Test number 01 (Solution)

Ex 01 :

$\boxed{a}$: We Compute the characteristic polynomial of the matrix :

$$A_3 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

From definition, we have :

$$P_{A_3}(x) = \det(A - x I_3)$$

$$= \begin{vmatrix} 1-x & 1 & 1 \\ 1 & 1-x & 1 \\ 1 & 1 & 1-x \end{vmatrix} \quad \begin{array}{l} C_1 \to C_1 - C_2 \\ C_2 \to C_2 - C_3 \end{array}$$

$$= \begin{vmatrix} -x & 0 & 1 \\ x & -x & 1 \\ 0 & x & 1-x \end{vmatrix}$$

$$= x^2 . \begin{array}{ccc} + & - & + \end{array} \begin{vmatrix} -1 & 0 & 1 \\ 1 & -1 & 1 \\ 0 & 1 & 1-x \end{vmatrix}$$

$$= x^2 \left[-(-1 + x - 1) + 1 \right] = x^2 (3 - x).$$

Hence, $P_{A_3}(x) = x^2(3-x)$.

$\boxed{b}$ We deduce the characteristic polynomial of the matrix:

$$A_n = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & \cdots & 1 \end{bmatrix} \in \mathcal{M}_n(\mathbb{R})$$

First, we compute the characteristic polynomial of the matrix A_4. In fact, we have:

$$P_{A_4}(x) = \begin{vmatrix} 1-x & 1 & 1 & 1 \\ 1 & 1-x & 1 & 1 \\ 1 & 1 & 1-x & 1 \\ 1 & 1 & 1 & 1-x \end{vmatrix}$$

$$= \begin{vmatrix} -x & 0 & 0 & 1 \\ x & -x & 0 & 1 \\ 0 & x & -x & 1 \\ 0 & 0 & x & 1-x \end{vmatrix}.$$

$$= x^3 \cdot \begin{array}{c} \begin{matrix} + & - & + & - \end{matrix} \\ \begin{vmatrix} -1 & 0 & 0 & 1 \\ 1 & -1 & 0 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 1-x \end{vmatrix} \end{array}$$

$$= x^3 \cdot (-1) \cdot \begin{array}{c} \begin{matrix} + & - & + \end{matrix} \\ \begin{vmatrix} -1 & 0 & 1 \\ 1 & -1 & 1 \\ 0 & 1 & 1-x \end{vmatrix} \end{array} + x^3 \cdot (-1) \cdot \begin{array}{c} \begin{matrix} + & - & + \end{matrix} \\ \begin{vmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{vmatrix} \end{array}$$

$$= -x^3 \left[-(-1 + x - 1) + 1 \right] - x^3 \left[1 \right]$$

$$= -x^3 (3 - x) - x^3 = x^3 (x - 4)$$

Thus,

$$P_{A_4}(x) = x^3 (x - 4).$$

Conclusion: We deduce that:

$$P_{A_n}(x) = \begin{cases} x^{n-1}(x - n), & \text{if } n \text{ is even} \\ x^{n-1}(n - x), & \text{if } n \text{ is odd} \end{cases}$$

even = pair

odd = impair

Ex 02: we find the characteristic polynomial of the matrix:

$$A = \begin{bmatrix} 7 & -6 & -2 \\ 2 & 0 & -1 \\ 2 & -3 & 2 \end{bmatrix}$$

By definition, we have:

$$P_A(x) = \begin{vmatrix} 7-x & -6 & -2 \\ 2 & -x & -1 \\ 2 & -3 & 2-x \end{vmatrix} \quad \begin{array}{c} L_1 \\ \downarrow \\ L_1 - 2L_2 \end{array}$$

$$= \begin{vmatrix} 3-x & -2(3-x) & 0 \\ 2 & -x & -1 \\ 2 & -3 & 2-x \end{vmatrix}$$

$$= (x-3) \cdot \begin{vmatrix} -1 & 2 & 0 \\ 2 & -x & -1 \\ 2 & -3 & 2-x \end{vmatrix}$$

$$L_2 \longrightarrow (-1)\, L_2 + L_3 .$$

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$$= \begin{vmatrix} 3-x & -2(3-x) & 0 \\ 2 & -x & -1 \\ 2 & -3 & 2-x \end{vmatrix}$$

$$= (x-3) \cdot \begin{vmatrix} -1 & 2 & 0 \\ 2 & -x & -1 \\ 2 & -3 & 2-x \end{vmatrix}$$

$$L_2 \longrightarrow (-1) L_2 + L_3 .$$

$$= (x-3) \cdot \begin{vmatrix} -1 & 2 & 0 \\ 0 & x-3 & -(x-3) \\ 2 & -3 & 2-x \end{vmatrix}$$

$$= (x-3)^2 \cdot \begin{vmatrix} \overset{+}{-1} & \overset{-}{2} & \overset{+}{0} \\ 0 & 1 & -1 \\ 2 & -3 & 2-x \end{vmatrix}$$

$$= (x-3)^2 \cdot \left[-(2-x-3) - 2(2) \right]$$

$$= (x-3)^3 \text{. That is,}$$

$$P_A(x) = (x-3)^3.$$

Ex 03: Let $A \in M_n(\mathbb{R})$ be a square matrix. We will prove that

$$P_{rA}(x) = r^n \cdot P_A\left(\frac{x}{r}\right), \text{ for } r \neq 0.$$

where;

$P_A \equiv$ is the characteristic polynomial of A.

$r \equiv$ is a nonzero real number.

In fact, by definition, we have:

$$p_{r.A}(x) = \det(r.A - x.I_n)$$

$$= \begin{vmatrix} ra_{11}-x & ra_{12} & \cdots & ra_{1n} \\ ra_{21} & ra_{22}-x & \cdots & ra_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ ra_{n1} & ra_{n2} & \cdots & ra_{nn}-x \end{vmatrix}$$

$$= \begin{vmatrix} r(a_{11}-\frac{x}{r}) & ra_{12} & \cdots & ra_{1n} \\ ra_{21} & r(a_{22}-\frac{x}{r}) & \cdots & ra_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ ra_{n1} & ra_{n2} & \cdots & r(a_{nn}-\frac{x}{r}) \end{vmatrix}$$

$= r^n . p_A\left(\frac{x}{r}\right)$, where $r \neq 0$.

This Completes the proof. $\square$

Ex 04: (i) Let A and B be two matrices such that:

$$A^2 = B^2 = (AB)^2 = I_n \quad \dots (*)$$

We show that $AB = BA$.

In fact, from $(*)$, we get:

$$\begin{cases} A = A^{-1} \\ B = B^{-1} \\ AB = (AB)^{-1} \end{cases}$$

On the other hand, we know that:

$$(AB)^{-1} = B^{-1} \cdot A^{-1}, \quad \text{and so}$$

$AB = B^{-1} \cdot A^{-1} = BA$. The proof is finished $\square$

(ii) Let A and B be two square matrices. We prove that if A is invertible, then

$$P_{AB}(x) = P_{BA}(x).$$

proof: By definition, we have:

$$P_{AB}(x) = \det(AB - xI)$$

$$= \det(ABAA^{-1} - xAA^{-1})$$

$$= \det[A(BA - xI)A^{-1}]$$

$$= \det(A) \cdot \det(BA - xI) \cdot \det(A^{-1})$$

Therefore, $P_{AB}(x) = \det(BA - xI) = P_{BA}(x)$.
The proof is finished $\square$.

Ex 05: Consider the matrix:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$$

We find the eigenvalues and Corresponding eigenvectors of A.

- eigenvalues ≡ Valeurs propres
- eigenvectors ≡ Vecteurs propres

First, we Compute the characteristic polynomial

$$P_A(x) = \det(A - xI_2)$$

$$= \begin{vmatrix} 1-x & 2 \\ 3 & 2-x \end{vmatrix} \quad C_1 \to C_1 - C_2$$

$$= \begin{vmatrix} -1-x & 2 \\ 1+x & 2-x \end{vmatrix}$$

$$= (1+x) \cdot \begin{vmatrix} -1 & 2 \\ 1 & 2-x \end{vmatrix}$$

$$= (1+x)(-2+x-2)$$

$$= (1+x)(x-4).$$

Thus, the eigenvalues of A are $\lambda_1 = -1$, $\lambda_2 = 4$.

- We find the eigenspaces E_{λ_1} and E_{λ_2}:

By definition, we have:

$$E_{\lambda_1} = \left\{ (x,y) \in \mathbb{R}^2 \,/\, \begin{array}{l} x+2y=-x \\ 3x+2y=-y \end{array} \right\}$$

$$= \left\{ (x,y) \in \mathbb{R}^2 \,/\, y=-x \right\}$$

$$= \left\{ (x,-x) \,;\, x \in \mathbb{R} \right\}$$

$= \text{Vect}\{(1,-1)\}$. That is, $v_1 = (1,-1)$.

We use the same manner to find v_2:

$$E_{\lambda_2} = \left\{ (x,y) \in \mathbb{R}^2 \,/\, \begin{array}{l} x+2y=4x \\ 3x+2y=4y \end{array} \right\}$$

$$= \left\{ (x,y) \in \mathbb{R}^2 \,/\, y=\frac{3}{2}x \right\}$$

$$= \left\{ \left(x,\frac{3}{2}x\right) \,;\, x \in \mathbb{R} \right\}$$

$$= \text{Vect}\left\{ \left(1,\frac{3}{2}\right) \right\} = \text{Vect}\{(2,3)\}.$$

Hence, $v_2 = (2,3)$.

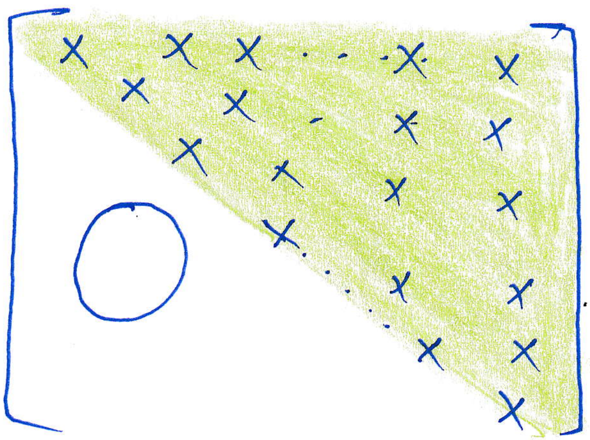
Ex 06 : We determine the eigenvalues and eigenvectors of the following matrices :

1 $A = \begin{bmatrix} 1 & 2 \\ 0 & 5 \end{bmatrix}$

We see that A is an upper triangular matrix. So its eigenvalues are :

$$\lambda_1 = 1 , \quad \lambda_2 = 5 .$$

Recall that an upper triangular matrix U is given by :

$U =$ 

- We Compute the eigenvectors of A.

$$E_{\lambda_1} = \left\{ (x, y) \in \mathbb{R}^2 \,\middle/\, \begin{aligned} x + 2y &= x \\ 5y &= y \end{aligned} \right\}$$

$$= \left\{ (x, y) \in \mathbb{R}^2 \,/\, y = 0 \right\}$$

$$= \left\{ (x, 0) \,;\, x \in \mathbb{R} \right\} = \mathrm{Vect}\left\{ (1, 0) \right\}.$$

Hence, $v_1 = (1,0)$.

- $E_{\lambda_2} = \left\{ (x,y) \in \mathbb{R}^2 \;\middle/\; \begin{array}{l} x + 2y = 5x \\ 5y = 5y \end{array} \right\}$

$= \left\{ (x,y) \in \mathbb{R}^2 \;/\; y = 2x \right\}$

$= \left\{ (x, 2x) \; ; \; x \in \mathbb{R} \right\}$

$= \left\{ x(1,2) \; ; \; x \in \mathbb{R} \right\}$

$= \mathrm{Vect}\left\{ (1,2) \right\}$. Hence, $v_2 = (1,2)$.

$\boxed{2}$ For the matrix:

$$A = \begin{bmatrix} 2 & 6 \\ 0 & 2 \end{bmatrix}$$

Since A is an upper triangular matrix, then the eigenvalues are : $\lambda = 2$ (double).

The Corresponding eigenspace is given by:

$E_{\lambda} = \left\{ (x,y) \in \mathbb{R}^2 \;\middle/\; \begin{array}{l} 2x + 6y = 2x \\ 2y = 2y \end{array} \right\}$

$= \left\{ (x,y) \in \mathbb{R}^2 \;/\; y = 0 \right\}$

$= \mathrm{Vect}\left\{ (1,0) \right\}$

That is, $v = (1,0)$.

3 For the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & -5 \end{bmatrix}$$

Since A is an upper triangular matrix, then the eigenvalues are:

- $\lambda_1 = 1$
- $\lambda_2 = 2$
- $\lambda_3 = -5$

Now, we find the corresponding eigenvectors:

$$E_{\lambda_1} = \left\{ (x, y, z) \in \mathbb{R}^3 \;\middle/\; \begin{aligned} x + 2y + 3z &= x \\ 2y + 3z &= y \\ -5z &= z \end{aligned} \right\}$$

$$= \left\{ (x, y, z) \in \mathbb{R}^3 \;/\; z = 0,\ y = 0 \right\}$$

$$= \left\{ (x, 0, 0)\ ;\ x \in \mathbb{R} \right\}$$

$$= \text{Vect}\left\{ (1, 0, 0) \right\}.$$ Hence, $v_1 = (1, 0, 0)$.

$$E_{\lambda_2} = \left\{ (x, y, z) \in \mathbb{R}^3 \;\middle/\; \begin{aligned} x + 2y + 3z &= 2x \\ 2y + 3z &= 2y \\ -5z &= 2z \end{aligned} \right\}$$

$$= \left\{ (x, y, z) \in \mathbb{R}^3 \;/\; z = 0,\ x = 2y \right\}$$

$$= \left\{ (2y, y, 0)\ ;\ y \in \mathbb{R} \right\} = \text{Vect}\left\{ (2, 1, 0) \right\}.$$

Hence, $v_2 = (2,1,0)$.

$$E_{\lambda_3} = \left\{ (x,y,z) \in \mathbb{R}^3 \;\middle/\; \begin{array}{l} x + 2y + 3z = -5x \\ 2y + 3z = -5y \\ -5z = -5z \end{array} \right\}$$

$$= \left\{ (x,y,z) \in \mathbb{R}^3 \;\middle/\; \begin{array}{l} z = \frac{-7}{3} y \\ x + 2y - 7y = -5x \end{array} \right\}$$

$$= \left\{ (x,y,z) \in \mathbb{R}^3 \;\middle/\; \begin{array}{l} y = \frac{6}{5} x \\ z = \frac{-42}{15} x \end{array} \right\}$$

$$= \left\{ \left(x, \frac{6}{5} x, \frac{-42}{15} x \right) ; x \in \mathbb{R} \right\}$$

$$= \mathrm{Vect} \left\{ \left(1, \frac{6}{5}, \frac{-42}{15} \right) \right\}$$

$$= \mathrm{Vect} \{ (5, 6, -14) \}.$$ Hence,

$v_2 = (5, 6, -14)$.

