Eigenvalues and Eigenvectors

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Throughout this chapter $\mathbb{K}$ denotes the field $\mathbb{R}$ or $\mathbb{C}$, and $\mathcal{M}_n(\mathbb{K})$ denotes the vector space of n by n matrices over $\mathbb{K}$.

Definition

Let A be an $n \times n$ square matrix. When $Ax = \lambda x$ has a non-zero vector solution x, then

- λ is called an **eigenvalue** of A.
- x is called an **eigenvector** of A corresponding to λ.
- The couple (λ, x) is called an **eigenpair** of A.

Notes: (i) eigenvector must be non-zero. **(ii)** But, eigenvalue λ can be zero, can be non-zero.
A vector $x \in E$ is an eigenvector of A if

- x is non-zero,
- there exists $\lambda \in \mathbb{K}$, $Ax = \lambda x$.

The **eigenspace** of A corresponding to λ is the subspace:

$$E_\lambda = \{v \in \mathbb{K}^n \; ; \; Av = \lambda v\}.$$

Note that E_λ is a vector subspace of $\mathbb{K}^n$. This is the **kernel** of the matrix $A - \lambda I_n$. So E_λ consists of all solutions v of the equation $Av = \lambda v$. In other words, E_λ consists of all eigenvectors with eigenvalue λ, together with the zero vector.

Example

Let $A = I_2$. Then any non-zero vector v of $\mathbb{R}^2$will be an eigenvector of A corresponding to eigenvalue $\lambda = 1$.

Example

Consider the matrix

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}.$$

Calculate the eigenvalues and eigenvectors of A.

Solution.

- First, we find the eigenvalues of A.

We start with calculating the characteristic polynomial of A. From definition, we obtain

$$\begin{aligned} p_A(x) &= \begin{vmatrix} 2-x & 1 \\ 1 & 2-x \end{vmatrix} \quad \begin{matrix} c_1 \\ \downarrow \\ c_1+c_2 \end{matrix} \quad \text{(the first column } c_1 \text{ becomes } c_1+c_2\text{)} \\ &= \begin{vmatrix} (3-x) & 1 \\ (3-x) & 2-x \end{vmatrix} = (3-x)\begin{vmatrix} 1 & 1 \\ 1 & 2-x \end{vmatrix} = (3-x)(2-x-1) \\ &= (3-x)(1-x). \end{aligned}$$

Hence, $p_A(x) = (1-x)(3-x)$, and so the eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = 3$.

- Second, we find the eigenvectors. By definition, the eigenspace E_{λ_1} is given by

$$\begin{aligned} E_{\lambda_1} &= \left\{(x,y) \in \mathbb{R}^2;\ \begin{array}{l} x+2y=x \\ 2x+y=y \end{array}\right\} \\ &= \left\{(x,y) \in \mathbb{R}^2;\ y=-x\right\} \\ &= Vect\left\{(1,-1)\right\}. \end{aligned}$$

  Thus, $v_1 = (1,-1)$.

- Using the same manner, the eigenspace E_{λ_2} is given by

$$\begin{aligned} E_{\lambda_2} &= \left\{(x,y) \in \mathbb{R}^2;\ \begin{array}{l} x+2y=3x \\ 2x+y=3y \end{array}\right\} \\ &= \left\{(x,y) \in \mathbb{R}^2;\ y=x\right\} \\ &= Vect\left\{(1,1)\right\}. \end{aligned}$$

  That is, $v_2 = (1,1)$.

Definition

- The **geometric multiplicity** for a given eigenvalue λ, denoted by $G_m(\lambda)$, is the dimension of the eigenspace E_λ. That is,

$$G_m(\lambda) = \dim E_\lambda.$$

- The **algebraic multiplicity** for a given eigenvalue λ, denoted by $A_m(\lambda)$, is the number of times the eigenvalue is repeated. For example, if the characteristic polynomial is $(x-1)^2(x-5)^3$ then for $\lambda = 1$ the algebraic multiplicity is 2 and for $\lambda = 5$ the algebraic multiplicity is 3.

Note that the algebraic multiplicity is greater than or equal to the geometric multiplicity. That is, we always have $A_m(\lambda) \geq G_m(\lambda)$.

Examples. Calculate eigenvalues and eigenvectors of the following matrices. Deduce the algebraic multiplicity and the geometric multiplicity of each eigenvalue of A.

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}.$$

Ans. We have $\lambda_1 = 4,\ v_1 = (2, 3)$ and $\lambda_2 = -1,\ v_2 = (1, -1)$.

$$A = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

Ans. We have $\lambda_1 = e^{i\theta},\ v_1 = (-i, 1)$ and $\lambda_2 = e^{-i\theta},\ v_2 = (i, 1)$.

