 \neq 0,$$

then the series diverges by the Divergent test.

$\mathcal{EXERCICE}$ 4 **Solution**.

(A)**:** Let $f_n : [a, +\infty\ ] \to \mathbb{R}\ \ x \mapsto f_n(x) = \ln\left(x + \dfrac{1}{n}\right), \quad \forall n \in \mathbb{N}^*$, $a > 0$. and $g_n(x) = \ln\left(\dfrac{nx+1}{xn}\right)$.

a_1**:** $f(x) = \lim\limits_{n\to\infty} f_n(x) = \lim\limits_{n\to\infty} \ln\left(x + \dfrac{1}{n}\right) = \ln(x)$. By simple calculation $g_n'(x) = \frac{-1}{x(nx+1)} < 0$, $\forall x \geq a$, then $g_n(x)$ is decreasing function $\forall x \geq a$.

(a_2)**:** We must prove that $\lim\limits_{n\to\infty} \sup\limits_{x\geq a} |f_n(x) - f(x)| = 0$. We have first

$$|f_n(x) - f(x)| = \left|\ln\left(x + \frac{1}{n}\right) - \ln(x)\right| = \left|\ln\left(\frac{nx+1}{xn}\right)\right| = |g_n(x)| = g_n(x).$$

from the above question $g_n(x)$ is decreasing function, then

$$\sup_{x\geq a} |f_n(x) - f(x)| = \sup_{x\geq a} g_n(x) = g_n(a) = \ln\left(\frac{na+1}{an}\right),$$

thus

$$\lim_{n\to\infty} \sup_{x\geq a} |f_n(x) - f(x)| = \lim_{n\to\infty} \ln\left(\frac{na+1}{an}\right) = 0.$$

Then the sequence of functions (f_n) converges uniformly to its limit $f(x) = \ln(x)$.

(B)**:** Consider the power series given by $\sum\limits_{n\geq 1} (-1)^n (2x-3)^n$.

(1) $\sum\limits_{n\geq 1} a_n(x-a) = \sum\limits_{n\geq 1} (-1)^n 2^n (x - 3/2)^n$, so $a_n = (-1)^n 2^n$ and $a = 3/2$.

(2) $R = \lim\limits_{n\infty} \left|\frac{a_n}{a_{n+1}}\right| = \lim\limits_{n\infty} \left|\frac{(-1)^n 2^n}{(-1)^{n+1} 2^{n+1}}\right| = \frac{1}{2}$. Then the interval of absolute convergence is given by $|x - 3/2| \leq 1/2 \Leftrightarrow 1 < x < 2$.
— First end point : $x = 1$, then $\sum\limits_{n\geq 1} (-1)^n (2x-3)^n = \sum\limits_{n\geq 1} (-1)^n (-1)^n = \sum\limits_{n\geq 1} (-1)^{2n} = \sum\limits_{n\geq 1} 1 =$.(Div)
— second end point : $x = 2$, then $\sum\limits_{n\geq 1} (-1)^n (2x-3)^n = \sum\limits_{n\geq 1} (-1)^n =$.(Div). Finally $1 < x < 2$.

(3) Since $1 < x < 2$, then $-1 < r = 2x - 3 < 1$, $\sum\limits_{n\geq 1} (-1)^n (2x-3)^n = \sum\limits_{n\geq 1} (-r)^n$ which is the sum of a geometric series $S = -r\frac{1}{1-(-r)} = \frac{-2x+3}{2x-2}$

(4) For $x \in [1, 2]$, $f(x) = \frac{-2x+3}{2x-2} = \sum\limits_{n\geq 1} (-1)^n (2x-3)^n$.

Benrabah A.