$\boxed{4}$ For the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

which is a lower triangular matrix, and so the eigenvalues are : $\lambda_1 = 1$ and $\lambda_2 = 2$.

- We compute the eigenvalues of A:

$$E_{\lambda_1} = \left\{ (x, y, z) \in \mathbb{R}^3 \;\middle|\; \begin{array}{l} x = x \\ x + 2y = y \\ x + 2z = z \end{array} \right\}$$

$$= \left\{ (x, y, z) \in \mathbb{R}^3 \;\middle|\; \begin{array}{l} x = -y \\ z = y \end{array} \right\}$$

$$= \left\{ (-y, y, y) \, ; \; y \in \mathbb{R} \right\}$$

$$= \text{Vect}\left\{ (-1, 1, 1) \right\}. \text{ Hence, } v_1 = (-1, 1, 1).$$

$$E_{\lambda_2} = \left\{ (x, y, z) \in \mathbb{R}^3 \;\middle|\; \begin{array}{l} x = 2x \\ x + 2y = 2y \\ x + 2z = 2z \end{array} \right\}$$

$$= \left\{ (x, y, z) \in \mathbb{R}^3 \;\middle|\; \begin{array}{l} x = 0 \\ y, z \in \mathbb{R} \end{array} \right\}$$

$$= \left\{ (0, y, z) \, ; \; y, z \in \mathbb{R} \right\}$$

$$= \left\{ y(0, 1, 0) + z(0, 0, 1) \, ; \; y, z \in \mathbb{R} \right\}$$

$$= \text{Vect}\left\{ (0, 1, 0), (0, 0, 1) \right\}$$

Then λ_2 has two eigenvectors:

$$v_2 = (0, 1, 0), \quad v_3 = (0, 0, 1).$$

5 For the matrix :

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

we see that A has one eigenvalue : $\lambda = 0$.

- The corresponding eigenspace :

$$E_\lambda = \{(x, y, z) \in \mathbb{R}^3 \;/\; y + z = 0\}$$
$$= \{(x, y, 3) \in \mathbb{R}^3 \;/\; z = -y\}$$
$$= \{(x, y, -y) \;;\; x, y \in \mathbb{R}\}$$
$$= \{x(1,0,0) + y(0,1,-1) \;;\; x, y \in \mathbb{R}\}$$
$$= \text{Vect}\{(1,0,0), (0,1,-1)\}$$

Thus, $\lambda = 0$ has two eigenvectors :

$$v_1 = (1,0,0),$$

$$v_2 = (0,1,-1).$$

6 For the matrix:

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 3 & 2 \end{bmatrix}$$

It is clear that A is a lower triangular matrix. That is, $\lambda = 2$ is the only eigenvalue.

Now, we compute the corresponding eigen-space: In fact, we have:

$$E_\lambda = \left\{ (x, y, z) \in \mathbb{R}^3 \;\middle/\; \begin{array}{l} 2x = 2x \\ x + 2y = 2y \\ 3y + 2z = 2z \end{array} \right\}$$

$$= \left\{ (x, y, z) \in \mathbb{R}^3 \;\middle/\; \begin{array}{l} x = 0 \\ y = 0 \end{array} \right\}$$

$$= \{ (0, 0, z) ; z \in \mathbb{R} \}$$

$$= \text{Vect}\{ (0, 0, 1) \}.$$ Hence, $v = (0, 0, 1)$.

7 For the matrix:

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

- The matrix A is neither upper triangular nor lower triangular. Thus, we must find $P_A(x)$ using the definition: $P_A(x) = \det(A - xI)$.

In fact, we have:

$$P_A(x) = \det(A - x I_3)$$

$$= \begin{vmatrix} 1-x & 1 & 0 \\ 1 & 1-x & 0 \\ 0 & 0 & 2-x \end{vmatrix}$$

$$= (1-x)\left[(1-x)(2-x) - 0\right] - (2-x)$$

$$= (2-x)\cdot\left[(1-x)^2 - 1\right]$$

$$= (2-x)(1-2x+x^2-1)$$

$$= x(2-x)(x-2) = -x(x-2)^2.$$

Thus, the eigenvalues of A are:

- $\lambda_1 = 0$
- $\lambda_2 = 2$ (double).

We compute the eigenvectors:

$$E_{\lambda_1} = \left\{ (x,y,z) \in \mathbb{R}^3 \;/\; \begin{array}{l} x + y = 0 \\ 2z = 0 \end{array} \right\}$$

$$= \left\{ (x,y,z) \in \mathbb{R}^3 \;/\; x = -y,\ z = 0 \right\}$$

$$= \left\{ (-y, y, 0) ;\ y \in \mathbb{R} \right\}$$

$$= \text{Vect}\left\{ (-1, 1, 0) \right\}.$$ Hence, $v_1 = (-1, 1, 0)$.

$$E_{\lambda_2} = \left\{ (x,y,z) \in \mathbb{R}^3 \;\middle|\; \begin{array}{l} x+y=2x \\ x+y=2y \\ 2z=2z \end{array} \right\}$$

$$= \{ (x,y,z) \in \mathbb{R}^3 \mid x = y \}$$

$$= \{ (x,x,z) ;\ x,z \in \mathbb{R} \}$$

$$= \{ x(1,1,0) + z(0,0,1) ;\ x,z \in \mathbb{R} \}$$

$$= \operatorname{Vect}\{ (1,1,0), (0,0,1) \}$$

Hence, the eigenvalues $\lambda = 2$ has 2 eigenvectors: $v_2 = (1,1,0)$ and $v_3 = (0,0,1)$.

$\boxed{8}$ For the matrix:

$$A = \begin{bmatrix} a & 2 & 3 \\ 0 & 2a & 8 \\ 0 & 0 & 3a \end{bmatrix}, \quad a \in \mathbb{R}.$$

We see that A is an upper triangular matrix, and hence $\lambda_1 = a$, $\lambda_2 = 2a$ and $\lambda_3 = 3a$ are the eigenvalues of A.

Remark. For $a = 0$, the matrix A has one eigenvalue; $\lambda = 0$.

there are two cases to consider:

Case 1: Assume that $a=0$. Hence,

$$E_\lambda = \left\{ (x,y,z) \in \mathbb{R}^3 \;\middle/\; \begin{array}{l} 2y+3z=0 \\ 8z=0 \end{array} \right\}$$

$$= \left\{ (x,y,z) \in \mathbb{R}^3 \;\middle/\; \begin{array}{l} z=0 \\ y=0 \end{array} \right\}$$

$$= \left\{ (x,0,0) \,;\, x \in \mathbb{R} \right\}$$

$$= \mathrm{Vect}\left\{ (1,0,0) \right\}.$$ That is, $v_1 = (1,0,0)$.

Case 2: Assume that $a \neq 0$. In this cas, the matrix A has three distinct eigenvalues:

$\lambda_1 = a$, $\lambda_2 = 2a$ and $\lambda_3 = 3a$.

- We find the eigenspace for each eigenvalue:

$$E_{\lambda_1} = \left\{ (x,y,z) \in \mathbb{R}^3 \;\middle/\; \begin{array}{l} ax+2y+3z=ax \\ 2ay+8z = ay \\ 3az = az \end{array} \right\}$$

$$= \left\{ (x,y,z) \in \mathbb{R}^3 \;\middle/\; \begin{array}{l} z=0 \quad (\text{because } a \neq 0) \\ y=0 \end{array} \right\}$$

$$= \left\{ (x,0,0) \,;\, x \in \mathbb{R} \right\} = \mathrm{Vect}\left\{ (1,0,0) \right\}$$

Hence,

$$v_1 = (1,0,0).$$

$$E_{\lambda_2} = \left\{ (x, y, z) \in \mathbb{R}^3 \;\middle/\; \begin{array}{l} ax + 2y + 3z = 2ax \\ 2ay + 8z = 2ay \\ 3az = 2az \end{array} \right\}$$

$$= \left\{ (x, y, z) \in \mathbb{R}^3 \;\middle/\; \begin{array}{l} z = 0. \quad (a \neq 0) \\ y = \frac{a}{2} x \end{array} \right\}$$

$$= \left\{ \left(\frac{2}{a} y, y, 0 \right) ; \; y \in \mathbb{R} \right\}$$

$$= \operatorname{Vect} \left\{ \left(\frac{2}{a}, 1, 0 \right) \right\}.$$

That is, $v_2 = \left(\frac{2}{a}, 1, 0 \right)$ or $(2, a, 0)$.

$$E_{\lambda_3} = \left\{ (x, y, z) \in \mathbb{R}^3 \;\middle/\; \begin{array}{l} ax + 2y + 3z = 3ax \\ 2ay + 8z = 3ay \\ 3az = 3az \end{array} \right\}$$

$$= \left\{ (x, y, z) \in \mathbb{R}^3 \;\middle/\; \begin{array}{l} y = \frac{8}{a} z \\ ax + \frac{16}{a} z + 3z = 3ax \end{array} \right\}$$

$$= \left\{ (x, y, z) \in \mathbb{R}^3 \;\middle/\; \begin{array}{l} y = \frac{8}{a} z \\ x = \frac{1}{2a} \left(\frac{16 + 3a}{a} \right) z \end{array} \right\}$$

$$= \left\{ \left(\frac{1}{2a^2} (16 + 3a) z, \frac{8}{a} z, z \right) ; \; z \in \mathbb{R} \right\}$$

$$= \operatorname{Vect} \left\{ \left(\frac{1}{2a^2} (16 + 3a), \frac{8}{a}, 1 \right) \right\}.$$

Test number 2

Ex 01: The definition of two similar matrices :

Definition: Let A, B be two n by n matrices. We say that A is **similar** to B if there exists an invertible matrix P such that :

$$A = P \cdot B \cdot P^{-1}.$$

Or, equivalently,

$$B = P^{-1} \cdot A \cdot P.$$

- We write $A \sim B$ when A is similar to B.

$\boxed{b}$ We prove that

$$A - \lambda I \sim B \overset{?}{\Rightarrow} A \sim \lambda I + B.$$

In fact, if $A - \lambda I \sim B$, then

$$A - \lambda I = P \cdot B \cdot P^{-1}$$ for some invertible matrix P. It follows that :

$$\begin{aligned} A &= \lambda I + P \cdot B \, P^{-1} \\ &= \lambda \underline{P P^{-1}} + P B P^{-1} \\ &= P (\lambda I + B) P^{-1} \end{aligned}$$

Thus, $A \sim \lambda I + B$.

That is, A is similar to $\lambda I + B$.

$\boxed{c}$ Assume that $A = PBP^{-1}$, where P is invertible.

If (λ, x) is an eigenpair of A $\overset{?}{\Longrightarrow}$ $(\lambda, P^{-1}x)$ is an eigenpair of B.

$$\left.\begin{array}{l}(\alpha, v) \text{ is an} \\ \text{eigenpair of } A\end{array}\right\} \Longrightarrow \left\{\begin{array}{l}\alpha = \text{eigenvalue} \\ v = \text{eigenvector}\end{array}\right.$$

Since $Ax = \lambda x$ and $A = PBP^{-1}$, then

$$\begin{aligned} B(P^{-1}x) &= P^{-1}AP(P^{-1}x) \\ &= P^{-1}A\underbrace{PP^{-1}}_{I}x = P^{-1}(Ax) \\ &= P^{-1}(\lambda x) \\ &= \lambda(P^{-1}x). \end{aligned}$$

Hence,

$$B(P^{-1}x) = \lambda(P^{-1}x).$$

That is, $(\lambda, P^{-1}x)$ is an eigenpair of B.

$\boxed{d}$ Let

$$f_r(x) = a_0 + a_1 x + \cdots + a_r x^r \quad \text{be}$$

a polynomial of degree r. We prove that if $B = P^{-1}AP$, then $f_r(B) = P^{-1} \cdot f_r(A) \cdot P$

Since $f_r(x) = a_0 + a_1 x + \cdots + a_r x^r$, then

$$f_r(B) = a_0 I + a_1 B + \cdots + a_r B^r$$

$$= a_0 I + a_1 (P^{-1} A P) + \cdots + a_r (P^{-1} A P)^r$$

$$= a_0 \cdot \underbrace{P^{-1} P}_{I} + a_1 P^{-1} A P + \cdots + a_r P^{-1} A^r P$$

$$= P^{-1} \left[a_0 \cdot I + a_1 \cdot A + \cdots + a_r A^r \right] \cdot P$$

$$= P^{-1} f_r(A) \cdot P^{-1}.$$

Remark. We have used the fact that

$$(P^{-1} A P)^i = P^{-1} A^i P, \text{ for } i \geq 1.$$

Ex 02: We show that similar matrices have the same eigenvalues.