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 5 \end{pmatrix}.$$

Ans. We have $\lambda_1 = 1,\ E_1 = Vect\left\{(1,0)\right\}$ and $\lambda_2 = 5,\ E_5 = Vect\left\{(1,2)\right\}.$

$$A = \begin{pmatrix} 2 & 6 \\ 0 & 2 \end{pmatrix}.$$

Ans. We have $\lambda = 2$ (double), $E_\lambda = Vect\left\{(1,0)\right\}.$

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & -5 \end{pmatrix}.$$

Ans. We have $\lambda_1 = 1,\ E_1 = Vect\left\{(1,0,0)\right\},\ \lambda_2 = 2,\ E_2 = Vect\left\{(2,1,0)\right\}$ and $\lambda_3 = -5,\ E_{-5} = Vect\left\{(5,6,-14)\right\}.$

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}.$$

Ans. We have $\lambda_1 = 1$, $E_{\lambda_1} = Vect\left\{(-1,1,1)\right\}$, $\lambda_2 = 2$ (the algebraic multiplicity of λ_2 is 2), $E_{\lambda_2} = Vect\left\{(0,1,0),(0,0,1)\right\}$. Also, the geometric multiplicity of λ_2 is 2.

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Ans. We have $\lambda = 0$ (triple eigenvalue), $E_\lambda = Vect\left\{(1,0,0),(0,1,-1)\right\}$. The eigenspace corresponding to $\lambda = 0$ is of dimension 2.

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 3 & 2 \end{pmatrix}.$$

Ans. We have $\lambda = 2$ (triple eigenvalue), $E_\lambda = Vect\left\{(0,0,1)\right\}$. The eigenspace corresponding to $\lambda = 2$ is of dimension 1.

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Ans. We have $\lambda_1 = 0$ (simple eigenvalue), $E_{\lambda_1} = Vect\left\{(-1, 1, 0)\right\}$ and $\lambda_2 = 2$ (double eigenvalue), $E_{\lambda_2} = Vect\left\{(0, 0, 1), (1, 1, 0)\right\}$. The eigenspace corresponding to λ_1 is of dimension 1 and the eigenspace corresponding to $\lambda_2 = 2$ is of dimension 2.

$$A = \begin{pmatrix} a & 2 & 3 \\ 0 & 2a & 8 \\ 0 & 0 & 3a \end{pmatrix}; \ a \in \mathbb{R}.$$

Ans. We have $\lambda_1 = a$ and $E_{\lambda_1} = Vect\left\{(1, 0, 0)\right\}$, $\lambda_2 = 2a$ and $E_{\lambda_2} = Vect\left\{\left(\frac{2}{a}, 1, 0\right)\right\}$, $\lambda_3 = 3a$ and $E_{\lambda_3} = Vect\left\{\left(\frac{1}{2a^2}(3a + 16), \frac{8}{a}, 1\right)\right\}$.

Corollary

Let (λ, x) be an eigenpair of A. Then $\left(\lambda^k, x\right)$ is an eigenpair of A^k.

Proof.

In fact, we see that

$$Ax = \lambda x \Rightarrow A^2 x = A(\lambda x) = \lambda A x = \lambda^2 x.$$

Therefore,

$$Ax = \lambda x \Rightarrow \forall\ k \geq 0 : A^k x = \lambda^k x.$$

The result is proved. □

Corollary

Let A be an invertible matrix and let (λ, x) be an eigenpair of A with $\lambda \neq 0$. Then $\left(\frac{1}{\lambda}, x\right)$ is an eigenpair of A^{-1}.

Proof.

By definition, we have

$$\begin{aligned} A^{-1}x &= A^{-1}(1.x) = A^{-1}\left(\frac{\lambda}{\lambda}.x\right) = \frac{1}{\lambda}A^{-1}(\lambda x) \\ &= \frac{1}{\lambda}A^{-1}(Ax) \quad (\text{since } Ax = \lambda x) \\ &= \frac{1}{\lambda}x. \end{aligned}$$

Thus, $A^{-1}x = \frac{1}{\lambda}x$. The proof is finished. □

Ex 01. Calculate the eigenvalues and eigenvectors of the following matrix:

$$A = \begin{pmatrix} -3 & 1 & -1 \\ -7 & 5 & -1 \\ -6 & 6 & -2 \end{pmatrix}.$$

Ans. $\lambda_1 = -2$, $v_1 = (1, 1, 0)$ and $\lambda_2 = 4$, $v_2 = (0, 1, 1)$.

Ex 02. Let $P \in \mathbb{GL}_n(\mathbb{R})$ and let D be the following diagonal matrix:

$$D = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}$$

Calculate the eigenpairs of D, then deduce the eigenpairs of the matrix PDP^{-1}.

Ex 03. Let $A \in \mathcal{M}_n(\mathbb{R})$ and $\alpha \in \mathbb{R}^*$. Prove that

$$v \text{ is an eigenvector of } A \Rightarrow \alpha v \text{ is also an eigenvector of } A.$$

Ex 04. Let $A \in \mathcal{M}_n(\mathbb{R})$ and λ_1, λ_2 be two eigenvalues of A with $\lambda_1 \neq \lambda_2$. Prove that

$$E_{\lambda_1} \cap E_{\lambda_2} = \{0_{\mathbb{R}^n}\}.$$

Recall that $E_\lambda = \{x \in \mathbb{R}^n \; ; \; Ax = \lambda x\}$.