Proof. Let $A, B \in \mathcal{M}_n(\mathbb{R})$ be two similar matrices. That is, $A = P B P^{-1}$ for some invertible matrix P. It suffices to prove that: $P_A(x) = P_B(x)$.
In fact, we have:

$$P_A(x) = \det(A - x I)$$

$$= \det(P B P^{-1} - x I)$$

$$= \det(P B P^{-1} - x P P^{-1})$$

$$= \det\left[P(B - xI)P^{-1}\right]$$

$$= \det(P)\cdot\det(B - xI)\cdot\det(P^{-1})$$

$$= \det(B - xI), \text{ since } \det(P^{-1}) = \frac{1}{\det(P)}$$

$$= P_B(x).$$

Second method:

First, let $Sp(A)$ the set given by:

$$Sp(A) = \{\lambda \in \mathbb{R} \mid \lambda \text{ is an eigenvalue of } A\}$$

Let $\lambda \in Sp(A)$

$$\Rightarrow 0 \in Sp(A - \lambda I)$$

on the other hand, we have

$$A - \lambda I = P(B - \lambda I)P^{-1}$$

$$\Rightarrow 0 \in Sp(B - \lambda I).$$

$$\Rightarrow \lambda \in Sp(B).$$

$$\Rightarrow Sp(A) \subset Sp(B).$$

We use the same manner to show that $Sp(B) \subset Sp(A)$. Hence, $Sp(A) = Sp(B)$.

Ex 03: We prove the following result:

$$A \sim B \Rightarrow e^{A} \sim e^{B}$$

$e^{A} \equiv$ the exponential of A.

Recall that for any square matrix $A \in M_n(\mathbb{R})$ we have:

$$e^{A} = I + \frac{A}{1!} + \frac{A^2}{2!} + \dots + \frac{A^k}{k!} + \dots$$

$$= \sum_{k=0}^{+\infty} \frac{A^k}{k!}$$

Since A is similar to B, then $A = P \cdot B \cdot P^{-1}$ for some invertible matrix P. Therefore,

$$e^{A} = \sum_{k=0}^{+\infty} \frac{(P B P^{-1})^k}{k!}$$

$$= \sum_{k=0}^{\infty} P \cdot \frac{B^k}{k!} \cdot P^{-1}$$

$$= P \cdot \left(\sum_{k=0}^{+\infty} \frac{B^k}{k!} \right) \cdot P^{-1}$$

$= P \cdot e^{B} \cdot P^{-1}$, and so $e^{A} \sim e^{B}$.

Ex 04 : Consider the two matrices

$$A = \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 2 \\ -2 & 5 \end{bmatrix}$$

We prove that A is similar to B (i.e, $A \sim B$).

From simple computation, we obtain :

$$P_A(x) = \begin{vmatrix} 2-x & 1 \\ 2 & 3-x \end{vmatrix}$$

$$= \begin{vmatrix} (1-x) & 1 \\ -(1-x) & 3-x \end{vmatrix} = (1-x) \cdot \begin{vmatrix} 1 & 1 \\ -1 & 3-x \end{vmatrix}$$

$$= (1-x)(3-x+1) = (1-x)(4-x).$$

For the matrix B, we also obtain :

$$P_B(x) = \begin{vmatrix} -x & 2 \\ -2 & 5-x \end{vmatrix} = \begin{vmatrix} 2-x & 2 \\ 3-x & 5-x \end{vmatrix}$$

$$= \begin{vmatrix} (4-x) & 2 \\ 2(4-x) & 5-x \end{vmatrix}$$

$$= (4-x) \cdot \begin{vmatrix} 1 & 2 \\ 2 & 5-x \end{vmatrix}$$

$$= (4-x)(5-x-4) = (4-x) \cdot (1-x).$$

Thus, $P_A(x) = P_B(x)$ and so $Sp(A) = Sp(B)$.
Since the eigenvalues of A (resp. B) are simple, then A and B are diagonalizable. That is, there exist two invertible matrices P and Q such that:

$$\begin{cases} A = P \cdot D P^{-1} \\ B = Q D Q^{-1} \end{cases}$$

It follows that

$$\begin{aligned} A &= P D P^{-1} \\ &= P \left(Q^{-1} B Q \right) P^{-1} \\ &= P Q^{-1} \cdot B \cdot Q P^{-1} \\ &= P Q^{-1} \cdot B \left(P Q^{-1} \right)^{-1} \end{aligned}$$

We put $R = P Q^{-1}$ which is invertible. Then

$$A = R \cdot B \cdot R^{-1} \Rightarrow A \sim B.$$

Remark: The product of invertible matrices is an invertible matrix.

- We find the matrix R.

We must compute P and Q.

- Simple Computation, we get:

For the matrix A:

$\lambda_1 = 1 \longrightarrow v_1 = (-1, 1)$

$\lambda_2 = 4 \longrightarrow v_2 = (1, 2)$

For the matrix B:

$\lambda_1 = 1 \longrightarrow v_1' = (2, 1)$

$\lambda_2 = 4 \longrightarrow v_2' = (1, 2)$

Therefore,

$$R = P \cdot Q^{-1} = \begin{bmatrix} -1 & 1 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} -1 & 1 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} \\ \frac{-1}{3} & \frac{2}{3} \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}$$

Ex 05: We prove the following result:

Let $A \in M_n(\mathbb{R})$ be a diagonalizable matrix which has one eigenvalue λ. Then $A = \lambda \cdot I$, where I is the identity matrix.

That is, $I = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}$.

proof: Since A is diagonalizable, then $A = PDP^{-1}$ for some invertible matrix P, where D is diagonal. Moreover, D includes the eigenvalues of A. Then,

$$A = P \cdot D \cdot P^{-1} = P \cdot \begin{bmatrix} \lambda & & & \\ & \lambda & & \\ & & \ddots & \\ & & & \lambda \end{bmatrix} \cdot P^{-1}$$

$$= \lambda \cdot P \cdot \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} \cdot P^{-1} = \lambda P P^{-1} = \lambda \cdot I.$$

Hence, $A = \lambda I$. The proof is finished.

Recall that any polynomial $p(x)$ of degree n can be factorized as follows:

$$p(x) = (x-\lambda_1)^{\alpha_1}(x-\lambda_2)^{\alpha_2} \cdots (x-\lambda_k)^{\alpha_k},$$

where $\alpha_1 + \alpha_2 + \cdots + \alpha_k = n$, λ_i are the roots of P. In this Case: α_i is Called:

$$\alpha_i \equiv \text{the Algebraic multiplicity of } \lambda_i$$

Example: Let $p(x) = x^3 - 3x + 2$.

It is clear that

$$p(x) = (x-1)^2 \cdot (x+2)$$

- The algebraic multiplicity of 1 is 2.
- " " " " -2 is 1

Notation: $A_m(\lambda)$ denotes the Algebraic multiplicity of λ.

- We also denote by $G_m(\lambda)$ for the dimension of E_λ. That is,

$$G_m(\lambda) = \dim E_\lambda.$$

From the Course, we have

$\big($A is diagonalizable$\big)$ iff $A_m(\lambda) = G_m(\lambda)$ for any $\lambda \in Sp(A)$.

Ex 06: We study the diagonalization of the matrix:

$$A = \begin{bmatrix} 1 & a & b & c \\ 0 & 1 & d & e \\ 0 & 0 & 2 & f \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

this is an upper triangular matrix.

The eigenvalues of A are:

- $\lambda_1 = 1$ with $A_m(\lambda_1) = 2$
- $\lambda_2 = 2$ with $A_m(\lambda_2) = 1$
- $\lambda_3 = 3$ with $A_m(\lambda_3) = 1$.

So, the matrix A is diagonalizable whenever $\dim E_{\lambda_1} = 2$.

We Compute E_{λ_1}:

$$E_{\lambda_1} = \left\{ (x, y, z, t) \in \mathbb{R}^4 \;\middle|\; \begin{array}{l} x + ay + bz + ct = x \\ y + dz + et = y \\ 2z + ft = z \\ 3t = t \end{array} \right\}.$$

$$= \left\{ (x, y, z, t) \in \mathbb{R}^4 \;\middle|\; \begin{array}{l} t = 0,\ z = 0 \\ ay = 0 \end{array} \right\}$$

we distinguish two cases:

Case 1: If $a \neq 0$, then

$$E_{\lambda_1} = \left\{ (x, y, z, t) \in \mathbb{R}^4 \;\middle|\; \begin{array}{l} t = 0,\ z = 0 \\ y = 0 \end{array} \right\}$$

$$= \{ (x, 0, 0, 0) ;\ x \in \mathbb{R} \}$$

$$= \mathrm{Vect}\{ (1, 0, 0, 0) \} \Rightarrow \dim E_{\lambda_1} = 1.$$

In this case, A is not diagonalizable.

Case 2: If $a = 0$, then

$$E_{\lambda_1} = \left\{ (x, y, z, t) \in \mathbb{R}^4 \;\middle|\; \begin{array}{l} z = 0 \\ t = 0 \end{array} \right\}$$

$$= \{ (x, y, 0, 0) ;\ x, y \in \mathbb{R} \}$$

$$= \{ x(1, 0, 0, 0) + y(0, 1, 0, 0) \mid x, y \in \mathbb{R} \}$$

$$= \mathrm{Vect}\{ (1, 0, 0, 0), (0, 1, 0, 0) \}$$

Hence, $\dim E_{\lambda_1} = 2 = A_m(\lambda_1)$

In this case, the matrix A is diagonalizable.

Conclusion: the matrix A is diagonalizable if and only if $a = 0$, $b, c, d, e, f \in \mathbb{R}$.

Ex 07: Let $A, B \in \mathcal{M}_n(\mathbb{R})$ be a diagonalizable matrix with $Sp(A) = \{-1, 1\}$. That is, A has two eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 1$.

We prove that $A^{-1} = A$.

Proof: Since $A = P \cdot D P^{-1}$ for some invertible matrix P, where

$$D = \begin{bmatrix} \pm 1 & & & \\ & \pm 1 & & \\ & & \ddots & \\ & & & \pm 1 \end{bmatrix}, \text{ then}$$

$A^{-1} = (P \cdot D \cdot P^{-1})^{-1} = P \cdot D^{-1} \cdot P^{-1}$. Moreover,

$$D^{-1} = \begin{bmatrix} \frac{1}{\pm 1} & & & \\ & \frac{1}{\pm 1} & & \\ & & \ddots & \\ & & & \frac{1}{\pm 1} \end{bmatrix} = \begin{bmatrix} \pm 1 & & & \\ & \pm 1 & & \\ & & \ddots & \\ & & & \pm 1 \end{bmatrix} = D$$

Thus, $A^{-1} = P \cdot D \cdot P^{-1} = A$.

The proof is finished $\square$.

Ex 08: Let A and B be two diagonalizable matrices such that

- $A = P D_1 P^{-1}$
- $B = P D_2 P^{-1}$

where P is invertible, D_1 and D_2 are diagonal.

• We prove that $AB = BA$.

proof: It follows from the hypothesis that:

$$\begin{aligned} AB &= P D_1 P^{-1} P D_2 P^{-1} \\ &= P D_1 D_2 P^{-1} \\ &= P D_2 D_1 P^{-1} \\ &= P D_2 P^{-1} P D_1 P^{-1} \\ &= (P D_2 P^{-1}) \cdot (P D_1 P^{-1}) \\ &= B \cdot A. \end{aligned}$$

The proof is finished $\square$.

Remark: we have used the fact that $D_1 D_2 = D_2 D_1$ since the matrices D_1 and D_2 are diagonal.

Ex 09: we study whether the matrix

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 4 & 1 & 2 \\ a & 0 & 3 \end{bmatrix}$$

is diagonalizable or not.

1. We find the characteristic polynomial:

$$P_A(x) = \begin{vmatrix} \overset{+}{3-x} & \overset{-}{0} & \overset{+}{0} \\ 4 & 1-x & 2 \\ a & 0 & 3-x \end{vmatrix}$$

$$= (3-x)\left[(1-x)(3-x) - 0\right]$$

$$= (3-x)^2(1-x).$$

Hence, $P_A(x) = (3-x)^2(1-x)$

2. The eigenvalues of A are:

- $\lambda_1 = 1$ with $A_m(\lambda_1) = 1$
- $\lambda_2 = 3$ with $A_m(\lambda_2) = 2$.

3. We compute the eigenspace E_{λ_2}:

$$E_{\lambda_2} = \left\{ (x,y,z) \in \mathbb{R}^3 \middle/ \begin{array}{l} 3x = 3x \\ 4x + y + 2z = 3y \\ ax + 3z = 3z \end{array} \right\}$$

$$= \left\{ (x,y,z) \in \mathbb{R}^3 \middle/ \begin{array}{l} y = 2x + z \\ ax = 0 \end{array} \right\}$$

There are two cases to consider:

Case 1: $a = 0$.

$$E_{\lambda_2} = \left\{ (x,y,z) \in \mathbb{R}^3 / y = 2x + z \right\}$$

$$= \{(x, 2x+z, z) \,;\, x, z \in \mathbb{R}\}$$
$$= \{x(1,2,0) + z(0,1,1) \,;\, x, z \in \mathbb{R}\}$$
$$= \text{Vect}\{(1,2,0), (0,1,1)\}$$

Thus, $\dim E_{\lambda_2} = 2 = A_m(\lambda_2)$, and so A is diagonalizable.

Case 2 : $a \neq 0$. In this case, we have:

$$E_{\lambda_2} = \left\{(x,y,z) \in \mathbb{R}^3 \,\middle/\, \begin{array}{l} x = 0 \\ y = z \end{array}\right\}$$
$$= \{(0,y,y) \,;\, y \in \mathbb{R}\}$$
$$= \text{Vect}\{(0,1,1)\}$$

Thus, $\dim E_{\lambda_2} = 1 \neq A_m(\lambda_2)$.

Therefore, A is not diagonalizable.

$\boxed{b}$ Let

$$A_\alpha = \begin{bmatrix} 2 & \alpha & 1 \\ 0 & 2 & 0 \\ 0 & 0 & \alpha \end{bmatrix}$$

We find the values $\alpha \in \mathbb{R}$ for which the matrix A_α is diagonalizable.

First, it is clear that A_α is an upper triangular matrix.

So, the eigenvalues of A_α are:

$$\lambda_1 = 2 \text{ and } \lambda_2 = \alpha.$$

There are two Cases:

Case 1: $\lambda_2 = 2$. In this Case, the matrix A_α has one eigenvalue, say $\lambda = 2$.

From exercise 05, if $A \neq \lambda I$, then A is not diagonalizable. But, we see that

$$A_\alpha = A_2 = \begin{bmatrix} 2 & 2 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \neq 2I$$

$\Rightarrow A_2$ is not diagonalizable.

Case 2: $\alpha \neq 2$: In this Case, the matrix A_α has two eigenvalues:

- $\lambda_1 = 2$ with $A_m(2) = 2$
- $\lambda_2 = \alpha$ with $A_m(\alpha) = 1$.

Let us Compute the eigenspace E_{λ_1}:

$$E_{\lambda_1} = \left\{ (x, y, z) \in \mathbb{R}^3 \;\middle|\; \begin{array}{l} 2x + \alpha y + z = 2x \\ 2y = 2y \\ \alpha z = 2z \end{array} \right\}$$

$$= \left\{ (x, y, z) \in \mathbb{R}^3 \;\middle|\; \begin{array}{l} z = 0 \\ \alpha \cdot y = 0 \end{array} \right\}$$

we distinguish two Cases:

Case 1: For $\alpha \neq 0$, we have:

$$E_{\lambda_1} = \left\{ (x, y, z) \in \mathbb{R}^3 \Big/ \begin{array}{l} z = 0 \\ y = 0 \end{array} \right\},$$

which gives $\dim E_{\lambda_1} = 1$, and so A_α is not diagonalizable.

Case 2: For $\alpha = 0$. In this Case, we have:

$$\begin{aligned} E_{\lambda_1} &= \left\{ (x, y, z) \in \mathbb{R}^3 \,/\, z = 0 \right\} \\ &= \left\{ (x, y, 0) \,;\, x, y \in \mathbb{R} \right\} \\ &= \text{Vect}\left\{ (1, 0, 0), (0, 1, 0) \right\} \end{aligned}$$

That is, $\dim E_{\lambda_1} = 2 = A_m(\lambda_1)$.

$\Rightarrow A_\alpha$ is diagonalizable.

Ex 10: Consider the matrix:

$$A = \begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix}, \quad c \in \mathbb{R}$$

we find the values c for which the matrix A is diagonalizable.

Since A is an upper triangular matrix, then the eigenvalues of A are $\lambda = 1$ (double)

• If $c \neq 0$ (See exercise 5), then A is not diagonalizable since $A \neq \lambda I$.

Second method:

We see that $\lambda = 1$ is the eigenvalue of A with multiplicity $A_m(\lambda) = 2$.

Let us find the corresponding eigenspace:

$$E_\lambda = \left\{ (x, y) \in \mathbb{R}^2 \;/\; \begin{array}{l} x + cy = x \\ y = y \end{array} \right\}$$

$$= \left\{ (x, y) \in \mathbb{R}^2 \;/\; c \cdot y = 0 \right\}$$

For $c \neq 0$, we have:

$$E_\lambda = \left\{ (x, y) \in \mathbb{R}^2 \;/\; y = 0 \right\}$$

$$= \text{Vect}\left\{ (1, 0) \right\}$$

$$\Rightarrow \dim E_\lambda = 1 \neq A_m(\lambda)$$

$$\Rightarrow A \text{ is not diagonalizable.}$$

For $c = 0$, we also have:

$$E_\lambda = \left\{ (x, y) \in \mathbb{R}^2 \;/\; \begin{array}{l} x = x \\ y = y \end{array} \right\}$$

$$= \left\{ (x, y) \,;\, x, y \in \mathbb{R} \right\}$$

$$= \left\{ x(1, 0) + y(0, 1) \,;\, x, y \in \mathbb{R} \right\}$$

$$= \text{Vect}\left\{ (1, 0), (0, 1) \right\} \Rightarrow A \text{ is diago-}$$

Remark. Any matrix of the form λI is diagonalizable. In fact, we have :

$$\lambda I = I \cdot (\lambda I) \cdot I^{-1} \qquad (I^{-1} = I).$$

Ex 11 : Consider the matrix

$$A(a) = \begin{bmatrix} 1 & 0 & 0 \\ a & -2 & 3 \\ 1 & -1 & 2 \end{bmatrix} ; \quad a \in \mathbb{R}.$$

- The eigenvalues of $A(a)$:

First, we compute the characteristic polynomial of $A(a)$. In fact, we have :

$$P_{A(a)}(x) = \begin{vmatrix} 1-x & 0 & 0 \\ a & -2-x & 3 \\ 1 & -1 & 2-x \end{vmatrix} \quad C_2 \to C_2 + C_3$$

$$= \begin{vmatrix} 1-x & 0 & 0 \\ a & 1-x & 3 \\ 1 & 1-x & 2-x \end{vmatrix}$$

$$= (1-x) \cdot \begin{array}{ccc} + & - & + \\ \end{array}$$
$$\left| \begin{array}{ccc} 1-x & 0 & 0 \\ a & 1 & 3 \\ 1 & 1 & 2-x \end{array} \right|$$

$$= (1-x)\left[(1-x)(2-x-3)\right]$$

$$= (1-x)^2(-1-x) = -(1+x)(1-x)^2.$$

- The eigenvalues of $A(a)$ are :

$\lambda_1 = -1$ with $A_m(\lambda_1) = 1$,

$\lambda_2 = 1$ with $A_m(\lambda_2) = 2$.

Now, we study the diagonalization of $A(a)$:

Let us Compute E_{λ_2} :

$$E_{\lambda_2} = E_1 = \left\{ (x,y,z) \in \mathbb{R}^3 \;\middle/\; \begin{array}{l} x = x \\ ax - 2y + 3z = y \\ x - y + 2z = z \end{array} \right\}$$

$$= \left\{ (x,y,z) \in \mathbb{R}^3 \;\middle|\; \begin{array}{l} z = -x + y \\ ax + 3z = 3y \end{array} \right\} \quad (*)$$

$$= \left\{ (x,y,z) \in \mathbb{R}^3 \;/\; (a-3)x = 0 \right\}$$

- If $a \neq 3$, then by $(*)$ we have

$$E_1 = \left\{ (x,y,z) \in \mathbb{R}^3 \;\middle/\; \begin{array}{l} x = 0 \\ z = y \end{array} \right\}$$

$= \text{Vect}\{(0,1,1)\} \Rightarrow A(a)$ is not diago-

• If $a = 3$, then by (*) we also have:

$$E_1 = \{(x, y, z) \in \mathbb{R}^3 \,/\, z = -x + y\}$$

$$= \{(x, y, -x+y) \,;\, x, y \in \mathbb{R}\}$$

$$= \{x(1,0,-1) + y(0,1,1) \,;\, x, y \in \mathbb{R}\}$$

$$= \text{Vect}\{(1,0,-1), (0,1,1)\}.$$

$\Rightarrow \dim E_1 = 2 = A_m(1).$

$\Rightarrow A(3)$ is diagonalizable.

Conclusion: The matrix $A(a)$ is diagonalizable if and only if $a = 3$.

• We diagonalize the matrix $A(3)$, i.e, we write the matrix $A(3)$ in the form:

$$A(3) = P \cdot D \cdot P^{-1}.$$

It sufficies to find E_{λ_1}:

$$E_{\lambda_1} = E_{-1} = \left\{(x, y, z) \in \mathbb{R}^3 \,/\, \begin{array}{l} x = -x \\ 3x - 2y + 3z = -y \\ x - y + 2z = -z \end{array}\right\}$$

$$= \left\{(x, y, z) \in \mathbb{R}^3 \,/\, \begin{array}{l} x = 0 \\ y = 3z \end{array}\right\}$$

$$= \{(0, 3z, z) \,;\, z \in \mathbb{R}\}$$

$$= \text{Vect}\{(0,3,1)\} \longrightarrow v_1 = (0,3,1).$$

Now, we put

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 3 \\ -1 & 1 & 1 \end{bmatrix}$$

and

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Simple Computations, we obtain :

$$P^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{3}{2} & -\frac{1}{2} & \frac{3}{2} \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

and so,

$$A(3) = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 3 \\ -1 & 1 & 1 \end{bmatrix}}_{P} \cdot \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}}_{D} \cdot \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ \frac{3}{2} & -\frac{1}{2} & \frac{3}{2} \\ -\frac{1}{2} & +\frac{1}{2} & -\frac{1}{2} \end{bmatrix}}_{P^{-1}}$$

This is the diagonalization of $A(3)$, from which we can compute $\sqrt{A(3)}$, $e^{A(3)}$, ...

Ex 12: Let

$$f : \mathbb{P}_3[x] \longrightarrow \mathbb{P}_3[x]$$
$$p \longmapsto f(p) = 3xp - (x^2-1)p'$$

p' is the derivative of p.

Let $\mathcal{B} = \{1, x, x^2, x^3\}$ be the Canonical basis of $\mathbb{P}_3[x]$.

① We Calculate $\mathcal{M}_f(\mathcal{B})$

$\mathcal{M}_f(B)$ = the matrix of f with respect the basis B.

First, we have:

- $f(1) = 3x = 0 + 3 \cdot x + 0 \cdot x^2 + 0 \cdot x^3$
- $f(x) = 1 + 2x^2 = 1 + 0 \cdot x + 2 \cdot x^2 + 0 \cdot x^3$
- $f(x^2) = 2x + x^3 = 0 + 2x + 0 \cdot x^2 + 1 \cdot x^3$
- $f(x^3) = 3x^2 = 0 + 0 \cdot x + 3x^2 + 0 \cdot x^3$

Hence,

$$\mathcal{M}_f(B) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3 & 0 & 2 & 0 \\ 0 & 2 & 0 & 3 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

② We study whether $\mathcal{M}_f(B)$ is diagonalizable or not.

⊛ characteristic polynomial

$$P(x) = (x-1)(x+1)(x-3)(x+3)$$

⊛ eigenvalues:

$$\lambda_1 = -1, \quad \lambda_2 = 1, \quad \lambda_3 = -3, \quad \lambda_4 = 3$$

⊛ eigenvectors:

$$v_1 = (1,-1,-1,1), \quad v_2 = (-1,-1,1,1)$$

$$v_3 = (-1,3,-3,1), \quad v_4 = (1,3,3,1)$$

The matrix $\mathcal{M}_f(B)$ is diagonalizable since the eigenvalues are simple.

After few computation, we obtain:

$$\mathcal{M}_f(B) = \underbrace{\begin{bmatrix} 1 & -1 & -1 & 1 \\ -1 & -1 & 3 & 3 \\ -1 & 1 & -3 & 3 \\ 1 & 1 & 1 & 1 \end{bmatrix}}_{P} \cdot \underbrace{\begin{bmatrix} -1 & & & \\ & 1 & & \\ & & -3 & \\ & & & 3 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} \frac{3}{8} & \frac{-1}{8} & \frac{-1}{8} & \frac{3}{8} \\ \frac{-3}{8} & \frac{-1}{8} & \frac{1}{8} & \frac{3}{8} \\ \frac{-1}{8} & \frac{1}{8} & -\frac{1}{8} & \frac{1}{8} \\ \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} \end{bmatrix}}_{P^{-1}}$$

This is the diagonalization formula of $\mathcal{M}_f(B)$ or f.

Test number 3

Ex01

1) the definition of exponential of a square matrix A:

Definition: We Know that:

$\forall x \in \mathbb{R}$:

$$e^x = \sum_{k=0}^{+\infty} \frac{x^k}{k!}, \quad \text{then}$$

for any square matrix A, we have:

$$e^A = \sum_{k=0}^{+\infty} \frac{A^k}{k!}$$

$$= I + \frac{A}{1!} + \frac{A^2}{2!} + \cdots + \frac{A^k}{k!} + \cdots$$

2) Let 0 = zero matrix. Then

$$e^0 = I + 0 + 0 + \cdots$$

$$= I, \text{ where } I \text{ is the unit matrix}$$

$$I = \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & \ddots \; 1 \end{bmatrix}$$

مثل الأعداد
$e^0 = 1$

3) Let $A \in M_n(\mathbb{R})$, and let $h \in \mathbb{R}$.
We prove that:

$$e^{h \cdot I} = e^{h} \cdot A$$

proof: By definition, we have

$$e^{hI} = I + \frac{(hI)}{1!} + \frac{(hI)^2}{2!} + \cdots + \frac{(hI)^k}{k!} + \cdots$$

$$= I + \frac{h}{1!} I + \frac{h^2}{2!} I + \cdots + \frac{h^k}{k!} I + \cdots$$

$$= \left(1 + \frac{h}{1!} + \frac{h^2}{2!} + \cdots + \frac{h^k}{k!} + \cdots\right) \cdot I$$

$$= e^{h} \cdot I.$$

The proof is finished.

Example: Let

$$A = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} = -2 \cdot I.$$

Then

$$e^{A} = e^{-2} \cdot I = e^{-2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} e^{-2} & 0 \\ 0 & e^{-2} \end{bmatrix}$$

Ex 02: Let A be the matrix

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$$

- We prove that A is diagonalizable.

Since A is an upper triangular matrix, then the eigenvalues are : $\lambda_1 = 1$, $\lambda_2 = 2$.

*) Now, we compute the eigenvectors :

$$E_1 = \left\{ (x,y) \in \mathbb{R}^2 \,/\, \begin{array}{l} x + y = x \\ 2y = y \end{array} \right\}$$

$$= \left\{ (x,y) \in \mathbb{R}^2 \,/\, y = 0 \right\}$$

$$= \text{Vect}\left\{ (1,0) \right\}.$$ Hence, $v_1 = (1,0)$.

Using the same manner, we obtain :

$$E_2 = \left\{ (x,y) \in \mathbb{R}^2 \,/\, \begin{array}{l} x + y = 2x \\ 2y = 2y \end{array} \right\}$$

$$= \left\{ (x,y) \in \mathbb{R}^2 \,/\, y = x \right\}$$

$$= \text{Vect}\left\{ (1,1) \right\}.$$ Hence, $v_2 = (1,1)$.

Since the eigenvalues of A are simple, i.e, $A_m(1) = A_m(2) = 1$, then the matrix A is diagonalizable.

Next, we put $D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$

ant let

$$P = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} v_1 & v_2 \end{bmatrix}$$

it is clear that

$$\begin{aligned} AP &= \begin{bmatrix} Av_1 & Av_2 \end{bmatrix} \\ &= \begin{bmatrix} \lambda_1 v_1 & \lambda_2 v_2 \end{bmatrix} \\ &= \begin{bmatrix} \lambda_1 & \lambda_2 \\ 0 & \lambda_2 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \end{aligned}$$

$= P \cdot D$. Therefore, $AP = PD$,

and so $A = PDP^{-1}$

- We Compute e^A:

we have

$$e^A = P \cdot e^D P^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e & 0 \\ 0 & e^2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} e & e^2 \\ 0 & e^2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} e & e^2 - e \\ 0 & e^2 \end{bmatrix}.$$

Exo3 : Let $A \in \mathcal{M}_n(\mathbb{R})$. We prove that the series

$$\sum_{k=0}^{+\infty} \frac{A^k}{k!}$$

is absolutely convergent.

For any $k \geq 0$, we have

$$\left\| \frac{A^k}{k!} \right\| \leq \frac{\|A\|^k}{k!}$$

We put

$$a_k = \frac{\|A\|^k}{k!}$$

Since

$$\lim_{k \to +\infty} \left| \frac{\frac{\|A\|^{k+1}}{(k+1)!}}{\frac{\|A\|^k}{k!}} \right| = \lim_{k \to +\infty} \frac{\|A\|}{k+1}$$

$$= 0 < 1$$

then $\displaystyle\sum_{k=0}^{\infty} \frac{\|A\|^k}{k!}$ is convergent.

On the other hand, we have:

$$\left\| \sum_{k=0}^{\infty} \frac{A^k}{k!} \right\| \leq \sum_{k=0}^{\infty} \frac{\|A\|^k}{k!}$$

Thus, $\displaystyle\sum_{k=0}^{\infty} \frac{A^k}{k!}$ is absolutely convergent.

Ex 04: Let A be a square matrix. We prove that:

$$\lim_{x \to 0} \frac{e^{xA} - I}{x} = A.$$

Solution: By definition, we have:

$$e^{xA} = I + \frac{xA}{1!} + \frac{(xA)^2}{2!} + \cdots$$

Then

$$\left\| e^{xA} - I - xA \right\| = \left\| \frac{(xA)^2}{2!} + \frac{(xA)^3}{3!} + \cdots \right\|$$

$$\leq \frac{\|xA\|^2}{2!} + \frac{\|xA\|^3}{3!} + \cdots$$

$$= e^{\|xA\|} - 1 - \|xA\|.$$

Therefore,

$$\left\| \frac{e^{xA} - I}{x} - A \right\| \leq \left(\frac{e^{|x| \cdot \|A\|} - 1}{|x|} - \|A\| \right) \to 0$$

as x tends to 0

and so $\left\| \frac{e^{xA} - I}{x} - A \right\| \xrightarrow[x \to 0]{} 0$

$$\Rightarrow \frac{e^{xA} - I}{x} \xrightarrow[x \to 0]{} A$$

Recall that

$$\frac{e^{x \cdot \alpha} - 1}{x} \xrightarrow[x \to 0]{} \alpha$$

Ex 05 : Consider the matrix

$$A = \begin{bmatrix} 1 & 3 & -3 \\ 1 & 3 & -1 \\ -2 & 2 & 0 \end{bmatrix}$$

① We find the eigenvalues and eigenvectors:
After simple computation, we get:

- $P_A(x) = (x-2)(x+2)(x-4)$
- The eigenvectors are :

$$\begin{cases} \lambda_1 = -2 \\ \lambda_2 = 2 \\ \lambda_3 = 4 \end{cases}$$

- The eigenvectors are :

$$\begin{cases} \lambda_1 \longrightarrow v_1 = (1, 0, 1) \\ \lambda_2 \longrightarrow v_2 = (0, 1, 1) \\ \lambda_3 \longrightarrow v_3 = (1, 1, 0) \end{cases}$$

② We deduce that A is diagonalizable.
That is, we can write the matrix A as:

$$A = P \cdot D \cdot P^{-1},$$ where

P is invertible and D is diagonal.

now, we put

$$P = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

After Computation, we get

$$P^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

③ We Compute e^A:

Since $A = P \cdot D \, P^{-1}$, then

$$e^A = P \cdot e^D \cdot P^{-1}, \text{ where}$$

$$e^D = \begin{bmatrix} e^{-2} & 0 & 0 \\ 0 & e^{2} & 0 \\ 0 & 0 & e^{4} \end{bmatrix}$$

Ex 06 : Consider the two matrices :

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

We verify that :

$$e^{A+B} \neq e^{A} \cdot e^{B} \neq e^{B} \cdot e^{A}.$$

- We have :

$$A + B = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$$

Then $$e^{A+B} = \begin{bmatrix} e^{2} & 0 \\ 0 & 1 \end{bmatrix}$$

- We Compute e^{A} :

The matrix A is diagonalizable, because it has two simple eigenvalues : $\lambda_1 = 1$, $\lambda_2 = 0$.

We have :

$$\underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}}_{P^{-1}}$$

and so

$$e^{A} = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & e \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

After Computation, we have:

$$e^{A} = \begin{bmatrix} e & e-1 \\ 0 & 1 \end{bmatrix}$$

- We Compute e^{B}:

Also, the matrix B is diagonalizable. In fact:

$$B = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$$

and hence,

$$e^{B} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} e^{0} & 0 \\ 0 & e^{1} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} e & 1-e \\ 0 & 1 \end{bmatrix}$$

Finally, we see that:

$$e^{A} \cdot e^{B} = \begin{bmatrix} e^{2} & 2e - e^{2} - 1 \\ 0 & 1 \end{bmatrix}$$

and

$$e^{B} \cdot e^{A} = \begin{bmatrix} e^{2} & e^{2} - 2e + 1 \\ 0 & 1 \end{bmatrix}$$

Ex 07 : Consider the matrix :

$$A = \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix}$$

we Calculate $\ln A$.

Few Computation, we obtain :

$$A = \underbrace{\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}}_{P} \cdot \underbrace{\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{3}{2} \end{bmatrix}}_{D} \cdot \underbrace{\begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}}_{P^{-1}}$$

and so

$$\operatorname{Ln} A = P \cdot \operatorname{Ln} D \cdot P^{-1}$$

$$= \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \operatorname{Ln} \frac{1}{2} & 0 \\ 0 & \operatorname{Ln} \frac{3}{2} \end{bmatrix} \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} \operatorname{Ln} 3 - \operatorname{Ln} 2 & \frac{1}{2} \operatorname{Ln} 3 \\ \frac{1}{2} \operatorname{Ln} 3 & \frac{1}{2} \operatorname{Ln} 3 - \operatorname{Ln} 2 \end{bmatrix}.$$

Ex 08: Let A be a diagonalizable matrix. That is, $A = P \cdot D P^{-1}$, where P is invertible and D is diagonal. Then

$$A^k = P D^k P^{-1} \quad \text{for } k \geq 0$$

$$\Rightarrow \lim_{k \to +\infty} A^k = \lim_{k \to +\infty} \left(P D^k P^{-1} \right)$$

$$= P \left(\lim_{k \to +\infty} D^k \right) P^{-1}.$$

- For example, if

$$A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{3}{4} \end{bmatrix}$$

At first, we compute the eigenvectors and the corresponding eigenvalues.

- $\lambda_1 = 1 \longrightarrow v_1 = (1, 1)$
- $\lambda_2 = \frac{1}{4} \longrightarrow v_2 = (-2, 1)$

We deduce that A is diagonalizable. That is, we can factorize A as:

$$A = P \cdot D \cdot P^{-1}, \text{ where } D = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{4} \end{bmatrix}$$

and

$$P = \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix}$$

Therefore,

$$A^k = P \cdot D^k \cdot P^{-1}, \text{ where } P^{-1} = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ -\frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1^k & 0 \\ 0 & \left(\frac{1}{4}\right)^k \end{bmatrix} \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ -\frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

Hence,

$$\lim_{k \to +\infty} A^k = P \left(\lim_{k \to +\infty} D^k \right) P^{-1}$$

$$= P \cdot \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} P^{-1}$$

$$= \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ -\frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix}.$$

Ex 09: Let

$$A = \begin{bmatrix} 3 & 1 \\ -2 & 0 \end{bmatrix}$$

We calculate A^k for $k \geq 0$:

First, after computation we get

$$A = \underbrace{\begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix}}_{P} \cdot \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}}_{D} \cdot \underbrace{\begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}}_{P^{-1}}$$

and so

$$A^k = P \cdot D^k \cdot P^{-1}$$

$$= \begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2^k \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}$$

It follows that:

$$A^k = \begin{bmatrix} 2^{k+1} - 1 & 2^k - 1 \\ 2 - 2^{k+1} & 2 - 2^k \end{bmatrix}, \quad k \geq 0.$$

In particular, for $k = 0$, we have:

$$A^0 = I.$$

Ex 10: Consider the matrix:

$$A = \begin{bmatrix} 9 & 0 & 0 \\ -5 & 4 & 0 \\ -8 & 0 & 1 \end{bmatrix}$$

We Compute A^k; $k \geq 0$:

The eigenvalues of A are:

$\lambda_1 = 9$, $\lambda_2 = 4$, $\lambda_3 = 1$.

The Corresponding eigenvectors are:

$v_1 = (-1, 1, 1)$, $v_2 = (0, 1, 0)$, $v_3 = (0, 0, 1)$.

Thus,

$$A = \underbrace{\begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}}_{P^{-1}}$$

and so $A^k = P D^k . P^{-1}$

$$= \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4^k & 0 \\ 0 & 0 & 9^k \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 9^k & 0 & 0 \\ 4^k - 9^k & 4^k & 0 \\ 1 - 9^k & 0 & 1 \end{bmatrix}, \quad k \geq 0.$$

Test N° 04

Ex 01: We solve the system of linear recurrence sequences:

$$(S)\dots\begin{cases} x_{n+1} = 3x_n - y_n, & x_0 = 1 \\ y_{n+1} = -x_n + 3y_n & y_0 = 2. \end{cases}$$

We can write the system (S) as follows:

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix}$$

$$X_{n+1} = A \cdot X_n$$

Therefore,

$$\begin{aligned} X_n &= A \cdot X_{n-1} \\ &= A(A \cdot X_{n-2}) = A^2 \cdot X_{n-2} \\ &= \dots \\ &= A^n \cdot X_0 \end{aligned}$$

Thus, $X_n = A^n \cdot X_0$ .... (*)

After Computation, we show that

$$\begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$A = P \cdot D \cdot P^{-1}$$

it follows that:

$$A^n = P \cdot D^n \cdot P^{-1}$$

$$= \begin{bmatrix} \frac{2^n + 4^n}{2} & \frac{2^n - 4^n}{2} \\ \frac{2^n - 4^n}{2} & \frac{2^n + 4^n}{2} \end{bmatrix}$$

From (*), we obtain:

$$X_n = \begin{bmatrix} x_n \\ y_n \end{bmatrix} = \begin{bmatrix} \frac{2^n + 4^n}{2} & \frac{2^n - 4^n}{2} \\ \frac{2^n - 4^n}{2} & \frac{2^n + 4^n}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{2} \cdot 2^n - \frac{1}{2} 4^n \\ \frac{3}{2} \cdot 2^n + \frac{1}{2} \cdot 4^n \end{bmatrix}$$

Thus, $x_n = 2^{n-1}(3 - 2^n)$ and $y_n = 2^{n-1}(3 + 2^n)$, $n \geq 0$

Ex 02: Consider the determinant

$$\Delta_n = \begin{vmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ 0 & -1 & 2 & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & -1 & 2 & -1 \\ & & & & -1 & 2 \end{vmatrix}, \quad n \text{ by } n$$

This is the determinant of a tridiagonal matrix.

① We show that $\Delta_n = 2\Delta_{n-1} - \Delta_{n-2}$.

In fact, we have:

$$\Delta_n = \begin{array}{c} \begin{matrix} + & - & + & - & \cdots \end{matrix} \\ \begin{vmatrix} 2 & -1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\ & & & & & & & & \\ 0 & 0 & 0 & 0 & 0 & \cdots & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 2 \end{vmatrix} \end{array}$$

Thus,

$$D_n = 2D_{n-1} + \underbrace{\begin{vmatrix} -1 & -1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 2 & -1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & 0 & -1 & 2 & & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 2 & -1 \\ 0 & 0 & 0 & 0 & & -1 & 2 \end{vmatrix}}_{\Delta}$$

Using an expansion according to the first Column of Δ we get:

$$\Delta = -D_{n-2}.$$

Therefore,

$$\boxed{D_n = 2D_{n-1} - D_{n-2}}$$

Now, we Calculate D_n in terms of n.

- Simple Computation, we have:

$$D_1 = \det[2] = 2$$

$$D_2 = \det\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = 3.$$

$$D_3 = \det \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} = 4.$$

Since
$$D_n = 2D_{n-1} - D_{n-2} \quad \cdots\cdots \quad (S)$$

in the matrix form, we get

$$\begin{bmatrix} D_n \\ D_{n-1} \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} D_{n-1} \\ D_{n-2} \end{bmatrix}$$

$$X_n = A \cdot X_{n-1} \quad \cdots\cdots \quad (*)$$

where,
$$X_2 = \begin{bmatrix} D_2 \\ D_1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

It follows from (*) that:

$$X_n = A^{n-2} \cdot X_2.$$ نتوقف عند X_2.

- We must Compute A^{n-2}.

$\boxed{I}$ Characteristic polynomial:

$$P_A(x) = \begin{vmatrix} 2-x & -1 \\ 1 & -x \end{vmatrix} \quad \begin{matrix} L_1 \\ \downarrow \\ L_1 - L_2 \end{matrix}$$

$$= \begin{vmatrix} 1-x & -(1-x) \\ 1 & -x \end{vmatrix}$$

$$= (1-x) \cdot \begin{vmatrix} 1 & -1 \\ 1 & -x \end{vmatrix}$$

$$= (1-x) \cdot \left[-x + 1 \right]$$

$$= (1-x)^2.$$

Hence, $P_A(x) = (1-x)^2$

- The Corresponding eigenvectors :

We see that the matrix A has one eigenvalue $h = 1$.

$$E_h = \left\{ (x,y) \in \mathbb{R}^2 \;/\; \begin{matrix} 2x - y = x \\ x = x \end{matrix} \right\}$$

$$= \left\{ (x,y) \in \mathbb{R}^2 \;/\; y = x \right\}$$

$$= \text{Vect}\left\{ (1,1) \right\}. \longrightarrow v = (1,1)$$

- The matrix A is not diagonalizable.

That is, we can not write it as : $A = PDP^{-1}$.

Then, we try to write it as:

$$A = P \cdot T \cdot P^{-1}, \text{ where}$$

P is invertible and T is upper triangular, i.e $T = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}$

We choose $v' = (1, 0)$ and put

$$P = \begin{array}{c} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \\ \begin{matrix} v & v' \end{matrix} \end{array} \Rightarrow P^{-1} \text{ exists.}$$

After computation $\Rightarrow P^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$

Also, we get

$$P^{-1} A P = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = T$$

Thus,

$$A = P \cdot T \cdot P^{-1}, \text{ and so}$$

$$A^k = P \cdot T^k \cdot P^{-1}$$

Next, we Compute T^k:

In fact

$$T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{D} + \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{N}$$

Remak: We use the Binomial formula to Compute T^k, where $T = D + N$ with $DN = ND$.

- The matrix N is nilpotent عديمة القوى because: $N^2 = 0$. Then

$$T^k = (D+N)^k = \sum_{i=0}^{k} C_k^i D^{k-i} \cdot N^i$$

$$= C_k^0 D^k N^0 + C_k^1 D^{k-1} \cdot N + 0$$

$$= D^n + k D^{k-1} \cdot N \qquad (D = I)$$

$$= I + k \cdot N$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & k \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$$

Hence, $T^k = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$, $k \geq 0$

Finally, we get

$$A^{n-2} = P \cdot T^{n-2} \cdot P^{-1}$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & n-2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} n-1 & 2-n \\ n-2 & 3-n \end{bmatrix}$$

Since $X_n = A^{n-2} \cdot X_2$, then

$$X_n = \begin{bmatrix} n-1 & 2-n \\ n-2 & 3-n \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

That is,

$$\begin{bmatrix} \Delta_n \\ \Delta_{n-1} \end{bmatrix} = \begin{bmatrix} 3(n-1) + 2(2-n) \\ 3(n-2) + 2(3-n) \end{bmatrix}$$

and so

$$\Delta_n = 3(n-1) + 2(2-n)$$
$$= n+1.$$

Ex 03: Let (x_n) be the sequence given by :

$$x_{n+2} = \frac{2}{\frac{1}{x_n} + \frac{1}{x_{n+1}}} \qquad \begin{array}{l} x_0 > 0 \\ x_1 > 0 \end{array}$$

نقوم بحساب $\lim_{n \to +\infty} x_n$ بدلالة x_0 و y_0

We evaluate the limit as n tends to ∞.

We put

$$y_n = \frac{1}{x_n}, \text{ then}$$

$$\frac{1}{y_{n+2}} = \frac{2}{y_n + y_{n+1}},$$

and so

$$2\,y_{n+2} = y_n + y_{n+1}$$

That is,

$$\boxed{y_{n+2} = \frac{1}{2} y_n + \frac{1}{2} y_{n+1}}$$

In the matrix form, we get

$$\begin{bmatrix} y_{n+2} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y_{n+1} \\ y_n \end{bmatrix}$$

and so

$$\begin{bmatrix} y_n \\ y_{n-1} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y_{n-1} \\ y_{n-2} \end{bmatrix}$$

$$X_n = A \cdot X_{n-1}$$
$$= \dots$$
$$= A^{n-1} \cdot X_1 \text{, where}$$

$$X_1 = \begin{bmatrix} y_1 \\ y_0 \end{bmatrix} = \begin{bmatrix} \frac{1}{x_1} \\ \frac{1}{x_0} \end{bmatrix}$$

- We must compute A^k, $k \geq 0$:

By few computation, we have

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & 0 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}}_{P} \cdot \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & -\frac{1}{2} \end{bmatrix}}_{D} \cdot \underbrace{\begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} \end{bmatrix}}_{P^{-1}}$$

and hence $A^k = P \cdot D^k \cdot P^{-1}$

Thus,

$$A^k = \begin{bmatrix} \frac{1}{3}\left(-\frac{1}{2}\right)^k + \frac{2}{3} & \frac{1}{3} - \frac{1}{3}\left(-\frac{1}{2}\right)^k \\ \frac{2}{3} - \frac{2}{3}\left(-\frac{1}{2}\right)^k & \frac{2}{3}\left(-\frac{1}{2}\right)^k + \frac{1}{3} \end{bmatrix}$$

and so

$$y_n = \frac{1}{x_1}\left[\frac{1}{3}\left(-\frac{1}{2}\right)^{n-1} + \frac{2}{3}\right] + \frac{1}{x_0}\left[\frac{1}{3} - \frac{1}{3}\left(\frac{-1}{2}\right)^{n-1}\right]$$

$$\Rightarrow \lim_{n \to +\infty} y_n = \frac{2}{3x_1} + \frac{1}{3x_0}$$

and hence

$$\lim_{n \to +\infty} x_n = \frac{3}{\frac{2}{x_1} + \frac{1}{x_0}}.$$

Ex 04: Consider the system of differential equations:

$$X' = A \cdot X, \quad \text{where}$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

Here, $$X' = \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}$$

Thus,

$$\begin{cases} x' = x \\ y' = x + 2y \\ z' = x + 2z \end{cases}$$

الدوال x ، y
و z بدلالة t.

① We find the eigenvalues and eigenvectors.

$$\begin{cases} \lambda_1 = 1 \longrightarrow v_1 = (-1, 1, 1) \\ \lambda_2 = 2 \longrightarrow \begin{cases} v_2 = (0, 1, 0) \\ v_3 = (0, 0, 1) \end{cases} \end{cases}$$

In this Case, the matrix A is diagonalizable

Thus,

$$X(t) = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2 + c_3 e^{\lambda_2 t} v_2.$$

That is,

$$X(t) = c_1 e^{t} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} + c_2 e^{0.t} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_3 e^{0.t} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

$c_1, c_2, c_3 \in \mathbb{R}$.

Example: Solve the differential system

$$\begin{cases} x' = x + 3y \\ y' = 3x + y \end{cases} \quad \dots \quad (S')$$

the system (S') Can be written as:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$X' = A \cdot X$$

- The eigenpairs of A are:

$$\lambda_1 = -2 \longrightarrow v_1 = (-1, 1)$$

$$\lambda_2 = 4 \longrightarrow v_2 = (1, 1)$$

- The matrix A is diagonalizable, and so

$$X(t) = c_1 \cdot e^{-2t} \cdot v_1 + c_2 \cdot e^{4t} v_2$$

$$= c_1 e^{-2t} \begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 \cdot e^{4t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

thus,

$$\begin{cases} x = -c_1 e^{-2t} + c_2 e^{4t} \\ y = c_1 e^{-2t} + c_2 e^{4t} \end{cases}, \quad c_1, c_2 = C^{te}$$

ثابتين

Ex 05: Using Cayley-Hamilton theorem we Calculate the inverse of the matrix:

$$A = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 0 \\ 2 & 0 & -1 \end{bmatrix}$$

First, we find the characteristic polynomial:

$$\begin{array}{c} \quad\; + \qquad\quad - \qquad\quad + \\ P_A(x) = \begin{vmatrix} 1-x & 1 & 0 \\ -1 & -x & 0 \\ 2 & 0 & -1-x \end{vmatrix} \end{array}$$

$$= x^3 + 1 \quad \text{(after Computation)}.$$

By Cayley-Hamilton theorem, we get

$$P_A(A) = 0 \equiv \text{zero matrix}.$$

$$\Rightarrow A^3 + I = 0$$

$$\Rightarrow A^3 = -I \Rightarrow A^2 = -A^{-1}$$

$$\Rightarrow \boxed{A^{-1} = -A^2}$$

$$A^{-1} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & -2 & -1 \end{bmatrix}$$

بعد الحساب

Ex 06: Let A be a square matrix.
Assume that the characteristic polynomial of A is given by:

$$P_A(x) = c_0 + c_1 x + \cdots + c_{n-1} x^{n-1} + c_n x^n,$$

where $c_0 \neq 0$.

For example, in Ex 05, we have:

$$P_A(x) = 1 + x^3, \text{ i.e } c_0 = 1 \neq 0.$$

Applying Cayley-Hamilton theorem, we get $P_A(A) = 0$. That is,

$$c_0 I + c_1 A + \cdots + c_{n-1} A^{n-1} + c_n A^n = 0.$$

It follows that

$$c_1 A + \cdots + c_{n-1} A^{n-1} + c_n A^n = -c_0 I$$

and hence,

$$c_1 I + c_2 A + \cdots + c_n A^{n-1} = -c_0 A^{-1}$$

Thus, $$A^{-1} = \frac{-1}{c_0}\left(c_1 I + c_2 A + \cdots + c_n A^{n-1}\right)$$

The proof is finished. □

Ex 07: Let $A \in \mathcal{M}_n(\mathbb{R})$ be a square matrix with $Sp(A) = \{\lambda\}$. That is, A has one eigenvalue λ. We show that

$$e^{tA} = e^{\lambda t} \cdot \sum_{k=0}^{n-1} (A-\lambda I)^k \frac{t^k}{k!}.$$

Proof: First, we have: $P_A(x) = (x-\lambda)^n$.

Then

$$e^{tA} = e^{tA + \lambda t I - \lambda t I}$$

$$= e^{\lambda t I + t(A-\lambda I)}$$

$$= e^{\lambda t I} \cdot e^{t(A-\lambda I)}$$

$$= e^{\lambda t} \cdot e^{t(A-\lambda I)} \qquad \left(e^{\alpha I} \cdot B = e^{\alpha} \cdot B\right)$$

$$= e^{\lambda t} \sum_{k=0}^{+\infty} \frac{(A-\lambda I)^k t^k}{k!}$$

$$= e^{\lambda t} \cdot \sum_{k=0}^{n-1} \frac{(A-\lambda I)^k}{k!} t^k$$

because by Cayley–Hamilton theorem,

$$P_A(A) = (A - \lambda I)^n = 0.$$

The proof is finished $\square$.

Now, we solve the system of differential equations:

$$\begin{cases} x' = 2x + y \\ y' = 2y \end{cases}$$

in the matrix form:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$X' = A \cdot X$$

The solution is given by:

$$X(t) = e^{tA} \cdot C, \quad \text{where } C = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$= e^{\lambda t} \Big[I + t(A - \lambda I) \Big]$$

$$= e^{2t} \left[\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + t \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right]$$

$$= e^{2t}\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} e^{2t} & t e^{2t} \\ 0 & e^{2t} \end{bmatrix}$$

Thus,

$$X(t) = \begin{bmatrix} c_1 e^{2t} + c_2 t e^{2t} \\ c_2 e^{2t} \end{bmatrix}$$

and so

$$\begin{cases} x = c_1 e^{2t} + c_2 t e^{2t} \\ y = c_2 e^{2t} , \quad c_1, c_2 \in \mathbb{R}. \end{cases}$$

- Solve the system of differential equations:

$$\underbrace{\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}}_{X'} = \underbrace{\begin{bmatrix} -4 & 1 & 1 \\ 1 & -1 & -2 \\ -2 & 1 & -1 \end{bmatrix}}_{A} \cdot \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{X}$$

The characteristic polynomial of A is given by (after computation).

$$P_A(x) = (x+2)^3.$$

Then A has one eigenvalue $\lambda = -2$.

and so:

$$e^{tA} = e^{\lambda t}\left[I + t(A-\lambda I) + \frac{t^2}{2}(A-\lambda I)^2\right]$$

The solution of $X' = AX$ is given by:

$$X(t) = e^{tA}.\,C, \quad \text{where } C = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix},$$

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- $A - \lambda I = A + 2I = \begin{bmatrix} -2 & 1 & 1 \\ 1 & 1 & -2 \\ -2 & 1 & 1 \end{bmatrix}$

- $(A-\lambda I)^2 = (A+2I)^2 = \begin{bmatrix} 3 & 0 & -3 \\ 3 & 0 & -3 \\ 3 & 0 & -3 \end{bmatrix}$

أكمل الحل !

The characteristic polynomial of A is given by (after computation).

$$f_A(x) = (x+2)^3.$$

Then A has one eigenvalue $\lambda = -2$.

and so:

$$e^{tA} = e^{\lambda t}\left[I + t(A-\lambda I) + \frac{t^2}{2}(A-\lambda I)^2\right]$$

The solution of $X' = AX$ is given by:

$$X(t) = e^{tA}.\, C, \quad \text{where } C = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix},$$

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- $A - \lambda I = A + 2I = \begin{bmatrix} -2 & 1 & 1 \\ 1 & 1 & -2 \\ -2 & 1 & 1 \end{bmatrix}$

- $(A-\lambda I)^2 = (A+2I)^2 = \begin{bmatrix} 3 & 0 & -3 \\ 3 & 0 & -3 \\ 3 & 0 & -3 \end{bmatrix}$

! أكمل الحل

• There is a few hard case when $A \in \mathcal{M}_3(\mathbb{R})$ with $Sp(A) = \{\lambda, \lambda, \mu\}$. For example:

$$A = \begin{bmatrix} 2 & -1 & 2 \\ 10 & -5 & 7 \\ 4 & -2 & 2 \end{bmatrix} \quad \cdots\cdots \quad (*)$$

If the matrix A is not diagonalizable, then the solution of $X' = AX$ is:

$$X(t) = e^{tA} \cdot C \, , \quad \text{where } C = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

Using Cayley-Hamilton theorem (see the Course) we can prove that,

$$e^{tA} = e^{\lambda t}\left[I + t(A - \lambda I)\right] + \frac{e^{\mu t} - e^{\lambda t}}{(\mu - \lambda)^2}(A - \lambda I)^2$$

$$- \frac{t\, e^{\lambda t}}{\mu - \lambda}(A - \lambda I)^2 \quad \cdots \quad (**)$$

In the matrix $(*)$, we have

- A is not diagonalizable
- $\lambda = 0$ (with multiplicity 2)
- $\mu = -1$ (simple). So, we must compute e^{tA} using the long formula $(**)$. For the proof see the Course.

Test N° 05

Ex 01 : We Calculate the minimal polynomial of the matrix ;

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Since A is an upper triangular matrix, then $P_A(x) = (1-x)^2$. Thus,

$$m_A(x) = \begin{cases} 1-x \\ \text{or} \\ (1-x)^2. \end{cases}$$

on the other hand, since $m_A(A) = 0$, then $m_A(x) = (1-x)^2 = P_A(x)$.

Recall that :

$$\Big(A \text{ is diagonalizable}\Big) \Longleftrightarrow \Big(\text{the roots of } m_A(x) \text{ are simple.}\Big)$$

• Let $A = \begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix}$, $c \neq 0$

Since $m_A(x) = (1-x)^2 \Rightarrow$ A is not diagonalizable.

Ex 02: We determine the minimal polynomial of the matrices:

[a]
$$A = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

- First, we see that $P_A(x) = x^3$. Then

$$m_A(x) = \begin{cases} x \\ \text{or} \\ x^2 \\ \text{or} \\ x^3 \end{cases}$$

Since
$$A^2 = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence,
$$\boxed{m_A(x) = x^2}$$

$\boxed{b}$

$$A = \begin{bmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 4 \end{bmatrix}$$

- Characteristic polynomial :

$$p_A(x) = (x-6)(x-3)^2 . \text{ Thus,}$$

$$m_A(x) = \begin{cases} (x-6)(x-3) \\ \text{or} \\ (x-6)(x-3)^2 . \end{cases}$$

But, $(A-6I)(A-3I) =$

$$\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence,

$$m_A(x) = (x-6)(x-3).$$

Recall that the minimal polynomial satisfies the following three conditions :

1. $m_A(x)$ divides $p_A(x)$ مهم
2. $m_A(x)$ and $p_A(x)$ have the same roots.
3. $m_A(A) = p_A(A) = 0$.

[c] Let

$$A = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$$

- Characteristic polynomial:

$$P_A(x) = (x-1)^2. \quad \text{Thus,}$$

$$m_A(x) = \begin{cases} x-1 \\ \text{or} \\ (x-1)^2. \end{cases}$$

Since $A - I = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \neq 0$, then

$m_A(x) = (x-1)^2$.

Ex 03: Assume that $m_A(x) = (x-a)(x-b)$, where $a, b \in \mathbb{R}$ $(a \neq b)$.

We show that A^n can be only written in terms of A and I. In fact:

$$m_A(x) = x^2 - (a+b)x + ab.$$

Thus, $A^2 - (a+b)A + ab.I = 0$.

$$\Rightarrow A^2 = (a+b)A - ab.I$$

إذن A^2 تكتب بدلالة A و I

Thus, A^2 is written in terms of A and I.

Now, assume that

$$A^n = \alpha_n . A + \beta_n . I, \text{ where } \alpha_n, \beta_n \in \mathbb{R}$$

Therefore,

$$A^{n+1} = A \cdot A^{n} = A(\alpha_n A + \beta_n \cdot I)$$

$$= \alpha_n A^2 + \beta_n \cdot A$$

$$= \alpha_n \left[(a+b)A - ab\,I \right] + \beta_n \cdot A$$

$$= \Big[\underbrace{\alpha_n (a+b) + \beta_n}_{\in \mathbb{R}} \Big] A - \underbrace{\alpha_n\, ab}_{\in \mathbb{R}} \cdot I$$

That is, A^{n+1} is written in terms of A and I.

The proof is finished $\square$.

Remark. In the proof of Ex 03, we have used the proof by induction. بالتراجع

Remark. If a matrix has the minimal polynomial:

$$m_A(x) = (x-a)(x-b),$$

then

$$P_A(x) = (x-a)^{\alpha} \cdot (x-b)^{\beta}, \text{ where}$$

$$\alpha + \beta = n.$$

Ex 04: We find the minimal polynomial of the matrix

$$A_n = \begin{bmatrix} \lambda & & & & \\ 1 & \lambda & & & \\ & 1 & \lambda & & \\ & & & \ddots & \\ & & & 1 & \lambda \end{bmatrix} \quad n \text{ by } n$$

- For $n = 2$:

$$A_2 = \begin{bmatrix} \lambda & 0 \\ 1 & \lambda \end{bmatrix}, \quad m_{A_2}(x) = (x-\lambda)^2$$

- For $n = 3$,

$$A_3 = \begin{bmatrix} \lambda & 0 & 0 \\ 1 & \lambda & 0 \\ 0 & 1 & \lambda \end{bmatrix}, \quad m_{A_3}(x) = (x-\lambda)^3$$

- For $n = 4$,

$$A_4 = \begin{bmatrix} \lambda & 0 & 0 & 0 \\ 1 & \lambda & 0 & 0 \\ 0 & 1 & \lambda & 0 \\ 0 & 0 & 1 & \lambda \end{bmatrix},$$

$$m_{A_4}(x) = (x-\lambda)^4.$$

By induction, we deduce that

$$m_{A_n}(x) = (x-\lambda)^n.$$

Ex 05: Consider the matrix

$$A = \begin{bmatrix} a & b & b \\ b & a & b \\ b & b & a \end{bmatrix}, \quad a, b \in \mathbb{R}.$$

We show that A is diagonalizable.

In fact, we see that:

$$A = a \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + b \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$= a \cdot I + b \cdot C_1, \text{ where}$$

$$C_1 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

- The characteristic polynomial of the matrix C is:

$$P_C(x) = (1+x)^2 (x-2)$$

This means that $m_C(x) = (1+x)(x-2)$ or $m_C(x) = (1+x)^2 (x-2)$.

But,

$$(I + C)(C - 2I) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Hence,

$m_C(x) = (1+x)(x-2)$ which has simple roots. Then C is diagonalizable. that is C can be written as:

$C = P \cdot D \cdot P^{-1}$, where P is invertible and D is diagonal. Therefore,

$$A = a \cdot I + b \cdot C$$

$$= a\, P \cdot P^{-1} + b\, P D P^{-1}$$

$$= P \left[\underbrace{a \cdot I + D}_{D'} \right] P^{-1} = P \cdot D' \cdot P^{-1}, \text{ where}$$

D' is diagonal.

Fact:

$(A \text{ is diagonalizable}) \Longleftrightarrow \left(\begin{array}{c} \text{The roots of } m_A(x) \\ \text{are all simple.} \end{array} \right)$

Ex 06: Consider the matrix

$$A_n = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & & & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix} \quad n \text{ by } n$$

We show that A_n is diagonalizable.

We have proved before that:

$$P_{A_n}(x) = \begin{cases} x^{n-1}(x-n), & \text{if } n \text{ is even (زوجي)} \\ x^{n-1}(n-x), & \text{if } n \text{ is odd (فردي)} \end{cases}$$

Here, we see that:

$$\begin{cases} A_n(A_n - nI) = 0 \\ \text{and} \\ A_n(nI - A_n) = 0. \end{cases}$$

For example:

$$A_n(A_n - nI) = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & & & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} 1-n & 1 & \cdots & 1 \\ 1 & 1-n & \cdots & 1 \\ \vdots & & & \vdots \\ 1 & 1 & & 1-n \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

Hence

$$m_{A_n}(x) = \begin{cases} x(x-n) \\ \text{or} \\ x(n-x) \end{cases}$$

and so A_n is diagonalizable.

Ex 07: Let

$$A = \begin{bmatrix} 2 & -1 \\ 1 & 4 \end{bmatrix}$$

We trigonalize the matrix A.

- It seems that A is not diagonalizable. But, we try to write it as:

$$A = P \cdot T \cdot P^{-1}, \text{ where}$$

P = is invertible

T = is an upper triangular matrix

مثلثية عليا

That is, A is similar to the matrix T.

• From simple computation, we have

$$P_A(x) = (x-3)^2.$$

This means that $\lambda = 3$ is the only eigenvalue of A with multiplicity 2. Hence A is not diagonalizable since $A \neq 3 \cdot I$.

Next, we find the corresponding eigenvector. In fact, we have:

$$E_\lambda = \left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; \begin{array}{l} 2x - y = 3x \\ x + 4y = 3y \end{array} \right\}$$

$$= \left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; y = -x \right\}$$

$$= \text{Vect}\left\{ (1,-1) \right\}.$$ Hence, $v_1 = (1,-1)$.

Let v_2 be a nonzero vector for which $\{v_1, v_2\}$ is a basis of $\mathbb{R}^2$. For example, we put $v_2 = (1,1)$, and let

$$P = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

Therefore,

$$P^{-1} A P = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

After Computation, we get

$$P^{-1} A P = \begin{bmatrix} 3 & -2 \\ 0 & 3 \end{bmatrix} = T.$$

That is, $A = P \cdot T \cdot P^{-1} \Rightarrow A$ is trigonalizable.

and so $$A^n = P T^n \cdot P^{-1}$$

• It suffices to compute T^n.

We write the matrix T as follows:

$$T = \begin{bmatrix} 3 & -2 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} + \begin{bmatrix} 0 & -2 \\ 0 & 0 \end{bmatrix}$$

$$T = D + N$$

where, $N^2 = 0$. Hence,

$$T^n = C_n^0 D^n + C_n^1 D^{n-1} \cdot N$$

$$= D^n + n D^{n-1} \cdot N$$

$$= \begin{bmatrix} 3^n & 0 \\ 0 & 3^n \end{bmatrix} + n \begin{bmatrix} 3^{n-1} & 0 \\ 0 & 3^{n-1} \end{bmatrix} \begin{bmatrix} 0 & -2 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 3^n & -2n \cdot 3^{n-1} \\ 0 & 3^n \end{bmatrix}, \quad n \geq 0.$$

Finally, we deduce that:

$$A^n = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 3^n & -2n.3^{n-1} \\ 0 & 3^n \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$= \begin{bmatrix} 3^n - n.3^{n-1} & -n.3^{n-1} \\ n.3^{n-1} & n.3^{n-1} + 3^n \end{bmatrix}$$

Ex 08: We trigonalize the matrix

$$A = \begin{bmatrix} 2 & -1 & 2 \\ 10 & -5 & 7 \\ 4 & -2 & 2 \end{bmatrix}$$

That is, we write A in the form:

$$A = P . T . P^{-1}$$

$\longrightarrow$: مثلثية عليا

- characteristic polynomial:

$$P_A(x) = x^2(x+1)$$

- eigenvectors

$$\lambda_1 = 0 \longrightarrow v_1 = (1, 2, 0)$$

$$\lambda_2 = -1 \longrightarrow v_2 = (-1, 1, 2)$$

λ_2 has one eigenvector $\Rightarrow$ A is not diagonalizable.

Now, we choose $v_3 = (0, 0, 1)$, and put

$$P = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

Thus,

$$P^{-1} A P = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & 0 \\ -\frac{2}{3} & \frac{1}{3} & 0 \\ \frac{4}{3} & -\frac{2}{3} & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 2 \\ 10 & -5 & 7 \\ 4 & -2 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 2 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 3 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = T.$$

$\Rightarrow$ A is trigonalizable.

Ex 09 We trigonalize the matrix

$$A = \begin{bmatrix} 7 & -6 & -2 \\ 2 & 0 & -1 \\ 2 & -3 & 2 \end{bmatrix}$$

- Characteristic polynomial

$$P_A(x) = (x-3)^3.$$

Thus, A has one eigenvalue $\lambda = 3$ with multiplicity 3.

- The Corresponding eigenvectors:

$$E_\lambda = \left\{ (x, y, z) \in \mathbb{R}^3 \;\middle/\; \begin{array}{l} 7x - 6y - 2z = 3x \\ 2x - z = 3y \\ 2x - 3y + 2z = 3z \end{array} \right\}$$

$$= \left\{ (x, y, z) \in \mathbb{R}^3 \;/\; z = 2x - 3y \right\}$$

$$= \left\{ (x, y, 2x - 3y), \; x, y \in \mathbb{R} \right\}$$

$$= \left\{ x(1, 0, 2) + y(0, 1, -3) \; ; \; x, y \in \mathbb{R} \right\}$$

$$= \text{Vect}\left\{ (1, 0, 2), (0, 1, -3) \right\}$$

Hence $\lambda = 3$ has two eigenvectors:

$$\begin{cases} v_1 = (1, 0, 2) \\ v_2 = (0, 1, -3) \end{cases}$$

Thus, A is not diagonalizable.

- We choose a vector v_3 such that $\{v_1, v_2, v_3\}$ is a basis of $\mathbb{R}^3$. For example,

$$v_3 = (1, 0, 0).$$

We put

$$P = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 2 & -3 & 0 \end{bmatrix}$$

it follows that:

$$P^{-1} A P = \begin{bmatrix} 0 & \frac{3}{2} & \frac{1}{2} \\ 0 & 1 & 0 \\ 1 & -\frac{3}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 7 & -6 & -2 \\ 2 & 0 & -1 \\ 2 & -3 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 2 & -3 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 0 & 4 \\ 0 & 3 & 2 \\ 0 & 0 & 3 \end{bmatrix} = T.$$

$\Rightarrow A = P \cdot T P^{-1} \Rightarrow A$ is trigonalizable.

$\Rightarrow A^n = P \cdot T^n \cdot P^{-1}$

- It suffices to compute T^n:

T is an upper triangular matrix $\Rightarrow T^n$ is computed by using the Binomial formula:

$$T = \underbrace{\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}}_{D} + \underbrace{\begin{bmatrix} 0 & 0 & 4 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}}_{N}$$

where $DN = ND$.

$$\Rightarrow T^n = (D+N)^n = \sum_{i=0}^{n} C_n^i D^{n-i} . N^i$$

- The matrix N is nilpotent ($N^2 = 0$)

Thus,

$$T^n = C_n^0 D^n + C_n^1 D^{n-1} . N$$

$$= D^n + n . D^{n-1} . N$$

$$= \begin{bmatrix} 3^n & 0 & 0 \\ 0 & 3^n & 0 \\ 0 & 0 & 3^n \end{bmatrix} + n \begin{bmatrix} 3^{n-1} & 0 & 0 \\ 0 & 3^{n-1} & 0 \\ 0 & 0 & 3^{n-1} \end{bmatrix} \begin{bmatrix} 0 & 0 & 4 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 3^n & 0 & 4n . 3^{n-1} \\ 0 & 3^n & 2n . 3^{n-1} \\ 0 & 0 & 3^n \end{bmatrix}, \quad n \geq 0.$$

Finally, we obtain:

$$A^n = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 2 & -3 & 0 \end{bmatrix} \begin{bmatrix} 3^n & 0 & 4n . 3^{n-1} \\ 0 & 3^n & 2n . 3^{n-1} \\ 0 & 0 & 3^n \end{bmatrix} \begin{bmatrix} 0 & \frac{3}{2} & \frac{1}{2} \\ 0 & 1 & 0 \\ 1 & -\frac{3}{2} & -\frac{1}{2} \end{bmatrix}$$

$$= \begin{bmatrix} 4n\cdot 3^{n-1} + 3^{n} & -2n\cdot 3^{n} & -2n\cdot 3^{n-1} \\ 2n\cdot 3^{n-1} & 3^{n}(1-n) & -n\cdot 3^{n-1} \\ 2n\cdot 3^{n-1} & -n\cdot 3^{n} & 3^{n} - n\cdot 3^{n-1} \end{bmatrix}$$

Ex 10: Consider the matrix:

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$\boxed{1}$ We verify that $A^2 = A + 2I$:

In fact:

$$A^2 = A\cdot A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

Moreover, we have:

$$A + 2I = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} + 2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

Thus, $A^2 = A + 2I$.

- Since $A^2 = A + 2I$, then $A^2 - A = 2I$ and so $A(A - I) = 2I$

$$\Rightarrow A - I = 2A^{-1}$$

$$\Rightarrow \boxed{A^{-1} = \frac{1}{2}(A - I)}$$

This is the expression of A^{-1} in terms of A and I.

[2] We use the proof by induction to show that:

$$A^n = a_n \cdot A + b_n \cdot I, \text{ where}$$

$a_n, b_n \in \mathbb{N}$ and

$$(S) \ldots \begin{cases} a_{n+1} = a_n + b_n \\ b_{n+1} = 2a_n \end{cases}, \quad n \geq 0.$$

- For $n = 1$, we have:

$$A^1 = 1 \times A + 0 \cdot I = a_1 A + b_1 I$$

- For $n = 2$, we also have:

$$A^2 = A + 2I = a_2 \cdot A + b_2 \cdot I$$

where:

$$\begin{cases} a_2 = a_1 + b_1 \\ b_2 = 2a_1 \end{cases}$$

Now, assume that $A^n = a_n \cdot A + b_n \cdot I$, where a_n, b_n satisfy the system (S'). Then

$$\begin{aligned} A^{n+1} &= A \cdot A^n \\ &= A \left[a_n \cdot A + b_n I \right] \\ &= a_n A^2 + b_n \cdot A \\ &= a_n (A + 2I) + b_n \cdot A \\ &= (a_n + b_n) \cdot A + 2a_n \cdot I \\ &= a_{n+1} \cdot A + b_{n+1} \cdot I \end{aligned}$$

The proof is finished $\square$.

3\] We write the system (S') in the matrix form :

$$(S') \Longleftrightarrow \begin{bmatrix} a_{n+1} \\ b_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} a_n \\ b_n \end{bmatrix}$$

$$X_{n+1} = A \cdot X_n$$

Therefore, $\boxed{X_n = A^{n-1} \cdot X_1}$

$$X_1 = \begin{bmatrix} a_1 \\ b_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

4\]

- We Compute A^k :

After Computation, we get

$$\begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix} = \underbrace{\begin{bmatrix} -1 & 1 \\ 2 & 1 \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} -\frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix}}_{P^{-1}}.$$

and so

$$A^k = \begin{bmatrix} \dfrac{(-1)^k + 2^{k+1}}{3} & \dfrac{2^k - (-1)^k}{3} \\ \dfrac{2^{k+1} - 2(-1)^k}{3} & \dfrac{2(-1)^k + 2^k}{3} \end{bmatrix}$$

it follows that

$$\begin{cases} a_n = \dfrac{2^n - (-1)^n}{3}, & n \geq 0 \\ b_n = \dfrac{2(-1)^n + 2^n}{3}, & n \geq 0 \end{cases}$$

$\boxed{5}$ We Calculate the minimal polynomial of A:

First, we have

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

- characteristic polynomial:

$P_A(x) = (x-2)(x+1)^2$. Then

$$m_A(x) = \begin{cases} (x-2)(x+1) \\ \text{or} \\ (x-2)(x+1)^2. \end{cases}$$

Since

$$(A-2I)(A+I) = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus, $m_A(x) = (x-2)(x+1)$.

$\Rightarrow$ We deduce that A is diagonalizable since the roots of $m_A(x)$ are simple.

$\boxed{6}$ We solve the system of differential equations:

$$\begin{cases} x' = y + z \\ y' = x + z \\ z' = x + y \end{cases} \quad \dots \quad (S')$$

- We must Compute the eigenvectors:

$$\begin{cases} \lambda_1 = -1 \longrightarrow \begin{cases} v_1 = (-1, 1, 0) \\ v_2 = (-1, 0, 1) \end{cases} \\ \lambda_2 = 2 \longrightarrow v_3 = (1, 1, 1) \end{cases}$$

Then

$$X(t) = c_1 \cdot e^{-t} v_1 + c_2 e^{-t} v_2 + c_3 e^{2t} v_3$$

$$= c_1 e^{-t} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + c_3 e^{2t} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},$$

where $c_1, c_2, c_3 \in \mathbb{R}$.

Hence,

$$\begin{cases} x = -c_1 e^{-t} - c_2 e^{-t} + c_3 e^{2t} \\ y = c_1 e^{-t} + c_3 e^{2t} \\ z = c_2 e^{-t} + c_3 e^{2t} \end{cases}$$

End

20 November 2021
By Bellaouar. Dj.

Some results with proof

1) If A is diagonalizable $\Rightarrow$ $\det(A) = \prod \lambda_i$.

$0 \notin Sp(A) \Leftrightarrow A^{-1}$ exists

2) (λ, x) eigenpair of $A \Rightarrow (e^{\lambda}, x)$ eigenpair of e^{A}.

3) $A = PDP^{-1} \Rightarrow e^{A} = P e^{D} P^{-1}$, $f(A) = P f(D) P^{-1}$

4) A nilpotent + diagonalizable $\Rightarrow A = 0$

5) A diagonalizable + $Sp(A) = \{\lambda\} \Rightarrow A = \lambda . I$.

6) A, B diagonalizable

with $\left.\begin{array}{l} A = P D_1 P^{-1} \\ B = P D_2 P^{-1} \end{array}\right\} \Rightarrow AB = BA.$

7) $A . e^{A} = e^{A} . A$, 8) A is idempotent $\Rightarrow e^{A} = I + (e-1).A$

9) $P_A(A) = 0$ (zero matrix): Cayley-Hamilton Theorem.

10) If A is strictly triangular $\Rightarrow$ A is nilpotent

11) $\sum \lambda_i = tr(A)$ and $\det(e^{A}) = e^{tr(A)}$.

12) N is nilpotent $\Rightarrow Sp(N) = \{0\}$

13) N is nilpotent $\Rightarrow I - N$ is invertible i.e $(I-N)^{-1}$ exists.

14) $A \cdot B = B \cdot A \Rightarrow e^{A+B} = e^{A} \cdot e^{B} = e^{B} \cdot e^{A}$.

15) $A \sim 2A \Rightarrow A$ is nilpotent 16) $(e^{A})^{-1} = e^{-A}$.

17) If $AB = BA$, then

$$(A+B)^n = \sum_{i=0}^{n} C_n^i A^i B^{n-i} = \sum_{i=1}^{n} C_n^i A^{n-i} B^i$$

(Binomial formula)

18) $A . (com(A))^t = \det(A) . I_n$

19) $\forall A \in \mathcal{M}_n(\mathbb{C})$: $A = P . T . P^{-1}$, where P is invertible, T is an upper triangular matrix.

20) (A is diagonalizable + $Sp(A) = \{-1, 1\}$) $\Rightarrow A^{-1} = A$

Test. Exam. Algebra III 23. 02. 2021

Ex 01: why? Finding powers of diagonalizable matrices is easy!

Let $A = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}$

Find an expression for A^k for any positive integer k.

Let $B = \begin{bmatrix} -2 & 2 & 1 \\ 0 & -2 & 1 \\ 0 & 0 & -2 \end{bmatrix}$ Compute B^k for $k \geq 1$.

Deduce the minimal polynomial of the matrix:

$$C = \begin{bmatrix} a & x & x & \cdots & x \\ & a & x & \cdots & x \\ & & \ddots & x & x \\ & 0 & & a & x \\ & & & & a \end{bmatrix} \longrightarrow a_{ij} > 0 \quad (i<j)$$

$a \neq 0$

Exo2: Give the proof of Cayley-Hamilton Theorem for a matrix $A \in \mathcal{M}_2(\mathbb{R})$.

(Use the Lemma: $A \cdot {}^t\mathrm{Com}(A) = \det(A) \cdot I_2$)

- Let A, B be two square matrices such that $A = P \cdot D_1 \cdot P^{-1}$ and $B = P\, D_2 \cdot P^{-1}$, where D_1, D_2 are diagonal. Show that:

$$e^{A+B} = e^{A} \cdot e^{B}$$

Good. Luck..

Test- Exam N° 02 04. 03. 2021

Ex 01: Consider the matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 5 & 0 & 0 \end{bmatrix}$$

Verify that $A^3 = 5I$, and deduce the expression of A^{-1}.

Ex 02: Let

$$A = \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 5 & 2 \\ 4 & 1 \end{bmatrix}$$

show that $A \not\sim B$.

Ex 03: Let $A, B \in M_n(\mathbb{R})$. prove the following results:

1. $A \sim B \Rightarrow A^k \sim B^k$ for $k \geq 1$.
2. $A \sim B \Rightarrow e^A \sim e^B$
3. $\det(A) = \prod \lambda_i$ (λ_i is an eigenvalue)
4. $\det(e^A) = e^{tr(A)}$ ($tr \equiv$ the trace)

Ex 04: Solve the system of recurrence sequences:

$$\begin{cases} x_{n+1} = a_{11} x_n + a_{12} y_n + c_1 \\ y_{n+1} = a_{21} x_n + a_{22} y_n + c_2 \end{cases} \quad x_0, y_0 \in \mathbb{R}.$$

Ex 05: Let

$$A = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 0 & 1 \\ 1 & -1 & 2 \end{bmatrix}$$

1. Find the minimal polynomial of A.
2. Let $n \in \mathbb{N}$. prove that $A^n = \alpha_n . A + \beta_n . I$, and find α_n, β_n in terms of n.

Good. Luck

Homework N° 03

Ex 01: Consider the matrix:

$$A = \begin{bmatrix} 3 & -1 \\ 2 & 0 \end{bmatrix}$$

prove that A is diagonalizable and find A^k.

Ex 02: Let

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 1 & 2 & 0 \\ -1 & 1 & 3 \end{bmatrix}$$

prove that

$$A^k = \begin{bmatrix} (-1)^k & 0 & 0 \\ \frac{2^k - (-1)^k}{3} & 2^k & 0 \\ \frac{(-1)^k - 2^k}{3} & 3^k - 2^k & 3^k \end{bmatrix}$$

Ex 03: Let (x_n), (y_n) be two sequences given by:

$$\begin{cases} x_{n+1} = 3x_n - y_n \;; & x_0 = 1 \\ y_{n+1} = 2x_n \;; & y_0 = -1 \end{cases}$$

Find x_n et y_n in terms of n.

End

Bellaouar. Dj.

Homework N° 04 : 21 | Jan 2021.

Ex 01: Consider the matrix :

$$A = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 0 \\ 2 & 0 & -1 \end{bmatrix}.$$

prove that $P_A(x) = x^3 + 1$, and deduce A^{-1}.

Ex 02: Let

$$A = \begin{bmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 4 \end{bmatrix}.$$

Compute the minimal polynomial of A.

Ex 03: Let $t \in \mathbb{R}^*$ and let

$$A_t = \begin{bmatrix} 0 & -t \\ t & 0 \end{bmatrix}.$$

Show that: $$e^{A_t} = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}.$$

Ex 04: Solve the system of linear differential equations :

$$\begin{cases} x' = -x + 2y \\ y' = 2x - y. \end{cases}$$

Good-luck;

passing Exam of Algebra III, 08-07-2021

Ex 01 : Trigonalize the following matrix:

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}$$

and Compute A^n for $n \geq 0$.

Ex 02 : study the diagonalization of the matrix:

$$A = \begin{bmatrix} 1 & a & b & c \\ 0 & 1 & d & e \\ 0 & 0 & -1 & f \\ 0 & 0 & 0 & 2 \end{bmatrix}, \quad a, b, c, d, e, f \in \mathbb{R}.$$

Ex 03 : Consider the n by n matrix given by:

$$A_n = \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & -1 \\ & & & -1 & 2 \end{bmatrix}$$

Compute $\det(A_n)$ in terms of n.

Ex 04 : Let $A \in \mathcal{M}_n(\mathbb{R})$. Prove the following properties:

1) $\det(A) = 0 \Rightarrow 0$ is an eigenvalue of A.
2) $A^2 = A \Rightarrow A$ is diagonalizable.
3) (A is diagonalizable + nilpotent) $\Rightarrow A = 0$.

Good luck.